C3.4 ALGEBRAIC GEOMETRY

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Contents

1. Preliminaries

1.1. COURSE POLICY and BOOK RECOMMENDATIONS

C3.4 Course policy: It is essential that you read your notes after each lecture.
You will notice that for most Part C courses, unlike previous years, each lecture builds on the previous. If you don't read the notes then within a lecture or two you may feel lost. For Part C courses, you should not expect every detail to be covered in lectures: often it is up to you to check statements as exercises.
The course assumes familiarity with algebra (or that you are willing to read up on it).
I'm afraid it would be unrealistic to expect commutative algebra to be taught as a subset of this 16-hour course. I write "Fact" if you are not required to read/know the proof (unless we prove it), and it usually refers to: algebra results, or difficult results, or results we don't have time to prove. Algebraic geometry is a difficult and extremely broad subject, and I will do my best to make it digestible. But this will not happen by itself: it requires effort on your part, thinking on your own about the notes, the examples, the exercises.

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RELEVANT BOOKS

Basic algebraic geometry
Reid, *Undergraduate algebraic geometry.* Start from Chp.II.3. (Available online from the author)
Fulton, *Algebraic Curves.* (Available online from the author)
Shafarevich, *Basic Algebraic Geometry.*
Harris, *Algebraic Geometry, A First Course.*
Gathmann, *Algebraic geometry.* (Online notes)

Background on algebra
Atiyah and MacDonald, *Introduction to commutative algebra.*
Reid, *Commutative algebra.*

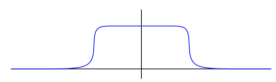
Beyond this course
Mumford, *The Red Book of varieties and schemes.*
Harshorne, *Algebraic geometry.*
Eisenbud and Harris, *Schemes.*
Griffiths and Harris, *Principles of Algebraic Geometry.* (This is complex alg.geom.)
Matsumura, *Commutative ring theory.*
Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry.*
Vakil, *Foundations of algebraic geometry.* (Online notes)

RELATED COURSES
Part C: C2.6 **Introduction to Schemes**, and C3.7 **Elliptic Curves**
It may help to look back at notes from Part B: **Algebraic Curves**, **Commutative algebra**.

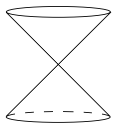
1.2. DIFFERENTIAL GEOMETRY versus ALGEBRAIC GEOMETRY

You may have encountered some differential geometry (DG) in other courses (e.g. B3.2 Geometry of Surfaces). Here are the key differences with algebraic geometry (AG):

(1) In DG you allow all **smooth functions**.
In AG you only allow **polynomials** (or **rational functions**, i.e. fractions poly/poly).
(2) DG is very **flexible**, e.g. you have *bump functions*:



smooth functions which are identically equal to 1 on a neighbourhood of a point, and vanish identically outside of a slightly larger neighbourhood of the point. In particular, two smooth functions which are equal on an open set need not be equal everywhere.
AG is very **rigid**: if a polynomial vanishes on a non-empty open set then it is the zero polynomial. In particular, two polynomials which are equal on a non-empty open set are equal everywhere. AG is however similar to studying holomorphic functions in complex differential geometry: non-zero holomorphic functions of one variable have isolated zeros, and more generally holomorphic functions which agree on a non-empty open set are equal.
(3) DG studies spaces $X \subset \mathbb{R}^n$ or $\mathbb{C}^n$ cut out by smooth equations.
AG studies $X \subset k^n$ cut out by polynomial equations over any field k. AG can study number theory problems by considering fields other than $\mathbb{R}$ or $\mathbb{C}$, e.g. $\mathbb{Q}$ or finite fields $\mathbb{F}_p$.
(4) DG cannot satisfactorily deal with singularities.
In AG, singularities arise naturally, e.g. $x^2 + y^2 - z^2 = 0$ over $\mathbb{R}$ has a singularity at 0.



AG has tools to study singularities.

(5) DG studies **manifolds**: a manifold is a topological space that locally looks like $\mathbb{R}^n$, so you can think of having a copy of a small Euclidean ball around each point. This is an especially nice topology: Hausdorff, metrizable, etc.
AG studies **varieties**. They are topological spaces, but their topology (**Zariski topology**) is not so nice. It is highly non-Hausdorff: for any irreducible[1] variety, any non-empty open set is dense, and any two non-empty open sets intersect in a non-empty open dense set! A variety is locally modeled on k^n. The points of k^n are in 1:1 correspondence with maximal ideals in $R = k[x_1, \dots, x_n]$. The collection of all maximal ideals of R is called $\mathrm{Specm}(R)$, the **maximal spectrum**. The irreducible closed subsets of k^n are in 1:1 correspondence with the prime ideals of R. The collection of all prime ideals of R is called $\mathrm{Spec}(R)$, the **spectrum**. AG can study very general spaces, called **schemes**: simply replace R by any commutative ring, and study spaces which are locally modeled on $\mathrm{Spec}(R)$. In AG studying varieties reduces locally to commutative algebra.

2. AFFINE VARIETIES

2.1. VANISHING SETS

k = algebraically closed field,[2] e.g. $\mathbb{C}$ but not $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{F}_p$.
Fact. k is an infinite set.
$k^n = \{a = (a_1, \dots, a_n) : a_j \in k\}$ is a vector space/k of dimension n.
We will work with the following k-algebra[3]

$$\boxed{R = k[x_1, \dots, x_n] = (\text{polynomial ring}/k \text{ in } n \text{ variables}).}$$

Definition. *$X \subset k^n$ is an* **affine (algebraic) variety** *if $X = \mathbb{V}(I)$ for some ideal[4] $I \subset R$, where*

$$\boxed{\mathbb{V}(I) = \{a \in k^n : f(a) = 0 \textit{ for all } f \in I\}}$$

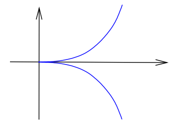
Remark. More generally we can define $\mathbb{V}(S)$ for any subset $S \subset R$. Notice $\mathbb{V}(S) = \mathbb{V}(I)$ for $I = \langle S \rangle$ the ideal generated by S.

EXAMPLES.

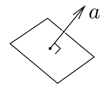
(1) $\mathbb{V}(0) = k^n$.
(2) $\mathbb{V}(1) = \emptyset = \mathbb{V}(R)$.
(3) $\mathbb{V}(x_1 - a_1, \dots, x_n - a_n) = \{\text{the point } (a_1, \dots, a_n)\} \subset k^n$.
(4) $\mathbb{V}(x_1) \subset k^2$ is the second coordinate axis.
(5) $\mathbb{V}(f) \subset k^n$ called **hypersurface**. Special cases:
$n = 2$: **affine plane curve**. E.g. **elliptic curves** over $\mathbb{C}$: $y^2 - x(x-1)(x-\lambda) = 0$ for $\lambda \neq 0, 1$, is a torus with a point removed (and it is a Riemann surface).
$n = 2, \deg f = 2$: **conic section**. E.g. the circle $x^2 + y^2 - 1 = 0$.

[1]A topological space X is **irreducible** if it is not the union of two proper closed sets.
[2]Recall this means k contains all the roots of any non-constant polynomial in $k[x]$. Thus the only irreducible polynomials are those of degree one, and every poly in $k[x]$ factorizes into degree 1 polys. It also means that for any algebraic field extension $k \hookrightarrow K$ then $k = K$. Recall a field extension is **algebraic** if any element of K satisfies a poly over k, for example any **finite field extension** (meaning $\dim_k K < \infty$) is algebraic.
[3]A k-algebra is a ring which is also a k-vector space, and the operations $+$, $\cdot$, and rescaling satisfy all the obvious axioms you would expect.
[4]Ideal means: $0 \in I$, $I + I \subset I$, $R \cdot I \subset I$.

$n = 2, \deg f = 3$: **cubic curve**. E.g. the cuspidal cubic $y^2 - x^3 = 0$.



Pictures are, strictly speaking, meaningless since we draw them over $k = \mathbb{R}$, which is not algebraically closed. Think of the picture as being the real part[1] of the picture for $k = \mathbb{C}$.
$\deg f = 1$: **hyperplane**: $a \cdot x = a_1 x_1 + \dots + a_n x_n = 0$ has normal $a \neq 0 \in k^n$.



Fact. k algebraically closed $\Rightarrow \mathbb{V}(I) = \emptyset$ iff $1 \in I$ (so iff $I = R$).
This fails for $\mathbb{R}$: $\mathbb{V}(x^2 + y^2 + 1) = \emptyset$ (real algebraic geometry is hard!)
EXERCISES.

(1) $I \subset J \Rightarrow \mathbb{V}(I) \supset \mathbb{V}(J)$. ("The more equations you impose, the smaller the solution set".)
(2) $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cdot J) = \mathbb{V}(I \cap J)$.
(3) $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I + J)$. (Note: $\langle I \cup J \rangle = I + J$.)

2.2. HILBERT'S BASIS THEOREM

Fact. Hilbert's Basis Theorem. $R = k[x_1, \dots, x_n]$ is a Noetherian ring.

Recall the following are equivalent definitions of **Noetherian ring** (intuitively a "small ring"):

(1) Every ideal is **finitely generated** (f.g.)

$$I = \langle f_1, \dots, f_N \rangle = Rf_1 + \dots + Rf_N.$$

(2) **ACC** (Ascending Chain Condition) on ideals:

$$I_1 \subset I_2 \subset \cdots \text{ ideals } \Rightarrow I_N = I_{N+1} = \cdots \text{ eventually all become equal.}$$

Note. (1) implies that affine varieties are cut out by *finitely* many polynomial equations. So affine varieties are intersections of hypersurfaces:

$$\mathbb{V}(I) = \mathbb{V}(f_1, \dots, f_N) = \mathbb{V}(f_1) \cap \dots \cap \mathbb{V}(f_N).$$

(2) implies that every ideal is contained in some **maximal ideal**[2] $\mathfrak{m}$ (as otherwise $I \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$ would contradict (2)).[3]
Exercise. R Noetherian $\Rightarrow R/I$ Noetherian.
Corollary. Any f.g. k-algebra A is Noetherian.
Proof. Let $f : R = k[x_1, \dots, x_n] \to A$, sending the x_i to a choice of generators for A. Then $R/I \cong A$ for $I = \ker f$ (first isomorphism theorem). $\square$

[1]You need to be careful with this. For example, the "circle" $x^2 + y^2 = 1$ over $k = \mathbb{C}$ also contains the hyperbola $x^2 - y^2 = 1$ by replacing y by iy. Also, disconnected pictures like $xy = 1$ over $\mathbb{R}$ become connected over $\mathbb{C}$ (why?).
[2]$\mathfrak{m} \neq R$ is an ideal and $R/\mathfrak{m}$ is a field.
[3]For any ring (commutative with 1), any proper ideal is always contained inside a maximal ideal. However, to prove this in general requires transfinite induction (Zorn's lemma), so in practice it is not clear how you would find the maximal ideal. Whereas for Noetherian rings, you know that the algorithm which keeps finding larger and larger ideals, $I \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$, will have to stop in finite time.

2.3. HILBERT'S WEAK NULLSTELLENSATZ

Fact. Hilbert's Weak Nullstellensatz. (k algebraically closed is crucial)
The maximal ideals of R are
$$\mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$$
for $a \in k^n$.
Warning. Fails over $\mathbb{R}$:
$$\mathfrak{m} = (x^2+1) \subset \mathbb{R}[x]$$
is maximal since $\mathbb{R}[x]/\mathfrak{m} \cong \mathbb{C}$ is a field. It is not maximal over $\mathbb{C}$:
$$(x^2+1) = ((x-i)(x+i)) \subset (x-i).$$

Remark. The evaluation homomorphism
$$\mathrm{ev}_a : R \to k, x_i \mapsto a_i, \text{ more generally } \mathrm{ev}_a(f) = f(a),$$
has $\ker \mathrm{ev}_a = \mathfrak{m}_a$, so
$$\mathfrak{m}_a = \{f \in R : f(a) = 0\}.$$

Proof. For $a = 0$, $k[x_1,\ldots,x_n] \to k$, $x_i \mapsto 0$ (so $f \mapsto$ the constant term of the polynomial f) obviously has kernel $(x_1,\ldots,x_n)$. For $a \neq 0$ do the linear change of coordinates $x_i \mapsto x_i - a_i$. □

Upshot.[1]
$$\begin{array}{rcl} \{\text{points of } k^n\} & \leftrightarrow & \{\text{maximal ideals of } R\} = \mathrm{Specm}(R), \text{ the } \textbf{maximal spectrum} \\ a & \mapsto & (x_1 - a_1, \ldots, x_n - a_n) = \mathfrak{m}_a \\ \left(\begin{array}{c}\text{points of the variety} \\ X = \mathbb{V}(I) \subset k^n\end{array}\right) & \leftrightarrow & \left\{\begin{array}{c}\text{maximal ideals } \mathfrak{m} \subset R \\ \text{with } I \subset \mathfrak{m}\end{array}\right\} = \mathrm{Specm}(R/I).\end{array}$$
Notice: if $I \not\subset \mathfrak{m}_a$ then some $f \in I$ satisfies $f(a) \neq 0$, so $a \notin \mathbb{V}(I)$.
Remark. Without assuming k algebraically closed, a max ideal $\mathfrak{m} \supset I$ defines a field extension
$$k \hookrightarrow R/\mathfrak{m} \cong K$$
where $R/\mathfrak{m} \cong K$ sends $x_i \mapsto a_i$. This defines a point $a \in \mathbb{V}(I) \subset K^n$, so it is a "$K$-point" solving our polynomial equations, but we don't "see" this point over k unless $a \in k^n \subset K^n$. For k algebraically closed, $k = K$ because $k \hookrightarrow K$ is an algebraic extension by the following Fact, so we "see" everything.
Key Fact. K f.g. k-algebra + K field $\Rightarrow K$ f.g. as a k-module[2] $\Rightarrow k \hookrightarrow K$ finite $\Rightarrow k \hookrightarrow K$ algebraic.
(Because the Key Fact implies the Weak Nullstellensatz via the Remark, the Key Fact is sometimes also called the Weak Nullstellensatz*).*
Example. $i \in \mathbb{V}(x^2+1) \subset \mathbb{C}$ but $\emptyset = \mathbb{V}(x^2+1) \subset \mathbb{R} \hookrightarrow \mathbb{C}$.

2.4. ZARISKI TOPOLOGY

The **Zariski topology** on k^n is defined by declaring[3] that the closed sets are the $\mathbb{V}(I)$.
The open sets are the
$$\begin{array}{rcl} U_I & = & k^n \setminus \mathbb{V}(I) \\ & = & k^n \setminus (\mathbb{V}(f_1) \cap \cdots \cap \mathbb{V}(f_N)) \\ & = & (k^n \setminus \mathbb{V}(f_1)) \cup \cdots \cup (k^n \setminus \mathbb{V}(f_N)) \\ & = & D(f_1) \cup \cdots \cup D(f_N)\end{array}$$
where the $D(f_i)$ are called the **basic open sets**, where
$$D(f) = U_f = k^n \setminus \mathbb{V}(f) = \{a \in k^n : f(a) \neq 0\}.$$

[1] For the last equality, recall:
$$\begin{array}{rcl} \{\text{ideals } J \subset R \text{ with } I \subset J\} & \leftrightarrow & \{\text{ideals } \overline{J} \subset R/I\} \\ J & \mapsto & \overline{J} = \{\overline{j} = j + I \in R/I : j \in J\} \\ J = \{j \in R : \overline{j} \in \overline{J}\} & \leftarrow & \overline{J}.\end{array}$$
[2] i.e. a k-vector space. *Clarification:* in an algebra you are allowed to multiply generators, in a module you are not.
[3] In fact it is the smallest topology such that polynomials are continuous and any point is a closed set.

