MODEL SOLUTIONS FOR PROBLEM SHEET 4

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(1) (a) *Show that if char(k) does not divide d then the hypersurface* $\mathbb{V}(x_0^d + \ldots + x_n^d) \subset \mathbb{P}^n$ *is nonsingular.*

Solution. Direct computation starting from the definition. Additional details: $d_pF = \sum dp_j^{d-1} \cdot (x_j - p_j)$, for this to vanish we would need $p_j = 0$ for all j (since $d \neq 0 \in k$), but $p = 0$ is not allowed in $\mathbb{P}^n$.

(b) *By computing the dimension of tangent spaces at various points show that the varieties* $\mathbb{V}(xy(x-y)) \subset \mathbb{A}^2$ *and* $\mathbb{V}(xy, yz, zx) \subset \mathbb{A}^3$ *are not isomorphic.*

Solution. Geometrically the first case consists of three lines meeting at 0 in $\mathbb{A}^2$, the second case consists of three lines meeting at 0 in $\mathbb{A}^3$. Away from the point where the lines meet, we get the same tangent space as that for $\mathbb{A}^1$, so $\dim = 1$. But at the point where they meet, the span of the tangent lines gives $\mathbb{A}^2$ in the first case, but $\mathbb{A}^3$ in the second case. Finally, to conclude, recall that the tangent space is intrinsic, independent of the embedding into affine space (in particular it is an isomorphism invariant).

Alternatively, a more computational approach: for $p = (p_1, p_2) \in \mathbb{A}^2$

$$T_p\mathbb{V}(xy(x-y)) = \mathbb{V}((2p_1p_2 - p_2^2)(x - p_1) + (p_1^2 - 2p_1p_2)(y - p_2))$$

and for $p = (p_1, p_2, p_3) \in \mathbb{A}^3$, the definition of $T_p\mathbb{V}(xy, yz, zx)$ is:

$$\mathbb{V}(p_2(x-p_1) + p_1(y-p_2), p_3(y-p_2) + p_2(z-p_3), p_3(x-p_1) + p_1(z-p_3)).$$

So $\dim T_{(0,0,0)}\mathbb{V}(xy, yz, zx) = 3$. If $F : \mathbb{V}(xy, yz, zx) \to \mathbb{V}(xy(x-y))$ was an isomorphism, this would induce an isomorphism

$$T_{(0,0,0)}\mathbb{V}(xy, yz, zx) \to T_{F(0,0,0)}\mathbb{V}(xy(x-y)),$$

but the latter has dimension ≤ 2.

(2) *Let* X *and* Y *be irreducible affine varieties. Suppose* $F : X \to Y$ *is a morphism of affine varieties which is a homeomorphism in the Zariski topology and assume*[1] *it induces isomorphisms* $F_x^* : \mathcal{O}_{Y,F(x)} \to \mathcal{O}_{X,x}$ *for every* $x \in X$.

(a) *Show that* $F^* : k[Y] \to k[X]$ *is injective.*

(b) *Suppose* $g \in k[X]$. *Show that, for all* $x \in X$, *there exists an open neighbourhood* V_x *of* $F(x)$ *and a regular function* $(V_x, h_x) \in \mathcal{O}_{Y,F(x)}$ *such that* $F^*(V_x, h_x) = (F^{-1}(V_x), g|_{F^{-1}(V_x)})$. *Show that* $h_x = h_{x'}$ *in* $V_x \cap V_{x'}$ *and conclude that there exists* $h \in k[Y]$ *such that* $F^*h = g$.

(c) *Now let* W *and* Z *be irreducible quasi-projective varieties, and suppose that* $G : W \to Z$ *is a morphism of quasi-projective varieties which is a homeomorphism in the Zariski topology, such that for every* $w \in W$ *the ring homomorphism* $G_w^* : \mathcal{O}_{Z,G(w)} \to \mathcal{O}_{W,w}$ *is an isomorphism. Show that* G *is an isomorphism.*

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[1] The second assumption is not automatic. Indeed recall the standard example from lectures: $\mathbb{A}^1 \to \mathbb{V}(y^2 - x^3) = \{(t^2, t^3) : t \in k\}$, $t \mapsto (t^2, t^3)$ is a homeomorphism, but it cannot induce isomorphisms on stalks, otherwise by this exercise the two varieties would have isomorphic coordinate rings which we know is false.

