C3.4 ALGEBRAIC GEOMETRY - EXERCISE SHEET 4

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(1) **Tangent Spaces.**
  (a) Show that if char(k) does not divide d then the hypersurface $\mathbb{V}(x_0^d + \ldots + x_n^d) \subset \mathbb{P}^n$ is nonsingular.
  (b) By computing the dimension of tangent spaces at various points show that the varieties $\mathbb{V}(xy(x-y)) \subset \mathbb{A}^2$ and $\mathbb{V}(xy, yz, zx) \subset \mathbb{A}^3$ are not isomorphic.

(2) **From infinitesimal to global.** Let X and Y be irreducible affine varieties. Suppose $F : X \to Y$ is a morphism of affine varieties which is a homeomorphism in the Zariski topology and assume[1] it induces isomorphisms $F_x^* : \mathcal{O}_{Y,F(x)} \to \mathcal{O}_{X,x}$ for every $x \in X$.
  (a) Show that $F^* : k[Y] \to k[X]$ is injective.
  (b) Suppose $g \in k[X]$. Show that, for all $x \in X$, there exists an open neighbourhood V_x of $F(x)$ and a regular function $(V_x, h_x) \in \mathcal{O}_{Y,F(x)}$ such that $F^*(V_x, h_x) = (F^{-1}(V_x), g|_{F^{-1}(V_x)})$. Show that $h_x = h_{x'}$ in $V_x \cap V_{x'}$ and conclude that there exists $h \in k[Y]$ such that $F^*h = g$.
  (c) Now let W and Z be irreducible quasi-projective varieties, and suppose that $G : W \to Z$ is a morphism of quasi-projective varieties which is a homeomorphism in the Zariski topology, such that for every $w \in W$ the ring homomorphism $G_w^* : \mathcal{O}_{Z,G(w)} \to \mathcal{O}_{W,w}$ is an isomorphism. Show that G is an isomorphism.

(3) **Function field.** Let X be an irreducible affine variety. Show that:
  (a) $\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ are subrings of $k(X)$.
  Hint. You need to explain first how to include $\mathcal{O}_X(U) \subset k(X)$ and $\mathcal{O}_{X,p} \subset k(X)$.
  (b) Restriction maps are inclusions: if $U \subset V$ then $\mathcal{O}_X(V) \subset \mathcal{O}_X(U) \subset k(X)$.
  (c) $\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V) \subset k(X)$.
  (d) $\mathcal{O}_X(U) = \bigcap \mathcal{O}_{X,p} \subset k(X)$, taking the intersection over all $p \in U$.
  (e) $\mathcal{O}_X(U) = \bigcap \mathcal{O}_X(D_h)$, taking the intersection over all $D_h \subset U$.

(4) **Rational and birational maps.**
  (a) Let $F : X \dashrightarrow Y$ be a rational map of quasi-projective varieties, with X irreducible. If (U, f) is a representation for F, with U affine, show that $\{(u, f(u)) : u \in U\} \subset U \times Y$ is a closed subvariety. Conclude that the projection from the graph[2] $\Gamma_F \to X$ is a birational equivalence.
  (b) Define $F : \mathbb{A}^2 \to \mathbb{A}^1$ by $F(x, y) = \frac{y}{x}$. Find the equation defining $\Gamma_F \subset \mathbb{A}^3$.
  (c) Show that the curve $\mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{A}^2$ is rational.[3]

(5) **Resolution of singularities.** Desingularise[4] $\mathbb{V}(y^2 - x^4 - x^5) \subset \mathbb{A}^2$. Draw a picture of the series of blow-ups.

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[1] The second assumption is not automatic. Indeed recall the standard example from lectures: $\mathbb{A}^1 \to \mathbb{V}(y^2 - x^3) = \{(t^2, t^3) : t \in k\}$, $t \mapsto (t^2, t^3)$ is a homeomorphism, but it cannot induce isomorphisms on stalks, otherwise by this exercise the two varieties would have isomorphic coordinate rings which we know is false.

[2] Recall that the *graph* Γ_F is defined to be the closure of $\{(u, f(u)) : u \in U\}$.

[3] Rational means birationally equivalent to $\mathbb{P}^1$. *Hint. Try blowing up the curve at* 0*, and consider the map that projects to the exeptional divisor.*

[4] Recipe: find the singularities, blow them up, analyse the result and if necessary blow up again.