Exercise. Affine varieties are compact:[1] any open cover of an affine variety X has a finite subcover.

Definition. Affine space $\mathbb{A}^n = \mathbb{A}^n_k$ *is the set* $\mathbb{A}^n = k^n$ *with the Zariski topology.*

Example. $\mathbb{A}^1_k = k$ has closed sets $\emptyset, k, \{\text{finite points}\}$, and open sets $\emptyset, k$, and (the complement of any finite set of points). It is not Hausdorff since any two non-empty open sets intersect. The open sets are dense (as the only closed set with infinitely many points is k, using that k is infinite).

Definition. *The* **Zariski topology** *on an affine variety* $X \subset \mathbb{A}^n$ *is the subspace topology, so the closed sets are* $\mathbb{V}(I+J) = X \cap \mathbb{V}(J)$ *for any ideal* $J \subset R$ *(equivalently,* $\mathbb{V}(S)$ *for ideals* $I \subset S \subset R$*). An* ***affine subvariety*** $Y \subset X$ *is a closed subset of* X.

2.5. VANISHING IDEAL

For any set $X \subset \mathbb{A}^n$, let
$$\boxed{\mathbb{I}(X) = \{f \in R : f(a) = 0 \text{ for all } a \in X\}}$$

EXAMPLES.
1. $\mathbb{I}(a) = \mathfrak{m}_a = \{f \in R : f(a) = 0\}$.
2. $\mathbb{I}(\mathbb{V}(x^2)) = \mathbb{I}(0) = (x) \subset k[x]$, so $\mathbb{I}(\mathbb{V}(I)) \neq I$ in general.

Exercises.
1. $X \subset Y \Rightarrow \mathbb{I}(X) \supset \mathbb{I}(Y)$.
2. $I \subset \mathbb{I}(\mathbb{V}(I))$.

Lemma 2.1. $\mathbb{V}(\mathbb{I}(\mathbb{V}(I))) = \mathbb{V}(I)$, *in particular* $\mathbb{V}(\mathbb{I}(X)) = X$ *for any affine variety* X.

Proof. Take $\mathbb{V}(\cdot)$ of exercise 2 above, to get $\mathbb{V}(\mathbb{I}(\mathbb{V}(I))) \subset \mathbb{V}(I)$.
Conversely, by contradiction, if $a \in \mathbb{V}(I) \setminus \mathbb{V}(\mathbb{I}(\mathbb{V}(I)))$ then there is an $f \in \mathbb{I}(\mathbb{V}(I))$ with $f(a) \neq 0$. But such an f vanishes on $\mathbb{V}(I)$, and $a \in \mathbb{V}(I)$. □

Corollary. *For affine varieties,* $X_1 = X_2 \Leftrightarrow \mathbb{I}(X_1) = \mathbb{I}(X_2)$.

2.6. IRREDUCIBILITY AND PRIME IDEALS

An affine variety X is **reducible** if $X = X_1 \cup X_2$ for proper closed subsets X_i (so $X_i \subsetneq X$). Otherwise, call X **irreducible**.[2]
Remark. Some books require varieties to be irreducible by definition, and call the general $\mathbb{V}(I)$ **affine algebraic sets**. We don't.

EXAMPLES.
1. $\mathbb{V}(x_1x_2) = \mathbb{V}(x_1) \cup \mathbb{V}(x_2)$ is reducible
2. **Exercise.** X irreducible $\Leftrightarrow$ any non-empty open subset is dense.
3. **Exercise.** X irreducible $\Leftrightarrow$ any two non-empty open subsets intersect.
4. In a Hausdorff topological space, only the empty set and one point sets are irreducible.

Theorem. $X = \mathbb{V}(I) \neq \emptyset$ *is irreducible* $\Leftrightarrow \mathbb{I}(X) \subset R$ *is a* **prime ideal**.[3]
Warning. $I \subset R$ need not be prime: $I = (x^2)$ is not prime but $\mathbb{I}(\mathbb{V}(x^2)) = (x)$ is prime.

Proof. If $\mathbb{I}(X)$ is not prime, then pick f_1, f_2 satisfying $f_1 \notin \mathbb{I}(X)$, $f_2 \notin \mathbb{I}(X)$, $f_1f_2 \in \mathbb{I}(X)$. Then
$$X \subset \mathbb{V}(f_1f_2) = \mathbb{V}(f_1) \cup \mathbb{V}(f_2)$$
so take $X_i = X \cap \mathbb{V}(f_i) \neq X$ (since $f_i \notin \mathbb{I}(X)$).
Conversely, if X is not irreducible, $X = X_1 \cup X_2$, $X_i \neq X$, so (by Lemma 2.1) there are $f_i \in \mathbb{I}(X_i) \setminus \mathbb{I}(X)$ but $f_1f_2 \in \mathbb{I}(X)$, so $\mathbb{I}(X)$ is not prime. □

[1] *Historically this property is called* ***quasi-compactness*** *rather than* compactness, *to remind ourselves that the topology is not Hausdorff.*
[2] So $X = X_1 \cup X_2$ for closed X_i implies $X_i = X$ for some i.
[3] $I \neq R$ is an ideal and R/I is an integral domain.

Notice, abbreviating $I = \mathbb{I}(X)$, $J = \mathbb{I}(Y)$,

$$\begin{aligned} \{\text{irreducible varieties } X \subset \mathbb{A}^n\} &\leftrightarrow \{\text{prime ideals } I \subset R\} = \mathrm{Spec}(R) \\ \{\text{irreducible subvarieties } Y = \mathbb{V}(J) \subset X = \mathbb{V}(I) \subset \mathbb{A}^n\} &\leftrightarrow \{\text{prime ideals } J \supset I \text{ of } R\} \\ &\leftrightarrow \{\text{prime ideals } \overline{J} \text{ of } R/I\} = \mathrm{Spec}(R/I). \end{aligned}$$

Remark. $\mathrm{Spec}(k) = \{0\}$ = just a point![1] So, in seminars, when someone writes $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(R/I)$ they are just saying "given a point in an affine variety…".

2.7. DECOMPOSITION INTO IRREDUCIBLE COMPONENTS

Theorem. *An affine variety can be decomposed into* **irreducible components***: that is,*

$$X = X_1 \cup X_2 \cup \cdots \cup X_N.$$

where the X_i are irreducible affine varieties, and the decomposition is unique up to reordering if we ensure $X_i \not\subset X_j$ for all $i \neq j$.

Proof. Proof of Existence. By contradiction, suppose it fails for X.
So $X = Y_1 \cup Y_1'$ for proper subvars.
So it fails for Y_1 or Y_1', WLOG Y_1.
So $Y_1 = Y_2 \cup Y_2'$ for proper subvars.
So it fails for Y_2 or Y_2', WLOG Y_2.
Continue inductively.
We obtain a sequence $X \supset Y_1 \supset Y_2 \supset \cdots$.
So $\mathbb{I}(X) \subset \mathbb{I}(Y_1) \subset \mathbb{I}(Y_2) \subset \cdots$.
So $\mathbb{I}(Y_N) = \mathbb{I}(Y_{N+1}) = \cdots$ eventually equal, since R is Noetherian (Hilbert Basis Thm).
So, by Lemma 2.1, $Y_N = \mathbb{V}(\mathbb{I}(Y_N))) = \mathbb{V}(\mathbb{I}(Y_{N+1})) = Y_{N+1}$ which is not proper. Contradiction.
Proof of Uniqueness. Suppose $X_1 \cup \cdots \cup X_N = Y_1 \cup \cdots \cup Y_M$, with $X_i \not\subset X_j$ and $Y_i \not\subset Y_j$ for $i \neq j$.
$X_i = (X_i \cap Y_1) \cup \cdots \cup (X_i \cap Y_M)$ contradicts X_i irreducible unless some $X_i \cap Y_\ell = X_i$.
So $X_i \subset Y_\ell$ for some ℓ.
Similarly, $Y_\ell \subset X_j$ for some j.
So $X_i \subset Y_\ell \subset X_j$, contradicting $X_i \not\subset X_j$ unless $i = j$.
So $i = j$ and so $X_i = Y_\ell$.
Given i, the ℓ is unique (due to $Y_i \not\subset Y_j$ for $i \neq j$) and vice-versa given ℓ there is a unique such i. □

Remark. The fact that R is a Noetherian ring implies that affine varieties are **Noetherian topological spaces**, i.e. given a descending chain

$$X \supset X_1 \supset X_2 \supset \cdots$$

of closed subsets of X, then $X_N = X_{N+1} = \cdots$ are eventually all equal.
Proof. Take $\mathbb{I}(\cdot)$ and use the ACC on ideals. So $\mathbb{I}(X_N) = \mathbb{I}(X_{N+1}) = \cdots$ are eventually equal. Then take $\mathbb{V}(\cdot)$ and use Lemma 2.1. □

2.8. IRREDUCIBLE DECOMPOSITIONS and PRIMARY IDEALS

This Section is not very central to the course. See the Appendix, Section 16.

2.9. $\mathbb{I}(\mathbb{V}(\cdot))$ AND $\mathbb{V}(\mathbb{I}(\cdot))$

Motivation. By Lemma 2.1, if X is a variety then

$$\mathbb{V}(\mathbb{I}(X)) = X.$$

Of course, the assumption was to be expected, since $\mathbb{V}(\cdot)$ is always closed, so for this equality to hold we certainly need X to be closed, i.e. a variety.
Under what assumption on an ideal I can we guarantee

$$\mathbb{I}(\mathbb{V}(I)) \stackrel{?}{=} I. \tag{2.1}$$

[1]Because the only ideals inside a field k are $0, k$.

The question really is, what is special about the ideals which arise as $\mathbb{I}(\mathbb{V}(\cdot))$? Observe that $\mathbb{I}(\mathbb{V}(I))$ is always a **radical ideal**: if it contains a power f^m then it must contain f. Indeed, if $f^m(a) = [f(a)]^m = 0 \in k$ then $f(a) = 0$. We show next that for any radical ideal I, (2.1) holds.

Definition. *The* **radical** $\sqrt{I}$ *of an ideal $I \subset R$ is defined by*

$$\sqrt{I} = \{f \in R : f^m \in I \text{ for some } m\}.$$

I is called a **radical ideal** *if $I = \sqrt{I}$.*

Example. $\mathbb{V}(x^3) = \{0\} \subset \mathbb{A}^1$ and $\mathbb{I}(\mathbb{V}(x^3)) = \langle x \rangle = \sqrt{\langle x^3 \rangle}$. So $\langle x \rangle$ is radical, but $\langle x^3 \rangle$ is not.

Exercise. Check that $\mathbb{V}(I) = \mathbb{V}(\sqrt{I})$.

Exercise. $I \subset R$ is radical $\Leftrightarrow$ R/I has no nilpotent[1] elements, i.e. R/I is a **reduced ring**.

Example. Any prime ideal is radical.

Motivation. The problem is that $\mathbb{V}(\cdot)$ forgets some information. One should really view $\mathbb{V}(x^3)$ as being $0 \in \mathbb{A}^1$ with a multiplicity 3 of vanishing. This idea is at the heart of the theory of **schemes**. Loosely, a scheme should be a "variety" together with a choice of a ring of functions. The ring of functions associated to (x^3) is $k[x]/x^3$, which is 3-dimensional, whereas for (x) it is $k[x]/x$, which is 1-dimensional. The "additional dimensions" can be thought of as an infinitesimal thickening of the variety, as it keeps track of additional derivatives. Roughly: $f = a + bx + cx^2 \in k[x]/x^3$ has $\partial_x f(0) = b$ and $\partial_x \partial_x f = 2c$, whereas $k[x]/x$ only "sees" $f \cong a \in k[x]/x$.

2.10. HILBERT'S NULLSTELLENSATZ

Theorem 2.2 (Hilbert's Nullstellensatz)**.**

$$\boxed{\mathbb{I}(\mathbb{V}(I)) = \sqrt{I}}$$

In particular, if I is radical then $\mathbb{I}(\mathbb{V}(I)) = I$.

Proof. We will prove this later. □

Corollary. *There are order-reversing[2] bijections*

$$\begin{aligned} \{\text{varieties}\} &\leftrightarrow \{\text{radical ideals}\} \\ \{\text{irreducible varieties}\} &\leftrightarrow \{\text{prime ideals}\} = \mathrm{Spec}(R) \\ \{\text{points}\} &\leftrightarrow \{\text{maximal ideals}\} = \mathrm{Specm}(R) \\ X &\mapsto \mathbb{I}(X) \\ \mathbb{V}(I) &\leftarrow\!\shortmid I. \end{aligned}$$

Proof. These are bijections because $\mathbb{V}(\mathbb{I}(\mathbb{V}(I))) = \mathbb{V}(I)$ by Lemma 2.1, and $\mathbb{I}(\mathbb{V}(I)) = I$ for radical ideals I by Theorem 2.2. □

The Nullstellensatz ("Zeros theorem") owes its name to the proof of the existence of common zeros for any set of polynomial equations (crucially, of course, k is algebraically closed):

Lemma 2.3. *For any proper ideal $I \subset R$, we have $\mathbb{V}(I) \neq \emptyset$.*

Proof. Pick a maximal ideal $I \subset \mathfrak{m} \subset R$. By Hilbert's weak Nullstellensatz, $\mathfrak{m} = \mathfrak{m}_a = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a \in k^n$. Hence $\mathbb{V}(I) \supset \mathbb{V}(\mathfrak{m}_a) = \{a\} \supset \mathbb{V}(R) = \emptyset$. □

Proof of the Nullstellensatz.
Easy direction: above we showed $\mathbb{I}(\mathbb{V}(I))$ is always radical, we know $I \subset \mathbb{I}(\mathbb{V}(I))$, so $\sqrt{I} \subset \mathbb{I}(\mathbb{V}(I))$.
Remains to show $\mathbb{I}(\mathbb{V}(I)) \subset \sqrt{I}$.
Given $g \in \mathbb{I}(\mathbb{V}(I))$.
Trick: let $I' = \langle I, yg - 1 \rangle \subset k[x_1, \ldots, x_n, y]$ (the idea being: we go to a new ring where $g = 0$ is impossible in $\mathbb{V}(I')$).

[1]$r \in R$ is *nilpotent* if $r^m = 0$ for some $m \in \mathbb{N}$.
[2]Recall $X \subset Y \Rightarrow \mathbb{I}(X) \supset \mathbb{I}(Y)$, $I \subset J \Rightarrow \mathbb{V}(I) \supset \mathbb{V}(J)$.