Solution. Cultural Remark: a high-tech viewpoint of this exercise is that we have sheaves on W, Z whose stalks are isomorphic, so the sheaves are isomorphic, and as it is also a homeomorphism on spaces we have an isomorphism of ringed spaces.

(a) F^* is injective since F is dominant (indeed F is surjective), using Ex.Sheet.1. Alternatively: F is a morphism of affine varieties, so F^* is the composition
$$k[Y] \hookrightarrow \mathcal{O}_{Y,F(x)} \to \mathcal{O}_{X,x} \supset k[X].$$
The image sits in $k[X]$ since the image of $f \in k[Y]$ is the composition of f with a polynomial map F. The composition is injective as it is a composition of injective maps.

(b) The first part follows from the fact that $g \in k[X] \subset \mathcal{O}_{X,x}$ and F_x^* is surjective. Since $F^*h_x = g|_{F^{-1}(V_x)}$, $F^*h_{x'} = g|_{F^{-1}(V_{x'})}$, we have the equality $F^*h_x = F^*h_{x'}$ on $F^{-1}(V_x \cap V_{x'})$. But F^* is injective, so $h_x = h_{x'}$ on $V_x \cap V_{x'}$.
Technical remark: strictly speaking, we only know F^* is injective on stalks $\mathcal{O}_{X,p}$, so we get $h_x = h_{x'}$ on a possibly smaller open $p \in U_p \subset V_x \cap V_{x'}$, but that is true for all $p \in V_x \cap V_{x'}$ hence $h_x = h_{x'}$ on all of $V_x \cap V_{x'}$.
Finally, these glue together to give a global regular function $h \in \mathcal{O}_Y(Y)$. But Y is affine, so $\mathcal{O}_Y(Y) = k[Y]$ holds.

(c) Choose an affine open cover of W and Z, such that G is given on these affine patches with regular maps. Apply part a) and b) for these morphisms separately, then we get that G induces an injective and surjective morphism between the coordinate rings of the affine charts, which is an isomorphism by the affine Nullstellensatz. So G is an isomorphism of quasi-projective varieties.
Technical remark: That argument takes for granted that G(affine open) is an affine open. This is somewhat tricky to prove using this course's methods. Here is a possible proof. Suppose $G(w) = z$. Pick an affine open $z \in B \subset Z$, then $G^{-1}(B) \subset W$ is open. Pick an affine open $w \in A \subset G^{-1}(B) \subset W$, then $z \in G(A) \subset B \subset Z$ is open. Since B is affine, we can pick a regular function $h : B \to k$ defining a basic open set $z \in D_h \subset G(A) \subset B \subset Z$. Notice $h \circ G : A \to k$ is regular (since by definition of morph of q.p.vars., G is a regular map on small affine opens, and we can shrink B if necessary). Then notice $G^{-1}(D_h) = G^{-1}(G(A) \cap (h \neq 0)) = A \cap (h \circ G \neq 0) = D_{h\circ G} \subset A$ is a basic open in A, containing w. Thus we finally found open affines around w, z with $G : D_{h\circ G} \to D_h$ homeo, so part (b) can be applied.

(3) *Let X be an irreducible affine variety. Show that:*

(a) *$\mathcal{O}_X(U)$ and $\mathcal{O}_{X,p}$ are subrings of $k(X)$.*
Hint. You need to explain first how to include $\mathcal{O}_X(U) \subset k(X)$ and $\mathcal{O}_{X,p} \subset k(X)$.

