C3.4 ALGEBRAIC GEOMETRY - EXERCISE SHEET 3

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(1) **Group actions**

(a) Let $G = \mathbb{Z}/3\mathbb{Z} = \{1, w, w^2\}$, where w is a cube root of unity. Let G act on $\mathbb{A}^2$ via $w(x, y) = (wx, wy)$. Can you recognise the categorical quotient of $\mathbb{A}^2$ by G? What do you get more generally for the analogous action of $\mathbb{Z}/d\mathbb{Z}$ on $\mathbb{A}^{n+1}$?

(b) Let $G = k^*$ act on $\mathbb{A}^2$ via $t(x, y) = (tx, y)$ for all $t \in k^*$. Find the categorical quotient Y of $\mathbb{A}^2$ by G, giving the map $\mathbb{A}^2 \to Y$ explicitly.

(2) **Dimension, Degree and Hilbert Polynomials.**

(a) For any variety[1] X, show that $\dim \mathcal{O}_{X,p} = \dim_p X$.

(b) Show that, if X is a reducible projective variety with equidimensional irreducible components X_i then $\deg X = \sum_i \deg X_i$.

(c) Find the Hilbert polynomial of $\nu_d(\mathbb{P}^n)$ (as a binomial coefficient), and verify that this projective variety has dimension n and degree d^n.

(d) Compute the degree[2] $\deg \nu_d(\mathbb{P}^1)$ in two ways: either directly, or by using the Hilbert polynomial.

(e) Let F be a homogeneous irreducible polynomial of degree d in $k[x_0, \ldots, x_n]$, and let $X := \mathbb{V}(F) \subset \mathbb{P}^n$. Find the Hilbert polynomial of X and deduce that, as expected, $\dim X = n - 1$ and $\deg X = d$.

(3) **Affine quasi-projective varieties, and Dimension.**

(a) Find an open affine cover of $\mathbb{A}^2 \setminus \{(0, 0)\}$.

(b) Show that $\mathrm{GL}(n, k)$ is an affine variety.[3]

(c) Let X be an affine variety and $f \in k[X]$. Show that f vanishes nowhere on X if and only if f is invertible in $k[X]$.

(4) **Localization.** Let R be a ring.[4]

(a) Let S be a multiplicative subset of R. Show that the localization $S^{-1}R$ is naturally isomorphic, as an R-algebra, to[5]

$$R[x_s : s \in S]/\langle s x_s - 1 : s \in S\rangle.$$

(b) Consider the elliptic curve $X \subset \mathbb{A}^2$ defined by $y^2 = x(x-1)(x-2)$. Find the localisation[6] of the coordinate ring $k[X]$ at the point $p = (1, 0) \in X$. Show that

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Date: This version of the notes was created on November 20, 2017.
These exercise sheets were inherited from Gergely Bérczi.

[1] If you like, you may assume (since we can pass to a local affine patch) that X is an affine variety and you may use the theorem that $\mathcal{O}_{X,p} = k[X]_{\mathfrak{m}_p}$ where $\mathfrak{m}_p$ is the maximal ideal corresponding to the point $p \in X$.

[2] Recall $\nu_d(\mathbb{P}^1)$ is the rational normal curve of degree d.

[3] in the sense that it is isomorphic (as a quasi-proj. variety) to a Zariski closed subset of an affine space.

[4] as usual, commutative with unit.

[5] intuitively we are formally inverting each element of S because we introduce a new abstract variable x_s for each $s \in S$, and we impose the relations $s x_s = 1$. *Hint. To prove you get an isomorphism, it is easiest to try to guess the definition of the inverse map, and show that the two composites give identity maps.*

[6] *Hint/Definition. You need to study* $k[X]_{\mathfrak{m}_p}$, *so localize at* $S = k[X] \setminus \mathfrak{m}_p$.

the (unique) maximal ideal of this local ring is generated by y. What does this mean geometrically?[7]

(c) Suppose $r \in R$ is zero in each local ring: $\frac{r}{1} = 0 \in R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m} \subset R$. Deduce[8] that $r = 0 \in R$.

(d) Deduce from (c) that for any affine variety X, if two functions $f, g : X \to k$ have equal local germs[9] at each point of X, then they are globally equal: $f = g$.

(5) **Structure sheaf: computing $\mathcal{O}_{X,p}$ and $\mathcal{O}_X(U)$.**

(a) Let $X = \mathbb{V}(xy) \subset \mathbb{A}^2$ be the union of the two axes. Show algebraically that $\mathcal{O}_{X,(0,1)} \cong k[y]_{(y-1)}$, and then explain geometrically why you are not surprised that this is the local ring of $\mathbb{A}^1$ at 1. Let $\mathfrak{m}_p$ be the maximal ideal of $\mathcal{O}_{X,p}$. Compare, as you vary $p \in X$, the dimensions of the k-vector spaces[10]

$$\mathfrak{m}_p/\mathfrak{m}_p^2.$$

Deduce that the k-algebra $\mathcal{O}_{X,(0,0)}$ cannot be generated by one function.

(b) Find $\mathcal{O}_X(U)$, where $X = \mathbb{P}^2$ and

$$U = X \setminus \mathbb{V}(x_0^2 + x_1^2 + x_2^2).$$

Hint. First compute $\mathcal{O}_X(U \cap U_i)$ for the usual cover of $\mathbb{P}^n$ by affine charts U_i. Then consider the restriction of functions to overlaps $U \cap U_i \cap U_j$.

Cultural Remark. $\mathfrak{m}_p/\mathfrak{m}_p^2$ is the *cotangent space* to X at p (its dual vector space is called *tangent space*). When the dimension of this vector space is not equal to the dimension of X, then p is called a *singularity*. Non-singular points are called *regular*. In algebra, for a Noetherian local ring A with maximal ideal $\mathfrak{m}$, one says that A is *regular* if $\dim A$ equals the vector space dimension $\dim_k \mathfrak{m}/\mathfrak{m}^2$ (in which case, that number equals the minimal number of generators needed to generate the ideal $\mathfrak{m}$).

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[7] i.e. interpret this result in terms of germs of functions at p defined on the curve.

[8] *Hint. Recall that the annihilator* $\operatorname{Ann} r = \{a \in R : ar = 0\}$ *is an ideal in* R.

[9] i.e. $f = g$ inside $\mathcal{O}_{X,p}$.

[10] For an ideal I, by I^2 one means the ideal generated by all $i \cdot j$, for $i, j \in I$.