Observe that $\mathbb{V}(I') = \emptyset \subset \mathbb{A}^{n+1}$.
By Lemma 2.3, $I' = k[x_1,\dots,x_n,y]$.
So $1 \in I'$.
So $1 = G_0(x_1,\dots,x_n,y)\cdot(yg-1) + \sum G_i(x_1,\dots,x_n,y)\cdot f_i$ for some polynomials G_j, and where f_i are the generators of $I = \langle f_1,\dots,f_N\rangle$.
For large ℓ, $g^\ell = F_0(x_1,\dots,x_n,gy)\cdot(yg-1) + \sum F_i(x_1,\dots,x_n,gy)\cdot f_i$ for some polynomials F_j (notice[1] the last variable is now gy instead of y).
Since y is a formal variable, we may[2] replace gy by 1, so $g^\ell = \sum F_i(x_1,\dots,x_n,1)\cdot f_i \in I$.
So $g \in \sqrt{I}$. □

2.11. FUNCTIONS

Motivating question: what maps $X \to \mathbb{A}^1$ do we want to allow?
Answer: any polynomial in the **coordinate functions** $x_i : a = (a_1,\dots,a_n) \mapsto a_i$.
The following are definitions (and notice the isomorphisms are k-algebra isos):

$$\begin{aligned}\mathrm{Hom}(\mathbb{A}^n,\mathbb{A}^1) &= \{\text{polynomial maps } \mathbb{A}^n \to \mathbb{A}^1, a \mapsto f(a), \text{ some } f \in R\}\\ &\cong R.\\ \mathrm{Hom}(X,\mathbb{A}^1) &= \{\text{restrictions to } X \text{ of such maps}\}\\ &\cong R/\mathbb{I}(X).\end{aligned}$$

Notice that the restricted maps do not change if we add $g \in \mathbb{I}(X)$ as $(f+g)(a) = f(a)$ for $a \in X$.
We may put a bar $\overline{f}$ over f as a reminder that we passed to the quotient, so $\overline{f+g} = \overline{f}$ if $g \in \mathbb{I}(X)$.
Remark. The above are isomorphisms because $f_1 = f_2$ as maps $\mathbb{A}^n \to \mathbb{A}^1$ iff $f_1 - f_2 \in \mathbb{I}(\mathbb{A}^n) = \{0\}$, similarly $f_1 = f_2$ as maps $X \to \mathbb{A}^1$ iff $f_1 - f_2 \in \mathbb{I}(X)$. That abstract polynomials can be identified with their associated functions relies on k being infinite[3] (which holds as k is algebraically closed). For the field $k = \mathbb{Z}/2$ there are four functions $k \to k$ whereas $k[x]$ contains infinitely many polynomials.

2.12. THE COORDINATE RING

Definition. The **coordinate ring** is the k-algebra generated by the coordinate functions $\overline{x}_i$,

$$k[X] = R/\mathbb{I}(X).$$

EXAMPLES.
1) $k[\mathbb{A}^n] = k[x_1,\dots,x_n] = R$.
2) $X = \{(a,a^2,a^3) \in k^3 : a \in k\} = \mathbb{V}(y-x^2, z-x^3)$, then[4] $k[X] = k[x,y,z]/(y-x^2,z-x^3)$.
3) $V = \text{(cuspidal cubic)} = \{(a^2,a^3) : a \in \mathbb{A}^1\} = \mathbb{V}(x^3-y^2)$, then[5] $k[V] = k[x,y]/(x^3-y^2)$.

Lemma 2.4 (The coordinate ring separates points)**.** *Given an affine variety X, and points $a,b \in X$, if $f(a) = f(b)$ for all $f \in k[X]$ then $a = b$.*

Proof. If $a \neq b \in X \subset \mathbb{A}^n$, some coordinate $a_i \neq b_i$, so $f = \overline{x}_i \in k[X]$ has $f(a) = a_i \neq b_i = f(b)$. □

[1]Example: if $y^3 + y = G(y)$ then multiply by g^3 to get: $g^3y^3 + g^3y = (gy)^3 + g^2(gy) = F(gy)$ where $F(z) = z^3 + g^2 z$.
[2]View the equation for g^ℓ as an equation in the variable $(gy-1)$ over R rather than in gy (this is a change of variables), then "putting $gy = 1$" is the same as saying "compare the order zero term of the polynomial over R in the variable $gy - 1$". Algebraically, the key is: $k[x_1,\dots,x_n] \hookrightarrow k[x_1,\dots,x_n,y]/(yg-1)$, $x_i \mapsto \overline{x}_i$ is an injective k-alg hom.
[3]*Hint.* If $f : \mathbb{A}^n \to k$ vanishes, fix $a_i \in k$, then $f(\lambda,a_2,\dots,a_n)$ is a poly in one variable λ with infinitely many roots.
[4]Strictly speaking, one needs to check that $I = (y-x^2,z-x^3)$ is a radical ideal, since $k[X]$ is the quotient of $k[x,y,z]$ by $\sqrt{I} = \mathbb{I}(X)$. Notice that $k[x,y,z]/(y-x^2,z-x^3) \cong k[t]$ via $x \mapsto t$, $y \mapsto t^2$, $z \mapsto t^3$, with inverse map given by $t \mapsto x$. Since $k[t]$ is an integral domain, it has no nilpotents, so I is radical (in fact we also proved I is prime). We remark that $\mathbb{I}(X) = (y-x^2,z-x^3)$ now follows by the Nullstellensatz: $\mathbb{V}(I) = X$ so $\mathbb{I}(X) = \mathbb{I}(\mathbb{V}(I)) = \sqrt{I} = I$.
[5]Again, we need to check $\mathbb{I}(V) = (x^3-y^2)$. Note that if $(\alpha,\beta) \in \mathbb{V}(x^3-y^2)$ we can pick $a \in k$ with $a^2 = \alpha$ (as k is alg.closed). Then $y^2 = a^6$ so $y = \pm a^3$, and we can get $+a^3$ by replacing a by $-a$ if necessary. So $\mathbb{V}(x^3-y^2) \subset V \subset \mathbb{V}(x^3-y^2)$, hence equality. We now show (x^3-y^2) is prime (hence radical). Since $k[x,y,z]$ is a UFD (so irreducible $\Leftrightarrow$ prime), it is enough to check that x^3-y^2 is irreducible. If it was reducible, then x^3-y^2 would factorize as a polynomial in x over the ring $k[y]$. So there would be a root $x = p(y)$ for a polynomial p. This is clearly impossible (check this).

2.13. MORPHISMS OF AFFINE VARIETIES

$F : \mathbb{A}^n \to \mathbb{A}^m$ is a **morphism** (or **polynomial map**) if it is defined by polynomials:

$$F(a) = (f_1(a),\dots,f_m(a)) \quad \text{for some } f_1,\dots,f_m \in R.$$

$F : X \to Y$ is a **morphism of affine varieties** if it is the restriction of a morphism $\mathbb{A}^n \to \mathbb{A}^m$ (here $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$), so

$$F(a) = (f_1(a),\dots,f_m(a)) \quad \text{for some } f_1,\dots,f_m \in k[X].$$

$F : X \to Y$ is an **isomorphism** if F is a morphism and there is an inverse morphism (i.e. there is a morphism $G : Y \to X$ such that $F \circ G = \mathrm{id}$, $G \circ F = \mathrm{id}$).
Example. $(\mathbb{V}(xy-1) \subset \mathbb{A}^2) \to \mathbb{A}^1$, $(x,y) \mapsto x$ is a morphism. Notice the image $\mathbb{A}^1 \setminus \{0\}$ is not a subvariety of $\mathbb{A}^1$.

Theorem. *For affine varieties $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ there is a 1:1 correspondence*

$$\begin{aligned}\mathrm{Hom}(X,Y) &\longleftrightarrow \mathrm{Hom}_{k\text{-alg}}(k[Y],k[X]) = \{k\text{-algebra homs } k[Y] \to k[X]\}\\ F = \varphi^* : X \to Y &\longleftrightarrow \varphi = F^* : k[Y] \to k[X]\\ &\longleftrightarrow \varphi = F^* : \mathrm{Hom}(Y,\mathbb{A}^1) \to \mathrm{Hom}(X,\mathbb{A}^1),\ g \mapsto F^*g = g \circ F\end{aligned}$$

where $k[X] = k[x_1,\dots,x_n]/\mathbb{I}(X)$, $k[Y] = k[y_1,\dots,y_m]/\mathbb{I}(Y)$ and

$$\begin{aligned}F^*(y_i) &= f_i(x_1,\dots,x_n) = \varphi(y_i)\\ \varphi^*(a) &= (\varphi(y_1)(a),\dots,\varphi(y_m)(a)) = (f_1(a),\dots,f_m(a)).\end{aligned}$$

Proof. The correspondence maps, in the two directions, are well-defined.[1] ✓
$(F^*)^*(a) = (F^*(y_1)(a),\dots,F^*(y_m)(a)) = (f_1(a),\dots,f_m(a)) = F(a)$, so $(F^*)^* = F$. ✓
$(\varphi^*)^*(y_i) = \varphi(y_i)$, so $(\varphi^*)^* = \varphi$. ✓ □

Remark. The maps φ^*, F^* are called **pull-backs** (or pull-back maps).

EXAMPLES.
1) $F : \mathbb{A}^1 \to V = \{(a,a^2,a^3) \in k^3 : a \in k\}$, $F(a) = (a,a^2,a^3)$ then

$$\begin{array}{rcl} k[\mathbb{A}^1] = k[t] & \xleftarrow{F^*} & k[V] = k[x,y,z]/(y-x^2,z-x^3)\\ t & \leftarrow\!\shortmid & x\\ t^2 & \leftarrow\!\shortmid & y\\ t^3 & \leftarrow\!\shortmid & z\end{array}$$

2) $F : \mathbb{A}^1 \to V = \{(a^2,a^3) : a \in \mathbb{A}^1\} =$(cuspidal cubic), $F(a) = (a^2,a^3)$ then

$$\begin{array}{rcl} k[\mathbb{A}^1] = k[t] & \xleftarrow{F^*} & k[V] = k[x,y]/(x^3-y^2)\\ t^2 & \leftarrow\!\shortmid & x\\ t^3 & \leftarrow\!\shortmid & y.\end{array}$$

Exercise. $F : X \to Y$ morph $\Rightarrow F^{-1}(\mathbb{V}(J)) = \mathbb{V}(F^*J) \subset X$ for any closed set $\mathbb{V}(J) \subset Y$. So morphisms are continuous in the Zariski topology.

EXERCISES.
1) $X \xrightarrow{F} Y \xrightarrow{G} Z \Rightarrow (G \circ F)^* = F^* \circ G^* : k[Z] \xrightarrow{G^*} k[Y] \xrightarrow{F^*} k[X]$.
2) $k[Z] \xrightarrow{\psi} k[Y] \xrightarrow{\varphi} k[X] \Rightarrow (\varphi \circ \psi)^* = \psi^* \circ \varphi^* : X \xrightarrow{\varphi^*} Y \xrightarrow{\psi^*} Z$.

Corollary. *For affine varieties,*

$$X \cong Y \Leftrightarrow k[X] \cong k[Y].$$

Proof. If $X \xrightarrow{F} Y$ has inverse G, $F \circ G = \mathrm{id}$ so $(F \circ G)^* = G^* \circ F^* = \mathrm{id}^* = \mathrm{id}$. Similarly for $G \circ F$.
If $k[Y] \xrightarrow{\varphi} k[X]$ has inverse ψ, $\varphi \circ \psi = \mathrm{id}$ so $(\varphi \circ \psi)^* = \psi^* \circ \varphi^* = \mathrm{id}^* = \mathrm{id}$. Similarly for $\psi \circ \varphi$. □

[1]In particular $\varphi^*(X) \subset Y \subset \mathbb{A}^m$, because $g(\varphi^*(a)) = \varphi(g)(a) = 0$ for all $g \in \mathbb{I}(Y)$ and $a \in X$, as $g = 0 \in k[Y]$.

EXAMPLES.
1) $V = \{(a,a^2,a^3) \in \mathbb{A}^3 : a \in \mathbb{A}^1\} \cong \mathbb{A}^1$ via $(a,a^2,a^3) \leftrightarrow a$, indeed $k[V] \cong k[t] \cong k[\mathbb{A}^1]$ via $x \leftrightarrow t$.
2) In the cuspidal cubic example above, F is a bijective morphism but it cannot be an isomorphism because F^* is not an isomorphism (it does not hit t in the image). The idea is that V has "fewer polynomial functions" than $\mathbb{A}^1$ due to the singularity at 0. Convince yourself that $k[t]$, $k[V]$ are not isomorphic k-algebras, so there cannot be any isomorphism $\mathbb{A}^1 \to V$ (stronger than just F failing).
3) **Exercise.** If $F : X \to Y$ is a surjective morphism of affine varieties, and X is irreducible, then Y is irreducible. Show that it suffices that F is **dominant**, i.e. has dense image.
Example. $Y = \{(t,t^2,t^3) : t \in k\}$ is irreducible as it is the image of $\mathbb{A}^1 \to Y$, $t \mapsto (t,t^3,t^3)$.

3. PROJECTIVE VARIETIES

3.1. PROJECTIVE SPACE

Notation:
$k^* = k \setminus \{0\}$ = units, i.e. the invertibles.
For V any vector space/k, define the **projectivisation** by

$$\mathbb{P}(V) = (V \setminus \{0\}) \,/\, (k^*\text{-rescaling action } v \mapsto \lambda v, \text{ for all } \lambda \in k^*).$$

Notice this always comes with a quotient map $\pi : V \setminus \{0\} \to \mathbb{P}(V)$, $v \mapsto [v]$, where $[v] = [\lambda v]$.
By picking a (linear algebra) basis for V, we can suppose $V = k^{n+1}$. We then obtain $\mathbb{P}^n = \mathbb{P}^n_k = \mathbb{P}(k^{n+1})$, called **projective space**, defined as follows

$$\begin{aligned} \mathbb{P}^n &= \mathbb{P}(k^{n+1}) \\ &= (\text{space of straight lines in } k^{n+1} \text{ through } 0) \end{aligned}$$

Write $[a_0, a_1, \ldots, a_n]$ or $[a_0 : a_1 : \cdots : a_n]$ for the equivalence class of $(a_0, a_1, \ldots, a_n) \in k^{n+1} \setminus \{0\}$, whose corresponding line in k^{n+1} is $k \cdot (a_0, \ldots, a_n) \subset k^{n+1}$. Via the rescaling action, we thus identify

$$[a_0 : \ldots : a_n] = [\lambda a_0 : \ldots : \lambda a_n] \quad \text{for all } \lambda \in k^*.$$

As before, we have a quotient map

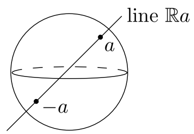
$$\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n, \ \pi(a) = [a].$$

The coordinates $x_0, \ldots, x_n$ of $k^{n+1} = \mathbb{A}^{n+1}$ are called **homogeneous coordinates** of $\mathbb{P}^n$, although notice they are not well-defined functions on $\mathbb{P}^n$: $x_i(a) = a_i$ but $x_i(\lambda a) = \lambda a_i$.

EXAMPLES.
1) For $k = \mathbb{R}$,

$$\mathbb{RP}^n = S^n / (\text{identify antipodal points } a \sim -a)$$

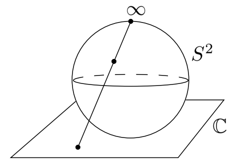
because the straight line in $\mathbb{R}^{n+1}$ corresponding to the given point of $\mathbb{RP}^n$ will intersect the unit sphere of $\mathbb{R}^{n+1}$ in two antipodal points.



2) For $k = \mathbb{C}$, $n = 1$,

$$\mathbb{CP}^1 = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\text{infinity}\} \cong S^2$$

the last isomorphism is the *stereographic projection*,



Above, identify $[1 : z]$ with $z \in \mathbb{C}$, and $[0 : 1]$ with ∞. Note $[a : b] = [1 : z]$ if $a \neq 0$, taking $z = b/a$, using rescaling by $\lambda = a^{-1}$. For $a = 0$, we get $[0 : b] = [0 : 1]$, rescaling by $\lambda = b^{-1}$ (note: $[0 : 0]$ is not an allowed point).

We can think of $\mathbb{P}^n$ as arising from "compactifying" $\mathbb{A}^n$ by hyperplanes, planes, and points at infinity:

$$\begin{aligned} \mathbb{P}^n &= \{[1 : a_1 : \cdots : a_n]\} \cup \{[0 : a_1 : \cdots : a_n]\} \\ &= \mathbb{A}^n \cup \mathbb{P}^{n-1} \\ &= \cdots \quad (\text{by induction}) \\ &= \mathbb{A}^n \cup \mathbb{A}^{n-1} \cup \cdots \cup \mathbb{A}^1 \cup \mathbb{A}^0 \end{aligned}$$

where $\mathbb{A}^0$ is the point $[0 : 0 : \cdots : 0 : 1]$.