(b) *Restriction maps are inclusions: if $U \subset V$ then $\mathcal{O}_X(V) \subset \mathcal{O}_X(U) \subset k(X)$.*

(c) *$\mathcal{O}_X(U \cup V) = \mathcal{O}_X(U) \cap \mathcal{O}_X(V) \subset k(X)$.*

(d) *$\mathcal{O}_X(U) = \bigcap \mathcal{O}_{X,p} \subset k(X)$, taking the intersection over all $p \in U$.*

(e) *$\mathcal{O}_X(U) = \bigcap \mathcal{O}_X(D_h)$, taking the intersection over all $D_h \subset U$.*

Solution.

(a) By the lecture notes, given $f \in \mathcal{O}_X(U)$, one can pass to a basic open set $D_h \subset U$ where $f = \frac{g}{h}$ holds for some $h, g \in k[X]$. Thus $\frac{g}{h} \in k(X) \equiv \operatorname{Frac} k[X]$. That this gives a well-defined inclusion $\mathcal{O}_X(U) \subset k(X)$ is because if two rational functions $\frac{g}{h}, \frac{g'}{h'} : X \dashrightarrow k$ both equal the rational function $f : X \dashrightarrow k$, then $gh' - g'h$ vanishes on a non-empty open (hence dense, by irreducibility) subset of X and hence everywhere (by continuity). Thus $gh' - g'h = 0 \in k[X]$, thus

$\frac{g}{h} = \frac{g'}{h'} \in k(X)$ by definition of $\operatorname{Frac} k[X]$. Thus $\mathcal{O}_X(U) \subset k(X)$, and that it is a subring follows immediately: e.g. for the product, given $f = \frac{g}{h}, f' = \frac{g'}{h'}$, then restriction to $D_{hh'}$ gives $f = \frac{gh'}{hh'}, f' = \frac{g'h}{hh'}$ and thus $f \cdot f' = \frac{gh'g'h}{hh'hh'} = \frac{gg'}{hh'}$.
Recall an element of $\mathcal{O}_{X,p}$ is some open set $p \in U \subset X$ together with a function $f : U \to k$ regular at p. The latter means $f = \frac{g}{h} : W \to k$ on some (possibly smaller) open $p \in W \subset U \subset X$, with $g, h \in k[X]$, with g never vanishing on W (by the lecture notes one can in fact shrink W further to ensure that $W = D_h$). Thus $f \in \mathcal{O}_X(W)$ and by above we include this into $k(X)$ by taking $\frac{g}{h}$. This association is well-defined: if we get $f = \frac{g}{h} : W \to k$ and $f' = \frac{g'}{h'} : W' \to k$, then they agree on the non-empty (hence dense) open $p \in W \cap W' \subset X$, and thus as before we deduce $gh' - g'h$ vanishes everywhere so $\frac{g}{h} = \frac{g'}{h'} \in k(X)$. The subring property is analogous to before.

(b) Given $f : V \to k$ regular at each point of V, the restricted function $f|_U : U \to k$ will be regular at each point of U (given $p \in U$, we obtain an open $p \in W \subset V$ where we have an equality of functions $f = \frac{g}{h} : W \to k$, with $g, h \in k[X]$ and h never vanishing on W, so the same holds on the open $p \in W \cap U \subset U$). One can also prove this as a consequence of (3d) below, but that's a bit round-about.

(c) The inclusion $\mathcal{O}_X(U \cup V) \subset \mathcal{O}_X(U) \cap \mathcal{O}_X(V)$ follows from (3b), and just says that a regular function $f : U \cup V \to k$ is also regular on each open U, V. For the other direction, we need to globalise: if $f \in \mathcal{O}_X(U), f' \in \mathcal{O}_X(V)$ are equal in $k(X)$, then obtain any local representations $f = \frac{g}{h} : W \to k$, $f' = \frac{g'}{h'} : W' \to k$ near $p \in U \cup V$, and since we intersect in $k(X)$ we are given that $\frac{g}{h} = \frac{g'}{h'} \in k(X)$. Thus $gh' = g'h \in k[X]$ and thus also as functions $X \to k$. Thus $f = \frac{gh'}{hh'} = \frac{g'h}{hh'} = f' : W \cap W' \to k$ (using that hh' never vanishes on $W \cap W'$). Thus the functions f, f' agree at all points of $U \cap V$, so they define a function $U \cup V \to k$. Regularity at each point of $U \cup V$ holds because each point lies in U or V or both, and f is regular on U and f' is regular on V. One can also prove this as a consequence of (3d).