3.2. HOMOGENEOUS IDEALS

Motivating example. Consider $f(x,y) = x^2 + y^3$, and $[a : b] \in \mathbb{P}^1$. It is not clear what $f[a : b] = 0$ means, since $[a : b] = [3a : 3b]$ but $f(a,b) = a^2 + b^3 = 0$ and $f(3a,3b) = 9a^2 + 27b^3 = 0$ are different equations. However, for the *homogeneous* polynomial $F(x,y) = x^2 y + y^3$, the equations $F(a,b) = a^2 b + b^3 = 0$ and $F(3a,3b) = 27(a^2 b + b^3) = 0$ are equivalent, so $F[a : b] = 0$ is meaningful.

Notation. $R = k[x_0, \ldots, x_n]$ (k algebraically closed)

Definition. $F \in R$ is a **homogeneous polynomial of degree** d if all the monomials $x_0^{i_0} \cdots x_n^{i_n}$ appearing in F have degree $d = i_0 + \cdots + i_n$. By convention, $0 \in R$ is homogeneous of every degree.

Notice any polynomial $f \in R$ decomposes uniquely into a sum of homogeneous polynomials

$$f = f_0 + \cdots + f_d,$$

where f_i is the homogeneous part of degree i, and d is the highest degree that arises.

Lemma 3.1. *For $f \in R$, if f vanishes at all points of the line $k \cdot a \subset \mathbb{A}^{n+1}$ (corresponding to the point $[a] \in \mathbb{P}^n$) then each homogeneous part of f vanishes at $[a]$.*

Proof. $0 = f(\lambda a) = f_0(a) + f_1(a)\,\lambda + \cdots + f_{d-1}(a)\,\lambda^{d-1} + f_d(a)\,\lambda^d$ is a polynomial/k in λ with infinitely[1] many roots. So it is the zero polynomial, i.e. the coefficients vanish: $f_i(a) = 0$, all i. □

Exercise. F is homogeneous of degree $d \Leftrightarrow F(\lambda x) = \lambda^d F(x)$ for all $\lambda \in k^*$.

Definition. $I \subset R$ is a **homogeneous ideal** if it is generated by homogeneous polynomials.
Exercise. $I \subset R$ is homogeneous $\Leftrightarrow$ for any $f \in I$, all its homogeneous parts f_i also lie in I.
Example. For $R = k[x,y]$, $(x^2 y + y^3) = k[x,y] \cdot (x^2 y + y^3)$ is homogeneous.
Example. $(x^2, y^3) = R \cdot x^2 + R \cdot y^3$ is homogeneous.
Non-example. $(x^2 + y^3)$ is not homogeneous: it contains $x^2 + y^3$ but not its hom.parts x^2, y^3.

Exercise.[2] Deduce that a homogeneous ideal is generated by finitely many homogeneous polys.

3.3. PROJECTIVE VARIETIES and ZARISKI TOPOLOGY

Definition. $X \subset \mathbb{P}^n$ is a **projective variety** if

$$X = \mathbb{V}(I) = \{a \in \mathbb{P}^n : F(a) = 0 \text{ for all homogeneous } F \in I\}$$

for some homogeneous ideal I.
Definition. The **Zariski topology** on $\mathbb{P}^n$ has closed sets precisely the projective varieties $\mathbb{V}(I)$.

[1] here we use that k is an infinite set, since k is algebraically closed.
[2] Recall the Hilbert Basis theorem, i.e. R is Noetherian.

The Zariski topology on a projective variety $X \subset \mathbb{P}^n$ is the subspace topology, so the closed subsets of X are $X \cap \mathbb{V}(J) = \mathbb{V}(I + J)$ for any homogeneous ideal J (equivalently, $\mathbb{V}(S)$ for homogeneous ideals $I \subset S \subset R$). A **projective subvariety** $Y \subset X$ is a closed subset of X.

EXAMPLES.

1) **Projective hyperplanes**: $\mathbb{V}(L) \subset \mathbb{P}^n$ where $L = a_0x_0 + \cdots + a_nx_n$ is homogeneous of degree 1 (a **linear form**). In particular, the ***i*-th coordinate hyperplane** is

$$H_i = \mathbb{V}(x_i) = \{[a_0 : \ldots : a_{i-1} : 0 : a_{i+1} : \ldots : a_n] : a_j \in k\}.$$

2) **Projective hypersurface**: $\mathbb{V}(F) \subset \mathbb{P}^n$ for a non-constant homogeneous polynomial $F \in R$. A **quadric** (**cubic**, **quartic**, etc.) is a projective hypersurface defined by a homogeneous polynomial of degree 2 (respectively 3, 4, etc.). For example, the **elliptic curves** $\mathbb{V}(y^2z - x(x-z)(x-cz)) \subset \mathbb{P}^2$ (where $c \neq 0, 1 \in k$) are cubics in $\mathbb{P}^2$.

3) (Projective) **linear subspaces**: the projectivisation $\mathbb{P}(V) \subset \mathbb{P}^n$ of any k-vector subspace $V \subset k^{n+1}$ is a projective variety. It is cut out by linear homogeneous polynomials. The case $\dim_k V = 1$ gives a point in $\mathbb{P}^n$. The case $\dim_k V = 2$ defines the (projective) **lines** in $\mathbb{P}^n$. Example: $V = \mathrm{span}_k(e_0, e_1) \subset k^3$ yields the line $\{[t_0 : t_1 : 0] \in \mathbb{P}^2 : t_0, t_1 \in k\} = \{[1 : t : 0] : t \in k\} \cup \{[0 : 1 : 0]\} \cong \mathbb{P}^1$.

Exercise. Using basic linear algebra in k^{n+1}, show that there is a unique line through any two distinct points in $\mathbb{P}^n$, and that any two distinct lines in $\mathbb{P}^n$ meet in exactly one point.

3.4. AFFINE CONE

For a projective variety $X \subset \mathbb{P}^n$, the **affine cone** $\hat{X} \subset \mathbb{A}^{n+1}$ is the union of the straight lines in k^{n+1} corresponding to the points of X. Thus, using the quotient map $\pi : \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$, $x \mapsto [x]$,

$$\widehat{X} = \{0\} \cup \pi^{-1}(X) \subset \mathbb{A}^{n+1} \text{ if } X \neq \emptyset, \text{ and } \widehat{\emptyset} = \emptyset \subset \mathbb{A}^{n+1}.$$

Exercise. If $\emptyset \neq X = \mathbb{V}(I) \subset \mathbb{P}^n$, for some homogeneous ideal $I \subset R$, then $\widehat{X}$ is the affine variety associated to the ideal $I \subset R$,

$$\widehat{X} = \mathbb{V}_{\text{affine}}(I) \subset \mathbb{A}^{n+1}.$$

Remark. $X = \emptyset$ only arises if $I \subset R$ does not vanish on any line in $\mathbb{A}^{n+1}$. By homogeneity of I, this forces $\mathbb{V}(I) \subset \mathbb{A}^{n+1}$ to be either $\emptyset$ or $\{0\}$, which by Nullstellensatz corresponds respectively to $I = R$ or $I = (x_0, \ldots, x_n)$. We want $\widehat{X} = \emptyset$ so $I = R$. The exercise would fail for the **irrelevant ideal**

$$I_{irr} = (x_0, \ldots, x_n).$$

Notice the maximal homogeneous ideal I_{irr} does not correspond to a point in $\mathbb{P}^n$ ([0] is not allowed).

In Section 3.3 we could have defined

$$\mathbb{V}(I) = \{a \in \mathbb{P}^n : f(\alpha) = 0 \text{ for all } f \in I, \text{ and all representatives } \alpha \in \mathbb{A}^{n+1} \text{ of } a\},$$

so here $\alpha \in \pi^{-1}(a)$ is any point on the line $k \cdot \alpha$ defined by a.

Exercise. Check this definition gives the same $\mathbb{V}(I)$, by using Lemma 3.1 (so $f(k \cdot \alpha) = 0$ forces all homogeneous parts of f to vanish at $a \in \mathbb{P}^n$).

Exercise.[1] Show that $\mathbb{V}(I) = \pi(\mathbb{V}_{\text{affine}}(I) \setminus 0)$.

3.5. VANISHING IDEAL

$R = k[x_0, \ldots, x_n]$.

For any set $X \subset \mathbb{P}^n$, define $\mathbb{I}^h(X)$ to be the homogeneous ideal generated by the homogeneous polys vanishing on X:

$$\mathbb{I}^h(X) = \langle F \in R : F \text{ homogeneous}, F(X) = 0 \rangle.$$

Exercise. If I is homogeneous, then $\mathbb{V}(\mathbb{I}^h(\mathbb{V}(I))) = \mathbb{V}(I)$ and $I \subset \mathbb{I}^h(\mathbb{V}(I))$.

Warning. $\mathbb{V}(I_{irr}) = \emptyset \subset \mathbb{P}^n$, but $\mathbb{I}^h(\emptyset) = R \neq \sqrt{I_{irr}} = I_{irr}$. Similarly, if $\sqrt{I} = I_{irr}$ then $\mathbb{V}(I) = \mathbb{V}(\sqrt{I}) = \emptyset$ and $\mathbb{I}^h(\mathbb{V}(I)) = R$. These are the only cases where the proj.Nullstellensatz fails (Sec.3.6).

[1] *Hint. Notice that* $\mathbb{V}(I) = X = \pi(\widehat{X} \setminus 0) = \pi(\mathbb{V}_{affine}(I) \setminus 0)$.

Lemma 3.2.

$$\begin{aligned}\mathbb{I}^h(X) &= \{f \in R : f(\alpha) = 0 \textit{ for every } \alpha \in \mathbb{A}^{n+1} \textit{ representing any point of } X \subset \mathbb{P}^n\} \\ &= \mathbb{I}(\widehat{X}).\end{aligned}$$

Proof. This follows by Lemma 3.1: $f \in \mathbb{I}^h(X) \Leftrightarrow f(X) = 0 \Leftrightarrow f(\widehat{X}) = 0 \Leftrightarrow f \in \mathbb{I}(\widehat{X})$. □

3.6. PROJECTIVE NULLSTELLENSATZ

Theorem (Projective Nullstellensatz)**.**

$$\boxed{\mathbb{I}^h(\mathbb{V}(I)) = \sqrt{I} \quad \text{for any homogeneous ideal } I \text{ with } \sqrt{I} \neq I_{irr}}$$

Proof. $\mathbb{V}_{\text{affine}}(I) \neq \{0\}$ by the affine Nullstellensatz, as $\sqrt{I} \neq I_{irr}$. So $X = \mathbb{V}(I) = \pi(\mathbb{V}_{\text{affine}}(I) \setminus 0) \subset \mathbb{P}^n$ is non-empty, so its affine cone is $\widehat{X} = \mathbb{V}_{\text{affine}}(I)$. Using Lemma 3.2 and the affine Nullstellensatz we obtain: $\mathbb{I}^h(X) = \mathbb{I}(\widehat{X}) = \mathbb{I}(\mathbb{V}_{\text{affine}}(I)) = \sqrt{I}$. □

Remark. From Section 3.4, if $X = \mathbb{V}(I) = \emptyset$, then $I =$ either R or I_{irr}, but $\mathbb{I}^h(X) = R$.

Theorem. *There are* $1:1$ *correspondences*

$$\begin{aligned}\{\textit{proj. vars. } X \subset \mathbb{P}^n\} &\leftrightarrow \{\textit{homogeneous radical ideals } I \neq I_{irr}\}\\ \{\textit{irred. proj. vars. } X \subset \mathbb{P}^n\} &\leftrightarrow \{\textit{homogeneous prime ideals } I \neq I_{irr}\}\\ \{\textit{points of } \mathbb{P}^n\} &\leftrightarrow \{\textit{“maximal” homogeneous ideals } I \neq I_{irr}\}\\ \emptyset &\leftrightarrow \{\textit{the homogeneous ideal } R\}\end{aligned}$$

where the maps are: $X \mapsto \mathbb{I}^h(X)$ *and* $\mathbb{V}(I) \leftarrow I$.

The point $p = [a_0 : \cdots : a_n] \in \mathbb{P}^n$ *corresponds*[1] *to the homogeneous ideal*

$$\mathfrak{m}_p = \langle a_ix_j - a_jx_i : \textit{all } i, j \rangle = \{\textit{homogeneous polys vanishing at } a\}$$

which amongst homogeneous ideals different from I_{irr} *is maximal with respect to inclusion.*

Remark. The maximal ideals of $k[x_0, \ldots, x_n]$ are $\langle x_i - a_i : \text{all } i \rangle$ in bijection with points $a \in \mathbb{A}^{n+1}$. These ideals are not homogeneous for $a \neq 0$. In fact, the only homogeneous maximal ideal is I_{irr} (the case $a = 0$). The points $p \in \mathbb{P}^n$ correspond to lines in $\mathbb{A}^{n+1}$, so they are prime but not maximal ideals. These are the homogeneous ideals $\mathfrak{m}_p \subset I_{irr} \subset k[x_0, \ldots, x_n]$ shown above.

3.7. OPEN COVERS

$U_i = \mathbb{P}^n \setminus H_i = \{[x] \in \mathbb{P}^n : x_i \neq 0\}$ is called the ***i*-th coordinate chart**.

Exercise.

$$\begin{aligned}\phi_i : U_i &\to \mathbb{A}^n\\ [x] = [\tfrac{x_0}{x_i} : \cdots : \tfrac{x_{i-1}}{x_i} : 1 : \tfrac{x_{i+1}}{x_i} : \cdots : \tfrac{x_n}{x_i}] &\to (\tfrac{x_0}{x_i}, \cdots, \tfrac{x_{i-1}}{x_i}, \tfrac{x_{i+1}}{x_i}, \cdots, \tfrac{x_n}{x_i})\end{aligned}$$

is a bijection, indeed a homeomorphism in the Zariski topologies.[2]

Consequence:

$X \subset \mathbb{P}^n$ projective variety $\Rightarrow X = \bigcup_{i=1}^n (X \cap U_i)$ is an open cover of X by affine varieties.

Example. $X = \mathbb{V}(x^2 + y^2 - z^2) \subset \mathbb{P}^2$.

$U_z = \{[x : y : 1] : x, y \in k\}$ (the complement of $H_z = \{[x : y : 0] : [x : y] \in \mathbb{P}^1\}$).

$X \cap U_z = \mathbb{V}(x^2 + y^2 - 1) \subset \mathbb{A}^2$ is a “circle”.

What is X outside of $X \cap U_z$?

[1] Notice the generators of $\mathfrak{m}_p$ are the 2×2 subdeterminants of the matrix with rows a and x, so the vanishing of the functions in $\mathfrak{m}_p$ say that x is proportional to a. Another way to look at this, is to pick an affine patch $U_i \cong \mathbb{A}^n$ containing p (so $a_i \neq 0$). Then homogenize the maximal ideal $\mathfrak{m}_{p,i} = \langle x_j - \frac{a_j}{a_i} : \text{all } j \neq i \rangle$ that you get for $p \in \mathbb{A}^n$.

[2] Hints: to show it is a bijection, just define a map ψ_i in the other direction such that $\psi_i \circ \phi_i$ and $\phi_i \circ \psi_i$ are identity maps. It remains to show continuity of ϕ_i, ψ_i. To show continuity, you need to check that preimages of closed sets are closed. So you need to describe the ideals whose vanishing sets give $\phi_i^{-1}(\mathbb{V}(J))$ and $\psi_i^{-1}(\mathbb{V}(I)) = \phi_i(\mathbb{V}(I))$. You will find that in one case, you need to homogenise polynomials with respect to the i-th coordinate, so $f \in J \subset k[\mathbb{A}^n]$ becomes $\widetilde{f} = x_i^{\deg f} f(\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i})$ (but omitting $\frac{x_i}{x_i}$), and in the other case you plug in $x_i = 1$ and relabel variables.

$X \cap H_z = \mathbb{V}(x^2+y^2)$ gives $[1:i:0], [1:-i:0] \in \mathbb{P}^2$ (the "points at infinity" of $X \cap U_z$).
Geometric explanation: change variables to $\widetilde{y} = iy$ then
$X \cap U_z = \mathbb{V}(x^2 - \widetilde{y}^2 - 1) \subset \mathbb{A}^2$ is a "hyperbola", with asymptotes $\widetilde{y} = \pm x$, so $y = \pm ix$ are the two lines corresponding to the two new points $[1:i:0]$, $[1:-i:0]$ at infinity.

3.8. PROJECTIVE CLOSURE and HOMOGENISATION

Given an affine variety $X \subset \mathbb{A}^n$, we can view $X \subset \mathbb{P}^n$ via:

$$X \subset \mathbb{A}^n \cong U_0 \subset \mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}.$$

The **projective closure** $\overline{X} \subset \mathbb{P}^n$ of X is the closure[1] of the set $X \subset \mathbb{P}^n$.

Remark. $X \cong X' \not\Rightarrow \overline{X} \cong \overline{X'}$.

Example. $\mathbb{V}(y-x^2)$, $\mathbb{V}(y-x^3)$ in $\mathbb{A}^2$ are $\cong \mathbb{A}^1$, but their projective closures are not iso (see Hwk).

Given a polynomial $f \in k[x_1,\dots,x_n]$ of degree d, write $f = f_0 + f_1 + \cdots + f_d$ where f_i are the homogeneous parts. Then the **homogenisation** of f is

$$\widetilde{f} = x_0^d f_0 + x_0^{d-1} f_1 + \cdots + x_0 f_{d-1} + f_d.$$

EXAMPLES.
1) $x^2+y^2=1$ in $\mathbb{A}^2$ becomes $x^2+y^2=z^2$ in $\mathbb{P}^2$.
2) $y^2 = x(x-1)(x-c)$ in $\mathbb{A}^2$ becomes the elliptic curve $y^2z = x(x-z)(x-cz)$ in $\mathbb{P}^2$.

Exercise. $X = \mathbb{V}(\widetilde{f}) \subset \mathbb{P}^n \Rightarrow X \cap U_0 = \mathbb{V}(f) \subset U_0 \cong \mathbb{A}^n$.

Exercise. For any $f, g \in R$, show that $\widetilde{fg} = \widetilde{f} \cdot \widetilde{g}$.

Exercise. You can also *dehomogenise* a homogeneous polynomial $F \in R$ by setting $x_0 = 1$, so $f = F(1, x_1, \dots, x_n)$. Check that $F = x_0^\ell \widetilde{f}$, some $\ell \geq 0$.

Question: $\overline{X} = \mathbb{V}(\widetilde{I}) \subset \mathbb{P}^n$ for some ideal $\widetilde{I} \subset k[x_0,\dots,x_n]$. Can we find an ideal $\widetilde{I}$ that works, from the given ideal $I \subset k[x_1,\dots,x_n]$ which defines $X = \mathbb{V}(I) \subset \mathbb{A}^n$?

Theorem 3.3. *We can*[2] *take $\widetilde{I}$ to be the* **homogenisation** *of I,*

$$\begin{aligned} \widetilde{I} &= \text{the ideal generated by homogenisations of all elements of } I \\ &= \langle \widetilde{f} : f \in I \rangle. \end{aligned}$$

Remark. *In general, it is not sufficient to homogenize only a set of generators of I (see the Hwk).*

Proof. X aff.var $\subset \mathbb{A}^n \equiv U_0 = (x_0 \neq 0) \subset \mathbb{P}^n$.

Claim. $\mathbb{V}(\widetilde{I}) = \overline{X} \subset \mathbb{P}^n$.

Step 1. $\overline{X} \subset \mathbb{V}(\widetilde{I})$.
Pf. It suffices to check that the homogeneous generators of $\widetilde{I}$ vanish on $\overline{X}$.
Let $G \in \widetilde{I}$ be the homogenisation of some $g \in I$.
$\Rightarrow G(1, a_1, \dots, a_n) = g(a_1, \dots, a_n) = 0$ for $(a_1, \dots, a_n) \in X = \mathbb{V}(I)$
$\Rightarrow G|_{U_0 \cap X} = G|_X = 0$ (viewing $X \subset U_0$, so $U_0 \cap X = X$)
$\Rightarrow X \subset \mathbb{V}(G)$
$\Rightarrow \overline{X} \subset \mathbb{V}(G)$ (note $\mathbb{V}(G)$ is already closed)
$\Rightarrow G|_{\overline{X}} = 0$.✓
Step 2. $\sqrt{\widetilde{I}} \supset \mathbb{I}^h(\overline{X})$. (We know secretly these are equal, see the Corollary below)
It suffices to show that homogeneous generators $G \in \mathbb{I}^h(\overline{X})$ are in $\sqrt{\widetilde{I}}$.
$\Rightarrow G|_{\overline{X}} = 0$.
$\Rightarrow G|_X = 0$. (Since $X \subset \overline{X} \cap U_0$, indeed equality holds by the above exercise)
$\Rightarrow f = G(1, x_1, \dots, x_n) \in \mathbb{I}(X)$.

[1] i.e. the smallest Zariski closed set of $\mathbb{P}^n$ containing X.
[2] The obvious choice is to take $I = \mathbb{I}(X)$ and $\widetilde{I}$ = homogenisation of $\mathbb{I}(X)$. However, the Theorem allows you also to start with a non-radical I: just homogenise and you get a (typically non-radical) $\widetilde{I}$ that works, so $\overline{X} = \mathbb{V}(\widetilde{I}) = \mathbb{V}(\sqrt{\widetilde{I}})$.

$\Rightarrow f^m \in I$, some m. (Using the Nullstellensatz $\sqrt{I} = \mathbb{I}(X)$)
$\Rightarrow$ homogenise: $\widetilde{f^m} = \widetilde{f}^m \in \widetilde{I}$.
$\Rightarrow$ Since[1] $G = x_0^\ell \widetilde{f}$, it follows that $G^m = x_0^{\ell m} \widetilde{f}^m \in \widetilde{I}$, so $G \in \sqrt{\widetilde{I}}$. ✓
Step 3. $\mathbb{V}(\widetilde{I}) \subset \overline{X}$.
Follows by Step 2: $\mathbb{V}(\widetilde{I}) = \mathbb{V}(\sqrt{\widetilde{I}}) \subset \mathbb{V}(\mathbb{I}^h(\overline{X})) = \overline{X}$. ✓ □

Exercise. How does the above proof simplify, if we start with $I = \mathbb{I}(X)$?

Lemma. *The homogenisation $\widetilde{I}$ of a radical ideal I is also radical.*

Proof. First, the easy case: suppose $G \in \sqrt{\widetilde{I}}$ is homogeneous.
Thus $G^m \in \widetilde{I}$ for some m, and we claim $G \in \widetilde{I}$.
$G^m(1, x_1, \dots, x_n) = (G(1, x_1, \dots, x_n))^m \in I$
$\Rightarrow f = G(1, x_1, \dots, x_n) \in I$, since I is radical.
$\Rightarrow$ homogenising, $\widetilde{f} \in \widetilde{I}$.
$\Rightarrow G = x_0^\ell \widetilde{f} \in \widetilde{I}$, some ℓ (just as in Step 2 of the previous proof). ✓
Secondly, the general case: $g \in \sqrt{\widetilde{I}}$.
$\Rightarrow g = G_0 + \cdots + G_d$ (decomposition into homogeneous summands).
$\Rightarrow g^m = (G_0 + \cdots + G_{d-1})^m + (\text{terms involving } G_d) + G_d^m \in \widetilde{I}$, some m.
$\Rightarrow G_d^m \in \widetilde{I}$, since $\widetilde{I}$ is homogeneous (G_d^m is the homogeneous summand of g^m of degree dm).
$\Rightarrow G_d \in \widetilde{I}$, by the easy case.
$\Rightarrow (g - G_d)^m = (G_0 + \cdots + G_{d-1})^m = g^m - (\text{terms involving } G_d) - G_d^m \in \widetilde{I}$.
$\Rightarrow$ by the same argument, $G_{d-1}^m \in \widetilde{I}$ so $G_{d-1} \in \widetilde{I}$. Continue inductively with $g - G_d - G_{d-1}$, etc.
$\Rightarrow G_0, \dots, G_d \in \widetilde{I}$, so $g \in \widetilde{I}$. ✓ □

Corollary. *In Theorem 3.3, if we take $I = \mathbb{I}(X)$ then $\widetilde{I}$ = (homogenisation of $\mathbb{I}(X)$) is radical and $\widetilde{I} = \mathbb{I}^h(\mathbb{V}(\widetilde{I}))$ by Hilbert's Nullstellensatz.*

3.9. MORPHISMS OF PROJECTIVE VARIETIES

Motivation. $\mathbb{P}^n$ is already a "global" object, covered by affine pieces. So it is not reasonable to define morphisms in terms of $\mathrm{Hom}(\mathbb{P}^n, \mathbb{A}^1)$. In fact $\mathrm{Hom}(\mathbb{P}^n, \mathbb{A}^1)$ ought to only consist of constant maps: $\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$, so restricting to $\mathbb{A}^n$ we ought to get $\mathrm{Hom}(\mathbb{A}^n, \mathbb{A}^1) \cong k[x_1, \dots, x_n]$, and these polynomials (if non-constant) will blow-up at the points at infinity which form $\mathbb{P}^{n-1} \subset \mathbb{P}^n$.

Definition. *For proj vars $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$, a* ***morphism*** *$F : X \to Y$ means: for every $p \in X$ there is an open neighbourhood $p \in U \subset X$, and* homogeneous *polynomials $f_0, \dots, f_m \in R$ of the same degree,*[2] *with*

$$F : U \to Y, \; F[a] = [f_0(a) : \cdots : f_m(a)].$$

Rmk 1. The fact that the degrees of the f_i are equal ensures that the map is well-defined: $F[\lambda a] = [f_0(\lambda a) : \cdots : f_m(\lambda a)] = [\lambda^d f_0(a) : \cdots : \lambda^d f_m(a)] = [f_0(a) : \cdots : f_m(a)] = F[a]$.
Rmk 2. When constructing such F, you must ensure the f_i do not vanish simultaneously at any a (and that F actually lands in $Y \subset \mathbb{P}^m$).
Rmk 3. An **isomorphism** means a bijective morphism whose inverse is also a morphism.

EXAMPLES.
1) The Veronese embedding $F : \mathbb{P}^1 \to \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$, $[s:t] \mapsto [s^2 : st : t^2]$ is a morphism.
We want to build an inverse morphism.
If $s \neq 0$ then $[s:t] = [s^2 : st]$.
If $t \neq 0$ then $[s:t] = [st : t^2]$.

[1] Example: $G = x_0^2(x_1^2 - x_0 x_1)$, $f = x_1^2 - x_1$, $\widetilde{f} = x_1^2 - x_0 x_1$ has lost the x_0^2 that appeared in G.
[2] recall, by convention, that the zero polynomial has every degree.

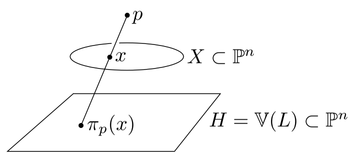
So define $G : \mathbb{V}(xz - y^2) \to \mathbb{P}^1$ by $[x : y : z] \mapsto [x : y]$ if $x \neq 0$, and $[x : y : z] \mapsto [y : z]$ if $z \neq 0$. This is a well-defined map, since on the overlap $x \neq 0, z \neq 0$ we have

$$[x : y] = [xz : yz] = [y^2 : yz] = [y : z].$$

It is now easy to check that $F \circ G = \mathrm{id}$, $G \circ F = \mathrm{id}$.

2) **Projection from a point**. Given a proj var $X \subset \mathbb{P}^n$, a hyperplane $H = \mathbb{V}(L) \subset \mathbb{P}^n$, and a point $p \notin X$ and $\notin H$, define

$$\pi_p : X \to H \cong \mathbb{P}^{n-1},\ x \mapsto (\text{the point in } H \text{ where the line through } x \text{ and } p \text{ hits } H).$$



Example. $p = [1 : 0 : \cdots : 0]$, $H = \mathbb{V}(x_0)$, then $\pi_p[x_0 : \cdots : x_n] = [0 : x_1 : \cdots : x_n]$.
Exercise. Show that by a linear change of coordinates on $\mathbb{A}^{n+1}$ the general case reduces to the Example. (*Hint. Use a basis* $\widetilde{p}, h_1, \ldots, h_n$ *where* $\widetilde{p} \in \mathbb{A}^{n+1}$ *represents* p*, and* h_j *is a basis for* H*.*)

3) **Projective equivalences**. An isomorphism $X \cong Y$ of projective varieties $X, Y \subset \mathbb{P}^n$ is a projective equivalence if it is the restriction of a linear isomorphism

$$\mathbb{P}^n \to \mathbb{P}^n,\ [x] \mapsto [Ax]$$

i.e. induced by a linear isomorphism $\mathbb{A}^{n+1} \to \mathbb{A}^{n+1}$, $x \mapsto Ax$, where $A \in GL(n+1, k)$. Since $[Ax] = [\lambda Ax]$ we only care about A modulo scalar matrices $\lambda\,\mathrm{id}$, so $A \in PGL(n+1, k) = \mathbb{P}(GL(n+1, k))$.

FACT.[1] The group $\mathrm{Aut}(\mathbb{P}^n)$ of isomorphisms $\mathbb{P}^n \to \mathbb{P}^n$ is precisely $PGL(n+1, k)$.

Example. $H_0 \cong H_1$ via $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \mathrm{Id} \end{smallmatrix}\right)$.
Example. Putting a projective linear subspace into standard form: if $f_1, \ldots, f_m$ are homogeneous linear polys which are linearly independent $\Rightarrow \mathbb{V}(f_1, \ldots, f_m) \cong \mathbb{V}(x_1, \ldots, x_m)$.
Non-example. $\mathbb{P}^2 \supset H_0 \cong \mathbb{P}^1 \cong \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$ but they are not projectively equivalent since their degrees are different (we discuss degrees in Sec.9.1).

3.10. GRADED RINGS and HOMOGENEOUS IDEALS

Recall $R = k[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} R_d$ where R_d = homogeneous polys of degree d, and $R_0 = k$, and by convention $0 \in R_d$ for all d. In particular, the irrelevant ideal is $I_{irr} = (x_0, \ldots, x_n) = \bigoplus_{d > 0} R_d$.

Definition. *Let A be a ring (commutative). An* $\mathbb{N}$-***grading*** *means*

$$A = \bigoplus_{d \geq 0} A_d$$

as abelian groups[2] *under addition, and the grading by d is compatible with multiplication:*

$$A_d \cdot A_e \subset A_{d+e}.$$

The elements in A_d are called the **homogeneous elements of degree** d.
Note every $f \in A$ is uniquely a finite sum $\sum f_d$ of homogeneous elements $f_d \in A_d$.
An **isomorphism of graded rings** $A \to B$ is an iso of rings which respects the grading ($A_d \to B_d$).
$I \subset A$ ideal, then define

$$I_d = I \cap A_d$$

which is a subgroup of A_d under addition.