(d) One inclusion, $\mathcal{O}_X(U) \subset \cap \mathcal{O}_{X,p}$, just says that "a function $f : U \to k$ regular on U is regular at each p" (which is true by definition). The other inclusion is less obvious, because we need to globalise by running the same argument as in (3c). Namely, we are given local functions $f_j = \frac{g_j}{h_j} : W_j \to k$ with $g_j, h_j \in k[X]$, h_j never vanishing on W_j. Since we are intersecting the $\mathcal{O}_{X,p}$ inside $k(X)$, we are also given that $\frac{g_j}{h_j} = \frac{g_i}{h_i} \in k(X)$, which means $g_j h_i = g_i h_j$ in $k[X]$, which in turn implies an equality of functions $g_j h_i = g_i h_j : X \to k$. This implies the equality of functions $\frac{g_j}{h_j} = \frac{g_i}{h_i} : W_j \cap W_i \to k$. Thus the $f_j : W_j \to k$ glue to give a well-defined function $f : U \to k$, since they agree on overlaps. That $f \in \mathcal{O}_X(U)$ is now just the statement that f is regular at each p, which we know since each p lies in some W_j where $f = \frac{g_j}{h_j}$ and $h_j \neq 0$ on W_j.

(e) Follows from (3d) by first decomposing $\mathcal{O}_X(D_h) = \cap \mathcal{O}_{X,q}$ intersecting over all $q \in D_h$.

(4) (a) *Let $F : X \dashrightarrow Y$ be a rational map of quasi-projective varieties, with X irreducible. If (U, f) is a representation for F, with U affine, show that $\{(u, f(u)) : u \in U\} \subset U \times Y$ is a closed subvariety. Conclude that the projection from the*

graph[2] $\Gamma_F \to X$ *is a birational equivalence.*

Solution. Assume $Y \subset \mathbb{P}^m$, so F (and f) are given with $m+1$ regular functions. Then $\{(u, f(u))|u \in U\} = \mathbb{V}((y_0 - f_0(x), \dots, y_m - f_m(x)))$. Now recall that $\Gamma_F = \overline{\{(u, f(u))|u \in U\}}$. The inverse of the projection $(u, y) \mapsto u$ is $u \mapsto (u, f(u))$ mapping to Γ_F, and well-defined on U.

(b) *Define* $F : \mathbb{A}^2 \to \mathbb{A}^1$ *by* $F(x, y) = \frac{y}{x}$. *Find the equation defining* $\Gamma_F \subset \mathbb{A}^3$.

Solution. $U = U_x$, then $\Gamma_F = \overline{\{(x, y, y/x) : x \neq 0\}} = \mathbb{V}(xz - y) \subset \mathbb{A}^3$.

(c) *Show that the curve* $\mathbb{V}(y^2 - x^2 - x^3) \subset \mathbb{A}^2$ *is rational.*[3]

Solution. We saw in the lectures that $X = \mathbb{V}(y^2 - x^2 - x^3)$ is birationally equivalent to its blow-up $B_0X = \mathbb{V}(zx - y, z^2 - x - 1) \subset \mathbb{A}^3$. Now the projection on the exceptional divisor gives the birational equivalence with $\mathbb{P}^1$, in coordinates: $\alpha : B_0X \to \mathbb{P}^1, (x, y, z) \mapsto [1 : z]$, whose inverse is ($x, y$ expressible with z on B_0X) $\alpha^{-1} : \mathbb{P}^1 \dashrightarrow B_0X, [a : b] \mapsto ((b/a)^2 - 1, (b/a)^3 - b/a, b/a)$, which is well defined where $a \neq 0$.