[1] Hartshorne, Chapter II, Example 7.1.1. This requires machinery beyond this course. You may have seen the case of holomorphic isomorphisms $\mathbb{P}^1 \to \mathbb{P}^1$ over $k = \mathbb{C}$: you get the Möbius maps $z \mapsto \frac{az+b}{cz+d}$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PGL(2, \mathbb{C})$.
[2] so $A_d \subset A$ is an additive subgroup and $A_d \cap A_e = \{0\}$ if $d \neq e$.

Definition. $I \subset A$ is a **homogeneous ideal** if[1]

$$I = \bigoplus_{d \geq 0} I_d.$$

EXERCISES.
1) I homogeneous $\Leftrightarrow$ I generated by homogeneous elements.
2) I homogeneous $\Leftrightarrow$ for every $f \in I$, also all homogeneous parts $f_d \in I$.
3) If I homogeneous,

$$I \text{ prime ideal } \Leftrightarrow \forall \textit{ homogeneous } f, g \in A, fg \in I \text{ implies } f \in I \text{ or } g \in I.$$

4) Sums, products, intersections, radicals of homogeneous ideals are homogeneous.
5) A graded, I homogeneous $\Rightarrow$ A/I graded, by declaring $(A/I)_d = A_d/I_d$
(So explicitly: $[\sum f_d] = \sum [f_d] \in A/I$ just inherits the grading from A).

3.11. HOMOGENEOUS COORDINATE RING

$R = k[x_0, \ldots, x_n]$ with grading determined by the usual grading of R (so $x_0, \ldots, x_n$ have degree 1). $X \subset \mathbb{P}^n$ a projective variety. The **homogeneous coordinate ring** $S(X)$ is the graded ring[2]

$$\boxed{S(X) = R/\mathbb{I}^h(X) = R/\mathbb{I}(\widehat{X}) = k[\widehat{X}]}$$

Example. $S(\mathbb{P}^n) = R = k[x_0, \ldots, x_n]$.
Example. $X = \mathbb{V}(yz - x^2) \subset \mathbb{P}^2$ (proj.closure of parabola $y = x^2$) then $S(X) = k[x, y, z]/(yz - x^2)$.
Remark. $f \in S(X)$ defines a function $f : \widehat{X} \to k$, but not $X \to k$ (due to rescaling).

Lemma 3.4. $S(X) \cong S(Y)$ *as graded k-algebras* $\Leftrightarrow$ $X \cong Y$ *via a projective equivalence.*

Proof. ($\Leftarrow$) Let $\varphi \colon \mathbb{P}^n \to \mathbb{P}^n$ be a linear iso inducing $Y \cong X$. So $\varphi^*(x_j) = \sum A_{ji} x_i$ is a linear poly in the homogeneous coords x_i of $\mathbb{P}^n$, where A is invertible. So $\varphi^* : S(X)_1 \to S(Y)_1$ is a vector space iso (the x_i span the vector spaces $S(X)_1$, $S(Y)_1$). This induces a unique[3] algebra iso $S(X) \to S(Y)$.
($\Rightarrow$) Given an iso $\psi : S(X) \cong S(Y)$, it restricts to a linear iso $S(X)_1 \to S(Y)_1$, $x_j \mapsto \sum A_{ji} x_i$. Suppose first the simple case that the x_i are linearly independent in $S(X)_1$, then the x_i are linearly independent also in $S(Y)_1$ (indeed $S(X)_1 = S(Y)_1 = k[x_0, \ldots, x_n]_1$). Then A is a well-defined invertible matrix. Thus $\varphi : \mathbb{P}^n \to \mathbb{P}^n$, $\varphi[a_0 : \ldots : a_n] = [\sum A_{0i} a_i : \ldots : \sum A_{ni} a_i]$ is a linear iso of $\mathbb{P}^n$ with $\varphi^* = \psi$, in particular $\varphi^* \mathbb{I}(X) \subset \mathbb{I}(Y)$ so $\varphi(Y) \subset X$, and $\varphi : Y \to X$ is the required proj.equiv.
Now the harder case when x_i are linearly dependent in $S(X)_1$. Notice these linear dependency relations are precisely $\mathbb{I}^h(X)_1$. Suppose $d = \dim_k \mathbb{I}^h(X)_1$. By pre-composing ψ by a linear equivalence of $\mathbb{P}^n$ we may assume $\mathbb{I}^h(X)_1 = \langle x_n, x_{n-1}, \ldots, x_{n-d+1} \rangle$. Then we can view $X \subset \mathbb{P}^{n-d}$ since the last d coordinates vanish on X, and $S(X)$ will not have changed up to isomorphism. As $\dim_k S(X)_1 = \dim_k S(Y)_1$, we can do the same for Y by post-composing ψ by another projective equivalence. Now we can apply the simple case to $X, Y \subset \mathbb{P}^{n-d}$ to obtain an invertible matrix $A \in GL(n - d + 1, k)$. Finally use $\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$ for a $d \times d$ identity matrix I to obtain the required projective equivalence for the original $X, Y \subset \mathbb{P}^n$ up to pre/post-composing with projective equivalences. □

Non-Example. $\mathbb{P}^2 \supset H_2 = X \cong \mathbb{P}^1 \cong Y = \nu_2(\mathbb{P}^1) \subset \mathbb{P}^2$ via $[x_0 : x_1 : 0] \mapsto [x_0^2 : x_0 x_1 : x_1^2]$, but $S(X) = k[x_0, x_1]$ and $S(Y) = k[y_0, y_1, y_2]/(y_0 y_2 - y_1^2)$ are not isomorphic as graded algebras: they contain a different[4] number of linearly independent generators of degree 1. Thus $\nu_2(\mathbb{P}^1)$ is (of course) not projectively equivalent to the hyperplane H_2.

Warning. $X \cong Y$ proj.vars $\not\Rightarrow \widehat{X} \cong \widehat{Y}$, so $S(X)$ is not an isomorphism-invariant of X.[5]
Example. $X = \mathbb{P}^1 \cong Y = \nu_2(\mathbb{P}^1) \subset \mathbb{P}^2$ via $[x_0 : x_1] \mapsto [x_0^2 : x_0 x_1 : x_1^2]$, but $S(X) = k[\widehat{X}] = k[x_0, x_1]$

[1] Recall $\bigoplus$ means that each $f \in I$ can be *uniquely* written as a finite sum $f = f_0 + \cdots + f_N$ with $f_d \in I_d$, some N.
[2] here $\widehat{X} \subset \mathbb{A}^{n+1}$ is the affine cone over X, see Section 3.4.
[3] e.g. $\varphi^*(x_0^2 x_3 + 7x_5^3) = \varphi^*(x_0)^2 \varphi^*(x_3) + 7\varphi^*(x_5)^3$.
[4] $k[\widehat{X}]$ has 2, e.g. x_0, x_1, and $k[\widehat{Y}]$ has 3, e.g. y_0, y_1, y_2. So $\dim_k S(X)_1 = 2$ and $\dim_k S(Y)_1 = 3$.
[5] Meaning, $X \cong Y$ does not imply $S(X) \cong S(Y)$, unlike the case of affine varieties: $\widehat{X} \cong \widehat{Y} \Leftrightarrow k[\widehat{X}] \cong k[\widehat{Y}]$.

and $S(Y) = k[\widehat{Y}] = k[y_0, y_1, y_2]/(y_0y_2 - y_1^2)$ are not isomorphic k-algebras because the first is a UFD but the second is not (consider the two factorisations $y_0y_2 = y_1^2$). Alternatively, one can[1] show that the affine cones $\widehat{X} = \mathbb{A}^2$, $\widehat{Y} = \mathbb{V}(xz - y^2) \subset \mathbb{A}^3$ are not isomorphic using methods from Section 13.

Harder exercise. An (ungraded) k-algebra isomorphism $S(X) \cong S(Y)$ implies $\widehat{X} \cong \widehat{Y}$, but in fact it also implies that $X \cong Y$ via a projective equivalence.[2]

4. CLASSICAL EMBEDDINGS

4.1. VERONESE EMBEDDING

Example 4.1. *The Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ is*

$$\nu_2 : \mathbb{P}^1 \hookrightarrow \mathbb{P}^2,\ [x_0 : x_1] \mapsto [x_0^2 : x_0x_1 : x_1^2].$$

The image $\nu_2(\mathbb{P}^1)$ is called the **rational normal curve of degree 2**,

$$\nu_2(\mathbb{P}^1) = \mathbb{V}(z_{(2,0)}z_{(0,2)} - z_{(1,1)}^2) \subset \mathbb{P}^2$$

labelling the homogeneous coordinates on $\mathbb{P}^2$ by $[z_{(2,0)} : z_{(1,1)} : z_{(0,2)}]$.

Example 4.2. *The image of $\nu_d : \mathbb{P}^1 \hookrightarrow \mathbb{P}^d, [x_0 : x_1] \mapsto [x_0^d : x_0^{d-1}x_1 : \ldots : x_1^d]$ is called* **rational normal curve of degree d**.

Motivation. Given a homogeneous polynomial in two variables, you can view its vanishing locus as the intersection of $\nu_d(\mathbb{P}^1)$ with a hyperplane. For example, $x_0^2x_1 - 8x_1^3 = 0$ is the intersection of $\nu_3(\mathbb{P}^1) \subset \mathbb{P}^3$ with the hyperplane $z_{(2,1)} - 8z_{(0,3)} = 0$ using coordinates $[z_{(3,0)} : z_{(2,1)} : z_{(1,2)} : z_{(0,3)}]$ on $\mathbb{P}^3$. The Veronese map, defined below, generalizes this to any number of variables.

Definition (Veronese embedding). *The* ***Veronese map*** *is*

$$\boxed{\nu_d : \mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{n+d}{d}-1},\ [x_0 : \ldots : x_n] \mapsto [\ldots : x^I : \ldots]}$$

running over all monomials $x^I = x_0^{i_0}x_1^{i_1}\cdots x_n^{i_n}$ of degree $d = i_0 + \cdots + i_n$, where you pick some ordering of the indices $I \subset \mathbb{N}^{n+1}$ whose sum of all entries equals d, e.g. lexicographic ordering.
The image of ν_d is called a ***Veronese variety***.

Remark 4.3 (Counting polynomials). *How many monomials are there in $n+1$ variables $x_0, x_1, \ldots, x_n$ of degree d? Draw $n + d$ points, e.g. for $n = 3, d = 4$:*

$$\bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet$$

Then choosing d of these points determines uniquely a monomial of degree d, e.g.

$$\star\ \bullet\ \star\ \star\ \bullet\ \star\ \bullet$$

means $x_0^1x_1^2x_2^1x_3^0$ (count up the stars to get the powers). So the number of monomials is $\binom{n+d}{d}$.

Remark 4.4 (Veronese surface). *The image of*

$$\nu_2 : \mathbb{P}^2 \hookrightarrow \mathbb{P}^5,\ [x_0 : x_1 : x_2] \mapsto [x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2]$$

is called **Veronese surface**.

[1] Proof: $\widehat{X} = \mathbb{A}^2$ is non-singular, but $\widehat{Y}$ has a singularity at 0 since the tangent space at (a, b, c) is defined by $c(x-a) - 2b(y-b) + a(z-c) = 0$, and at $(a, b, c) = 0 \in \mathbb{A}^3$ this equation is identically zero. So $T_0\widehat{Y} = \mathbb{A}^3 \not\cong \mathbb{A}^2 \cong T_p\widehat{X}$.

[2] If the ambient dimensions n, m are not the same, then one gets a linear injection $\mathbb{A}^{n+1} \hookrightarrow \mathbb{A}^{m+1}$, but one can extend that to a linear isomorphism $\mathbb{A}^{m+1} \to \mathbb{A}^{m+1}$ by inserting additional variables.

Theorem 4.5.

$$\begin{aligned}\mathbb{P}^n \cong \mathrm{Image}(\nu_d) &= \mathbb{V}(z_Iz_J - z_Kz_L : I + J = K + L)\\ &= \bigcap_{I+J=K+L} (\textit{quadrics } \mathbb{V}(z_Iz_J - z_Kz_L)) \subset \mathbb{P}^{\binom{n+d}{d}-1}\end{aligned}$$

where we run over all multi-indices I, J, K, L of type $(i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $i_0 + \cdots + i_n = d$. Moreover, the ideal $\langle z_Iz_J - z_Kz_L : I + J = K + L\rangle$ is radical.[1]

Example. *For $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$, the equation $z_{(2,0)}z_{(0,2)} - z_{(1,1)}z_{(1,1)} = 0$ is the familiar $xz - y^2 = 0$.*

Proof. That image(ν_d) satisfies the equations $z_Iz_J - z_Kz_L = 0$ is obvious since $z_Iz_J = x^Ix^J = x^{I+J}$. Conversely, we find an explicit inverse morphism for ν_d. Fix $J = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ with $d - 1 = i_0 + \cdots + i_n$, and denote $J_\ell = (j_0, \ldots, j_\ell + 1, \ldots, j_n)$ (so we add one in the ℓ-th slot of I, and these indices now add up to d). Define

$$\varphi_J : \cap(\text{those quadrics}) \dashrightarrow \mathbb{P}^n,\ [\ldots : z_I : \ldots] \mapsto [z_{J_0} : z_{J_1} : \ldots : z_{J_n}]$$

which is a well-defined morphism except on the closed set where all $z_{J_\ell} = 0$.

Example to clarify. *For $\nu_2(\mathbb{P}^1)$, $J = (0, 1)$, $\varphi_J : [z_{(2,0)} : z_{(1,1)} : z_{(0,2)}] \mapsto [z_{(1,1)} : z_{(0,2)}]$ corresponds to the map $[x^2 : xy : y^2] \mapsto [xy : y^2] = [x : y]$ which is defined for $y \neq 0$, and notice $y = (x, y)^J$.*

The φ_J, as we vary J, agree on overlaps. Indeed for another such J', notice $J_\ell + J'_{\ell'} = J_{\ell'} + J'_\ell$ (this equals $J + J'$ plus add 1 in the two slots ℓ, ℓ'), hence $z_{J_\ell}z_{J'_{\ell'}} = z_{J_{\ell'}}z_{J'_\ell}$, and thus[2] $\varphi_J([z]) = \varphi_{J'}([z])$.

We claim φ_J is an inverse of ν_d wherever φ_J is defined. The key observation is: $x^{J_\ell} = x^J \cdot x_\ell$. Notice $\varphi_J \circ \nu_d([x]) = [x^{J_0} : \ldots : x^{J_n}] = [x_0 : \ldots : x_n]$ (rescale by $1/x^J$).

Now consider $\nu_d \circ \varphi_J([z_I])$. Abbreviate $x_j = z_{J_j}$, then $\varphi_J([z_I]) = [x_0 : \ldots : x_n]$ and $\nu_d \circ \varphi_J([z_I]) = [x^I]$, and one can[3] check this equals $[z_I]$. ∎

Theorem. $Y \subset \mathbb{P}^n$ *proj.var.* $\Rightarrow \mathbb{P}^n \supset Y \cong \nu_d(Y) \subset \mathbb{P}^m$ *is a proj.subvar.*

Proof. This is immediate: $\nu_d : \mathbb{P}^n \to \mathbb{P}^m$ is a homeomorphism onto a closed set (hence a *closed embedding*), so it sends closed sets to closed sets. We give below another, explicit, proof.

Key Trick: $\mathbb{V}(F) = \mathbb{V}(x_0F, x_1F, \ldots, x_nF) \subset \mathbb{P}^n$ since $x_0, \ldots, x_n$ cannot all vanish simultaneously.
So: $Y = \mathbb{V}(F_1, \ldots, F_N)$ for some F_i homog. of various degrees.
By Trick: $Y = \mathbb{V}(G_1, \ldots, G_M)$ for some G_i homog. of *same* degree $= c \cdot d$.
So: $G_i = H_i \circ \nu_d$ for some H_i homog. of same degree c.
So: $\mathbb{P}^n \supset Y = \mathbb{V}(G_1, \ldots, G_m) \xrightarrow{\nu_d} \mathbb{V}(H_1, \ldots, H_M) \subset \mathbb{P}^m$,
indeed: $\{a \in \mathbb{P}^n : G_i(a) \equiv H_i(\nu_d(a)) = 0\ \forall i\} \longrightarrow \{b \in \mathbb{P}^m : H_i(b) = 0\ \forall i\}$ via $a \mapsto \nu_d(a) = b$.
So $\nu_d(Y) = \nu_d(\mathbb{P}^n) \cap \mathbb{V}(H_1, \ldots, H_M)$. ∎

[1] *Non-examinable proof.* Trick from 3.8: the homogenisation of a radical ideal is radical. So it suffices to check it is a radical ideal on an affine patch. Example for $\nu_2 : \mathbb{P}^1 \to \mathbb{P}^2$: on the affine patch $z_1 \neq 0$ we can put $z_1 = 1$, so $z_3 = z_2^2$ and $k[z_2, z_3]/(z_3 - z_2^2) \cong k[z_2]$ is an integral domain, so the ideal is radical. General case: on the affine patch $z_{(d,0,\ldots,0)} = 1$, by the other non-examinable footnote all $z_I = x^I$ are determined by the $x_\ell = z_{J_\ell}$ for $J = (d-1, 0, \ldots, 0)$, $\ell = 0, \ldots, n$, and the x_ℓ are independent. So $k[z_I : \text{all } I]/\langle z_Iz_J - z_Kz_L : I + J = K + L\rangle \cong k[x_0, \ldots, x_n]$ which is an integral domain.

[2] In general, $[x_0 : \ldots : x_n] = [y_0 : \ldots : y_n] \in \mathbb{P}^n \Leftrightarrow x, y$ are proportional $\Leftrightarrow$ all 2×2 minors of the matrix $(x|y)$ vanish.

[3] *Non-examinable.* This is messy to check. We first need to check that $z_{(0,\ldots,0,d,0,\ldots,0)}$ cannot all vanish simultaneously. Suppose by contradiction that they do. We know some z_I is non-zero (since $[z_I] \in$ projective space). By reordering the indices (symmetry), WLOG $i_0 \geq i_1 \geq \cdots \geq i_n$ with $i_0 + \cdots + i_n = d$. Also, WLOG, this is the non-zero z_I with largest occurring maximal index i_0 (so $z_K = 0$ if K has any indices k_j larger than i_0). We claim $i_0 = d$, hence $I = (d, 0, \ldots, 0)$, so $z_I = z_{(d,0,\ldots)} = 0$, contradiction. Proof: if $i_0 \neq d$, then $i_1 \geq 1$ and $z_Iz_I = z_Kz_{K'}$ where $K = (i_0 + 1, i_1 - 1, i_2, \ldots)$, $K' = (i_0 - 1, i_1 + 1, i_2, \ldots)$. But $z_K = 0$ since $i_0 + 1 > i_0$, forcing $z_I = 0$, contradiction✓ Now, WLOG by reordering indices and then rescaling, $z_{(d,0,\ldots)} = 1$. It suffices to check $\nu_d \circ \varphi_J([z_I]) = [x^I]$ for a specific choice of J (since the various φ-maps agree on overlaps). We pick $J = (d-1, 0, \ldots)$. So $x_0 = z_{(d,0,\ldots)}$, $x_1 = z_{(d-1,1,0,\ldots)}$, $x_2 = z_{(d-1,0,1,0,\ldots)}$, etc. It is now a straightforward exercise to check that, using the quadratic equations "$z_Iz_J = z_Kz_L$" one obtains $x^I = z_{(d,0,\ldots)}^{d-1}z_{(i_0d+i_1(d-1)+i_2(d-1)+\ldots+i_n(d-1)-d(d-1), i_1, i_2, \ldots, i_n)} = z_{(i_0, i_1, \ldots)} = z_I$. As a warm-up, try checking first that $x_1x_2 = z_{(d,0,\ldots)}z_{(d-2,1,1,0,\ldots)} = z_{(d-2,1,1,0,\ldots)}$.

Example 4.6. *For* $\nu_2 : \mathbb{P}^2 \to \mathbb{P}^5$ *and* $Y = \mathbb{V}(x_0^3 + x_1^3) \subset \mathbb{P}^2$,

$$Y = \mathbb{V}(x_0(x_0^3 + x_1^3), x_1(x_0^3 + x_1^3), x_2(x_0^3 + x_1^3)) = \mathbb{V}(G_1, G_2, G_3)$$

for example: $G_1 = x_0(x_0^3 + x_1^3) = (x_0^2)^2 + (x_0x_1)x_1^2 = H_1 \circ \nu_2$ *taking* $H_1 = z_{(2,0,0)}^2 + z_{(1,1,0)}z_{(0,2,0)}$.
So $\nu_2(Y) = \nu_2(\mathbb{P}^2) \cap \mathbb{V}(H_1, H_2, H_3) \subset \mathbb{P}^5$.

Example. Let $X \subset \mathbb{P}^n$ be a projective variety. Consider a basic open set

$$D_F = X \setminus \mathbb{V}(F),$$

where $F = \sum a_I x^I$ is a homogeneous polynomial of degree d. Abbreviate $N = \binom{n+d}{d} - 1$. Then D_F can be identified with an affine variety in $\mathbb{A}^N$ as follows. By the same argument as in the Motivation above, $\nu_d(\mathbb{V}(F))$ lies in the hyperplane $H = \mathbb{V}(\sum a_I z_I) \subset \mathbb{P}^N$. Then, observe that we can identify

$$\nu_d(D_F) = \nu_d(X) \setminus H \subset \mathbb{P}^N \setminus H \cong \mathbb{A}^N$$

(you can use a linear isomorphism to map H to the standard hyperplane H_0, then recall $\mathbb{P}^N \setminus H_0 = U_0 \cong \mathbb{A}^N$ is a homeomorphism).
Explicit example. $X = \mathbb{V}(x) = [0:1] \in \mathbb{P}^1$, $F = x^2 + y^2$. Then $\nu_2(\mathbb{V}(F)) \subset \mathbb{V}(X+Z) \subset \mathbb{P}^2$ since $\nu_2([x:y]) = [X:Y:Z] = [x^2 : xy : y^2] \in \mathbb{P}^2$. Also, $X = \mathbb{V}(xx, yx)$ (Key Trick above), so

$$\nu_2(D_F) = \mathbb{V}(XZ - Y^2, X, Y) \setminus \mathbb{V}(X+Z) \subset \mathbb{P}^2.$$

Change coordinates: $a = X + Z$, $b = Y$, $c = Z$. So $\nu_2(D_F) \cong \mathbb{V}(a-c, b) \setminus \mathbb{V}(a) \subset U_0 = (a \neq 0) \cong \mathbb{A}^2$ (using coords b, c after rescaling so that $a = 1$) we obtain the affine variety (a point!) $b = 0, c = 1$.

4.2. SEGRE EMBEDDING

Below, we haven't actually defined what $\mathbb{P}^n \times \mathbb{P}^m$ means as a projective variety (we do *not* use the product topology, see Hwk). So it does not make sense to talk about "morphism" yet. In reality, we are *defining* the variety $\mathbb{P}^n \times \mathbb{P}^m$ as being the image of $\sigma_{n,m}$ in $\mathbb{P}^{\text{large power}}$. See Section 6.2.

Definition (Segre embedding[1])**.**

$$\begin{array}{rcl}\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m = \mathbb{P}(k^{n+1}) \times \mathbb{P}(k^{m+1}) & \hookrightarrow & \mathbb{P}(k^{n+1} \otimes k^{m+1}) \cong \mathbb{P}^{(n+1)(m+1)-1} = \mathbb{P}^{nm+n+m} \\ ([v],[w]) & \mapsto & [v \otimes w]\end{array}$$

More explicitly, in terms of the standard bases, $(\sum x_i e_i, \sum y_j f_j) \mapsto [\sum x_i y_j\, e_i \otimes f_j]$, *thus:*

$$([x_0 : \cdots : x_n], [y_0 : \cdots : y_m]) \mapsto [x_0y_0 : x_0y_1 : \cdots : x_0y_m : x_1y_0 : x_1y_1 : \cdots : x_ny_1 : \cdots : x_ny_m]$$

using the lexicographic ordering. The **Segre variety** *is* $\Sigma_{n,m} = \sigma_{n,m}(\mathbb{P}^n \times \mathbb{P}^m) \subset \mathbb{P}^{nm+n+m}$

Example. $\sigma_{1,1} : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$, $([x:y],[a:b]) \mapsto [xa : xb : ya : yb]$, so the image is defined by the equation $XW - YZ = 0$ using $[X : Y : Z : W]$ on $\mathbb{P}^3$.
You can think of $k^{n+1} \otimes k^{m+1} \cong \mathrm{Mat}_{(n+1)\times(m+1)}$ as matrices (the coefficient of $e_i \otimes f_j$ being the (i,j)-entry), then $\sigma_{n,m}([x],[y])$ is the matrix product of the column vector x and the row vector y, giving the matrix $[z_{ij}] = [x_i y_j]$.
Example. In the previous example, for $\sigma_{1,1}$, the matrix is $\left[\begin{smallmatrix} xa & xb \\ ya & yb\end{smallmatrix}\right] = \left[\begin{smallmatrix} X & Y \\ Z & W\end{smallmatrix}\right] \in \mathbb{P}(\mathrm{Mat}_{2\times 2})$.

Theorem 4.7.

$$\begin{array}{rcl}\Sigma_{n,m} & = & \mathbb{V}(\textit{all } 2\times 2 \textit{ minors of the matrix } (z_{ij})) \subset \mathbb{P}(\mathrm{Mat}_{(n+1)\times(m+1)}) \\ & = & \mathbb{V}(z_{ij}z_{k\ell} - z_{kj}z_{i\ell} : 0 \leq i < k \leq n, 0 \leq j < \ell \leq m)\end{array}$$

[1]Recall the **tensor product** of two k-vector spaces $V \otimes W$ is a vector space of dimension $\dim V \cdot \dim W$ with basis $v_i \otimes w_j$ where v_i, w_j are bases for V, W. So $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}$. You can extend the symbol $\otimes$ to all vectors by declaring that $(\sum \lambda_i v_i) \otimes (\sum \mu_j w_j) = \sum (\lambda_i \mu_j) v_i \otimes w_j$. Notice therefore that $0 \otimes w = 0 = v \otimes 0$, so do not confuse this with the product $V \times W$ which has dimension $\dim V + \dim W$, e.g. $\mathbb{R}^n \times \mathbb{R}^m \cong \mathbb{R}^{n+m}$.
Exercise. Prove $V^* \otimes W \cong \mathrm{Hom}(V,W)$ for finite dimensional vector spaces V, W, where V^* is the dual of V.

Proof. **Exercise.** Hint: use that the columns of a matrix are proportional iff all 2×2 minors vanish. An explicit inverse of $\sigma_{n,m}$ is:

$$\sigma_{n,m} : \Sigma_{n,m} \xrightarrow{\pi_{\text{col}} \times \pi_{\text{row}}} \mathbb{P}^n \times \mathbb{P}^m$$

where $\pi_{\text{col}} : \Sigma_{n,m} \to \mathbb{P}^n$ is the projection to any (non-zero) column (the images are the same since the columns are proportional). Similarly, $\pi_{\text{row}} : \Sigma_{n,m} \to \mathbb{P}^m$ is the projection to any (non-zero) row. □

4.3. GRASSMANNIANS AND FLAG VARIETIES

Definition (Grassmannian)**.** *The* **Grassmannian** *(of d-planes in k^n) is*

$$\mathrm{Gr}(d,n) = \{\textit{all } d\textit{-dimensional vector subspaces } V \subset k^n\}$$

where $1 \leq d < n$. *For example,* $\mathbb{P}^n = \mathrm{Gr}(1, n+1)$.
The **Flag variety** $\mathrm{Flag}(d_1, \ldots, d_s, n)$ *is*

$$\mathrm{Flag}(d_1, \ldots, d_s, n) = \{\textit{all flags of vector subspaces } V_1 \subset \cdots \subset V_s \subset k^n, \dim V_i = d_i\}.$$

Remark 4.8. *We can identify*

$$\mathrm{Gr}(d,n) = \{d \times n \textit{ matrices of rank } d\}/GL_k(d)$$

by identifying the d-plane $V \in \mathrm{Gr}(d,n)$ with the matrix whose rows are any choice of basis v_i for $V \subset k^n$. Two such choices of bases $v_i, \widetilde{v}_i$ are related by a change of basis matrix $g \in GL_k(d)$: $\widetilde{v}_i = \sum g_{ij} v_j$ (so above, $GL_k(d)$ acts by left-multiplication on $d \times n$ matrices). More abstractly: $\mathrm{Aut}(V) \cong GL_k(d) = \{d \times d$ *invertible matrices over* $k\}$.

4.4. PLÜCKER EMBEDDING

Definition 4.9 (Plücker embedding)**.** *The Plücker map is defined by*[1]

$$\begin{array}{rcl} Gr(d,n) & \hookrightarrow & \mathbb{P}(\Lambda^d k^n) \cong \mathbb{P}^{\binom{n}{d}-1} \\ V & \mapsto & k \cdot (v_1 \wedge \cdots \wedge v_d) \quad \textit{where } v_i \textit{ is a basis for } V.\end{array}$$

Exercise 4.10. *Show that explicitly the Plücker map is*

$$\begin{array}{rcl}\mathrm{Gr}(d,n) = \{d \times n \textit{ matrices of rank } d\}/\mathrm{GL}_d(k) & \hookrightarrow & \mathbb{P}^{\binom{n}{d}-1} \\ {[d \times n \textit{ matrix } A]} & \mapsto & [\textit{all } d \times d \textit{ minors } \Delta_{i_1,\ldots,i_d} \textit{ of } A]\end{array}$$

($\Delta_{i_1,\ldots,i_d}$ is the determinant of the matrix whose columns are the $i_1, \ldots, i_d$-th columns of A).