(5) *Desingularise*[4] $\mathbb{V}(y^2 - x^4 - x^5) \subset \mathbb{A}^2$. *Draw a picture of the series of blow-ups.*

Solution. $X = \mathbb{V}(y^2 - x^4 - x^5)$.
Singular locus: the Jacobian at $p = (x, y)$ is $J_pF = (-4x^3 - 5x^4, 2y) = 0$ iff $(x, y) = (0, 0), (-4/5, 0)$, but $(-4/5, 0)$ is not on the curve. Blowing up at $(0, 0)$ we get $B_0X = \overline{\{((x, y), [a : b]) : y^2 - x^4 - x^5 = 0, xb = ya, (x, y) \neq (0, 0)\}}$. On the affine chart $\mathbb{A}^3 = \mathbb{A}^2 \times \mathbb{A}^1 \subset \mathbb{A}^2 \times \mathbb{P}^1$ where $a \neq 0$ we introduce the affine coordinate $z = b/a$ and the equations are $y = xb, (xb)^2 - x^4 - x^5 = 0$, determining two components: $\mathbb{V}(x, y)$ (the exceptional divisor) and $\mathbb{V}(y - xb, b^2 - x^2 - x^3 = 0\}$.
Side-remark: notice that the exceptional divisor will always show up inside $\pi^{-1}(X)$ since it is $\pi^{-1}(0)$, but we only care about the points of the exceptional divisor that are needed when taking the closure of $\pi^{-1}(X) \setminus \pi^{-1}(0)$. In the case at hand, we only need one extra point: $B_0X \cap E = \{((0, 0), [1 : 0])\}$).

This is a curve in $\mathbb{A}^3$, isomorphic (via the projection) to $\mathbb{V}(b^2 - x^2 - x^3) \subset \mathbb{A}^2_{x,b}$. Since B_0X does not contain points of the form $((x, y), [0, 1])$, we get that B_0X is affine,

$$B_0X = \mathbb{V}(b^2 - x^2 - x^3) \subset \mathbb{A}^2_{x,b}.$$

This is a singular plane cubic so repeat the blow-up at the origin.
The blow-up is:

$$\{((x, b), [X, B]) \in \mathbb{A}^2 \times \mathbb{P}^1 : b^2 - x^2 - x^3 = 0,\ xB - bX = 0\}$$

On the $X \neq 0$ patch, we use coordinates $[1, B]$ on the $\mathbb{P}^1$: $b = xB$, and $(xB)^2 - x^2 - x^3$ becomes $B^2 - 1 - x = 0$ when[5] we seek the proper transform of B_0X. Taking the closure, we obtain

$$\begin{aligned} B_0B_0X &= \{((x, b), [X, B]) \in \mathbb{A}^2 \times \mathbb{P}^1 : B^2 - X^2 - X^2x = 0,\ xB = bX\} \\ &= \{(x \cdot (1, B), [1, B]) \in \mathbb{A}^2 \times \mathbb{P}^1 : B^2 - 1 - x = 0\}. \end{aligned}$$

The birational map to $\mathbb{P}^1$ is defined by projecting to the $\mathbb{P}^1$ factor.
We remark that B_0B_0X is $\mathbb{V}(b - xB, B^2 - 1 - x) \subset \mathbb{A}^3_{x,b,B}$ and thus is isomorphic to

[2] Recall that the *graph* Γ_F is defined to be the closure of $\{(u, f(u)) : u \in U\}$.

[3] Rational means birationally equivalent to $\mathbb{P}^1$. *Hint. Try blowing up the curve at* 0, *and consider the map that projects to the exceptional divisor.*

[4] Recipe: find the singularities, blow them up, analyse the result and if necessary blow up again.

[5] if $x = 0$ then $b = 0$, but we are considering the preimage of $(x, b) \neq (0, 0)$ and then taking the closure.

the plane quadric $\mathbb{V}(y^2 - x - 1) \subset \mathbb{A}^2_{x,y}$.

Pictures: after the first blow-up: $b^2 - x^2 - x^3$ looks like the letter $\propto$, where the two strands meet at the origin. Before that blow-up, we had $y^2 - x^4 - x^5$, which looks like an $\propto$ that has been squashed at the origin (so it looks flat along the x-direction near the origin). After the second blow-up, $y^2 - x - 1$ looks like the letter $\subset$ (a parabola).