Non-examinable Fact. The image of the Plücker map is $\mathbb{V}(\text{Plücker relations}) \subset \mathbb{P}(\Lambda^d k^n)$. We now describe the relations.[2] Let $z_{i_1 i_2 \ldots i_d}$ be the homogeneous coordinates on $\mathbb{P}(\Lambda^d k^n)$, i.e. $z_{i_1 i_2 \ldots i_d}$ is the coefficient of the basis vector $e_{i_1} \wedge \cdots \wedge e_{i_d} \in \Lambda^d k^n$, where $i_1 < \cdots < i_d$. The Plücker relations are:

[1]Recall the d-th **exterior product** $\Lambda^d W$ of a k-vector space W is a k-vector space of dimension $\binom{\dim W}{d}$ generated by the symbols $w_{i_1} \wedge \cdots \wedge w_{i_d}$ where $i_1 < \cdots < i_d$, where w_i is a basis for W. One can extend the wedge-symbol to all vectors by declaring it to be alternating: $w_i \wedge w_j = -w_j \wedge w_i$ (in particular $w_i \wedge w_i = 0$), and multi-linear:

$$(\sum \lambda_i w_i) \wedge (\sum \mu_j w_j) = \sum_{i,j} \lambda_i \mu_j\, w_i \wedge w_j = \sum_{i<j} (\lambda_i \mu_j - \mu_i \lambda_j)\, w_i \wedge w_j.$$

Exercise. *Given any vectors $v_1, \ldots, v_d \in W$, let $V = \mathrm{span}(v_1, \ldots, v_d)$. Then for any $g \in \mathrm{Aut}(V)$, show that*

$$(gv_1) \wedge \cdots \wedge (gv_d) = (\det g)\, v_1 \wedge \cdots \wedge v_d.$$

If you think carefully, you'll notice this is the *definition* of determinant!
So definition 4.9 makes sense: i.e. the choice of basis v_i for V does not affect the line $k \cdot (v_1 \wedge \cdots \wedge v_d) \in \mathbb{P}(\Lambda^d k^n)$.
[2]Equivalently, recall the homogeneous coordinate ring of $\mathbb{P}(\Lambda^d k^n)$ is the polynomial ring in the variables denoted by $e_{i_1} \wedge \cdots \wedge e_{i_d}$, with strictly increasing indices, where e_j is the standard basis for k^n. Then the Plücker relations are the quadratic polynomial relations, given by:

$$(v_1 \wedge \cdots \wedge v_d) \cdot (w_1 \wedge \cdots \wedge w_d) = \sum_{i_1 < \cdots < i_\ell} (v_1 \wedge \cdots \wedge v_{i_1 - 1} \wedge w_1 \wedge v_{i_1+1} \wedge \cdots \wedge v_d) \cdot (v_{i_1} \wedge \cdots \wedge v_{i_\ell} \wedge w_{\ell+1} \wedge w_{\ell+2} \wedge \cdots \wedge w_d) \in S(\mathbb{P}(\Lambda^d k^n))$$

$$z_{i_1 i_2 \dots i_d} \cdot z_{j_1 j_2 \dots j_d} = \sum_{1 \le \ell < d} \sum_{r_1 < r_2 < \dots < r_\ell} z_{i_1 i_2 \dots i_{r_1 - 1} \mathbf{j_1} i_{r_1+1} \dots i_{r_2-1} \mathbf{j_2} i_{r_2+1} \dots i_{r_\ell - 1} \mathbf{j_\ell} i_{r_\ell+1} \dots i_d} \cdot z_{\mathbf{i_{r_1}} \mathbf{i_{r_2}} \dots \mathbf{i_{r_\ell}} j_{\ell+1} j_{\ell+2} \dots j_d}$$

On the right we interchanged the positions of $j_1, \dots, j_\ell$ with those of $i_{r_1}, \dots, i_{r_\ell}$, in that order. Notice we do not allow $\ell = d$ (the case $r_1 = 1, \dots, r_d = d$). On the right, we typically must reorder the indices on the z-variables to be strictly increasing: the convention is that $z_{\dots i \dots j \dots} = -z_{\dots j \dots i \dots}$ when we swap two indices (this equals zero if two indices are equal). E.g. $z_{32} = -z_{23}$ and $z_{22} = 0$.

Example 4.11. $\mathrm{Gr}(2,4)$*: the standard basis for* k^4 *is* e_1, e_2, e_3, e_4*, so a basis for* $\Lambda^2 k^4$ *is* $e_i \wedge e_j$ *for* $1 \le i < j \le 4$*, explicitly:*

$$e_1 \wedge e_2, \quad e_1 \wedge e_3, \quad e_1 \wedge e_4, \quad e_2 \wedge e_3, \quad e_2 \wedge e_4, \quad e_3 \wedge e_4.$$

Their coefficients define coordinates $[z_{12} : z_{13} : z_{14} : z_{23} : z_{24} : z_{34}]$ *for* $\mathbb{P}(\Lambda^2 k^4) \cong \mathbb{P}^5$*, for example* $6e_1 \wedge e_4 - 3e_2 \wedge e_4$ *has coordinates* $[0:0:6:0:-3:0]$*. Then we get*

$$\mathrm{Gr}(2,4) = \mathbb{V}(z_{12}z_{34} - z_{32}z_{14} - z_{13}z_{24}) = \mathbb{V}(z_{12}z_{34} - z_{13}z_{24} + z_{23}z_{14}) \subset \mathbb{P}^5.$$

In the notation of the previous footnote, in the homogeneous coordinate ring $S(\mathbb{P}(\Lambda^2 k^4))$ *we have*

$$(e_1 \wedge e_2) \cdot (e_3 \wedge e_4) = (e_3 \wedge e_2) \cdot (e_1 \wedge e_4) + (e_1 \wedge e_3) \cdot (e_2 \wedge e_4).$$

Exercise 4.12. *What are the Plücker relations written explicitly in terms of the* $d \times d$ *minors* $\Delta_{i_1,\dots,i_d}$*? (e.g. check that in the example* $\mathrm{Gr}(2,4)$ *you just need one relation:* $\Delta_{12}\Delta_{34} - \Delta_{13}\Delta_{24} + \Delta_{23}\Delta_{14} = 0$*.)*

Similarly, using the Plücker maps, for flag varieties:

$$\mathrm{Flag}(d_1, \dots, d_s, n) \hookrightarrow \mathbb{P}^{\binom{n}{d_1}-1} \times \dots \times \mathbb{P}^{\binom{n}{d_s}-1}.$$

The Zariski topology on Gr and Flag is defined as the subspace topology via the Plücker embeddings.

Remark 4.13. *All the embeddings above, over* $\mathbb{R}$ *(respectively over* $\mathbb{C}$*), are in fact smooth (respectively holomorphic) when viewing the spaces as smooth (respectively complex) manifolds.*

Lemma 4.14. *The Grassmannian* $\mathrm{Gr}(d,n)$ *is an irreducible variety.*

Proof. Let $W = \mathrm{span}(e_1, \dots, e_d) = k^d \oplus 0 \subset k^n$. Given $V = \mathrm{span}(v_1, \dots, v_d) \in \mathrm{Gr}(d,n)$ complete this to a basis $v_1, \dots, v_n$, then $A \in GL_n(k)$ with columns v_i will map W to V. This defines a surjective polynomial map $GL_n(k) \to \mathrm{Gr}(d,n), A \mapsto A(W)$, where we can view $GL_n(k)$ as an affine variety by identifying it with $\mathbb{V}(z \cdot \det - 1) \subset k^{n^2+1}$ via $A \mapsto (A, [\det A]^{-1})$ (here z is a new variable that formally inverts the determinant). By the final example 3 in Sec.2.13, it remains to show $GL_n(k)$ is irreducible. This is easy to check,[1] since $GL_n(k)$ is dense in k^{n^2}, and k^{n^2} is irreducible. ∎

Exercise. Show that $\mathrm{Flag}(d_1, \dots, d_s, n)$ is irreducible by a similar argument.

where we sum over all choices except $\ell = d$, and these hold for all $v_i \in k^n$, $w_j \in k^n$ (notice that if you expand these out, using the alternating multi-linear property of $\wedge$, then they become quadratic polynomial relations in the variables $e_{i_1} \wedge \dots \wedge e_{i_d}$). For a minimal set of relations, you just need the above for all v_i, w_j picked amongst the standard basis vectors e_j (so explicitly: $v_1 = e_{j_1}, \dots, v_d = e_{j_d}$ with $j_1 < \dots < j_d$ and similarly for the w's).

[1] In general, if $U \subset X$ is a dense open set of an irreducible affine variety X, then U is irreducible. Indeed, if $U = (C_1 \cap U) \cup (C_2 \cap U)$ for closed $C_1, C_2 \subset X$, then $X = \overline{U} = C_1 \cup C_2$, forcing $C_i = X$ for some i, so $U = C_i \cap U$. Finally, notice that relatively closed subsets $\mathbb{V}(I) \cap GL_n(k)$ for $GL_n(k) \subset k^{n^2}$ correspond precisely to relatively closed sets when viewing $GL_n(k) \subset k^{n^2+1}$. This is because given any poly f for k^{n^2+1}, $(\det)^N f$ cuts out the same subset in $GL_n(k)$ as f does, and it cuts out the same subset if we also replace all occurrences of $z \cdot \det$ in $(\det)^N f$ by 1. So WLOG the equations f used to define a relatively closed subset of $GL_n(k) \subset k^{n^2+1}$ can be chosen not to involve z.

5. EQUIVALENCE OF CATEGORIES

5.1. REDUCED ALGEBRAS

For any ring A, $f \neq 0 \in A$ is **nilpotent** if $f^m = 0$ for some m.
A is a **reduced ring** if it has no nilpotents.

Lemma. *A/I is reduced $\Leftrightarrow$ I is radical.*

Proof. If A/I is reduced: $f^m \in I \Leftrightarrow f^m = 0 \in A/I \Leftrightarrow f = 0 \in A/I \Leftrightarrow f \in I$.
If I is radical: $f^m = 0 \in A/I \Leftrightarrow f^m = 0 \in I \Leftrightarrow f \in I \Leftrightarrow f = 0 \in A/I$. ∎

Upshot:[1]

$$\begin{array}{ccc} \{\text{affine algebraic varieties}\} & \to & \{\text{f.g. reduced } k\text{-algebras}\} \\ (X \subset \mathbb{A}^n) & \mapsto & k[X] = R/\mathbb{I}(X) \\ ? & \leftarrow\!\shortmid & A. \end{array}$$

A f.g. $\Rightarrow$ one can pick generators $\alpha_1, \dots, \alpha_n$ (some n)
$\Rightarrow$ determines[2] a k-algebra hom $f : R = k[x_1, \dots, x_n] \to A$, $x_i \mapsto \alpha_i$
$\Rightarrow$ $I = \ker f \subset R$ is radical (since A is reduced)
$\Rightarrow$ $A \cong R/I$, so choose $X = \mathbb{V}(I)$.

Note. A different choice of generators can give a completely different embedding $X \subset \mathbb{A}^m$, some m. Due to this choice, the correct way to phrase the above "correspondence", between varieties and algebras, is as an equivalence of categories, which we now explain.

5.2. WARM-UP: EQUIVALENCE OF CATEGORIES IN LINEAR ALGEBRA

We assume some familiarity with very basic category theory terminology.

Category 1: $\mathcal{C}$
Objects:[3] k^n
Morphisms: $\mathrm{Hom}(k^n, k^m) = \mathrm{Mat}_{m \times n}(k)$ (matrices).

Category 2: $\mathcal{D}$
Objects: finite dimensional vector spaces over k.
Morphisms: $\mathrm{Hom}(V, W) = \{k\text{-linear maps } V \to W\}$.

Linear algebra courses secretly prove that the functor

$$\begin{array}{ccl} F : \mathcal{C} & \to & \mathcal{D} \\ k^n & \mapsto & k^n \\ (\text{matrix}) & \mapsto & (\text{linear map given by left multiplication by that matrix}) \end{array}$$

is an equivalence of categories. It is not an isomorphism of categories since there is no inverse functor $\mathcal{D} \to \mathcal{C}$. There is an obvious object to associate to V, namely $V \mapsto k^{\dim V}$, but at the level of morphisms in order to define a linear isomorphism $\mathrm{Hom}(V,V) \to \mathrm{Mat}_{\dim V \times \dim V}(k)$ we would need to choose a basis for V.

Define $G : \mathcal{D} \to \mathcal{C}$ as follows:
Pick a basis $v_1, \dots, v_n$ for each vector space V (heresy!)
For k^n we stipulate that we choose the standard basis $e_1, \dots, e_n$.
Then $G : \mathrm{Hom}(V, W) \mapsto \mathrm{Mat}_{m \times n}(k)$ (where $m = \dim W$, $n = \dim V$) is defined by sending φ to the matrix for φ in the chosen bases for V, W.
$G \circ F = \mathrm{id}_{\mathcal{C}}$ by construction, but

$$F \circ G : V \to k^n \xrightarrow{\mathrm{id}} k^n, \ \mathrm{Hom}(V, W) \to \mathrm{Mat}_{m \times n} \xrightarrow{\mathrm{id}} \mathrm{Mat}_{m \times n}$$

[1] f.g. = finitely generated.
[2] recall, a k-algebra hom is the identity map on k (since it is k-linear and $1 \mapsto 1$), so by linearity and multiplicativity it suffices to define the hom on generators.
[3] since it is just a symbol, one could also just label the objects by $n \in \mathbb{N}$, and $\mathrm{Hom}(n, m) = \mathrm{Mat}_{m \times n}(k)$.

is not $\mathrm{id}_{\mathcal{D}}$, so G is not an inverse for F. But for an equivalence of categories, we just need there to be a natural isomorphism $F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$.

Define $F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ by sending[1]

$$V \mapsto (\text{morphism } FG(V) = k^n \to \mathrm{id}(V) = V \text{ given by } e_i \mapsto v_i).$$

In general, to find/define G is a nuisance. So one uses the following FACT:

Lemma 5.1 (Criterion for Equivalences of Categories)**.**
A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if it is full, faithful, and essentially surjective.

Explanation:
Full means $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(FX,FY)$ is surjective;
Faithful means $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(FX,FY)$ is injective.
So fully faithful means you have isomorphisms at the level of morphisms.
Essentially surjective means: any $Z \in \mathrm{Ob}(\mathcal{D})$ is isomorphic to FX for some X.
(in the above example, any vector space V is isomorphic to some k^n, indeed take $n = \dim V$).

Exercise. Prove the Lemma.

5.3. Equivalence: AFFINE VARIETIES AND F.G. REDUCED k-ALGEBRAS

Theorem. *There is an equivalence of categories*[2]

$$\begin{array}{rcl} \{\textit{affine algebraic varieties and morphs of aff.vars.}\} & \leftrightarrow & \{\textit{f.g. reduced } k\textit{-algs and homs of } k\textit{-algs}\}^{op} \\ X & \overset{\mathcal{T}}{\mapsto} & k[X] \\ (X \overset{F}{\to} Y) & \overset{\mathcal{T}}{\mapsto} & (F^* : k[X] \leftarrow k[Y]). \end{array}$$

Proof. $\mathcal{T}$ is a well-defined functor. ✓
$\mathcal{T}$ is faithful: because $(F^*)^* = F$. ✓
$\mathcal{T}$ is full: given a k-alg hom $\varphi : k[X] \leftarrow k[Y]$, take $F = \varphi^*$ then $F^* = (\varphi^*)^* = \varphi$. ✓
$\mathcal{T}$ is essentially surjective: given a f.g. reduced k-alg A, *choose* generators $\alpha_1, \ldots, \alpha_n$ for A. Define

$$I_A = \ker\left(k[x_1,\ldots,x_n] \to A,\ x_i \mapsto \alpha_i\right). \tag{5.1}$$

Then $A \cong k[x_1,\ldots,x_n]/I_A = k[X_A]$ for $X_A = \mathbb{V}(I_A)$, using $\mathbb{I}(X_A) = \sqrt{I_A} = I_A$ as A is reduced. ✓ □

Remark. The proof of Lemma 5.1, in this particular example, would construct a functor $G : A \mapsto X_A = \mathbb{V}(I_A)$ and $G : (\varphi : A \leftarrow B) \mapsto (\varphi^* : X_A \to X_B)$. Then mimic Section 5.2.

Specm notation: if A is a finitely generated reduced k-algebra, then we've shown that there is an affine variety X_A (unique up to isomorphism) whose coordinate ring is isomorphic to A. Write

$$\mathrm{Specm}\, A$$

for this affine variety. Section 15 will discuss Specm properly. For now, recall that $\mathrm{Specm}(A)$ as a set consists of the maximal ideals of A, which indeed represent the geometric points of X_A. However, to realise this as an affine variety (i.e. with a choice of embedding $X_A \subset \mathbb{A}^n$ into some $\mathbb{A}^n$) we had to make a choice of generators for A.

[1]The fact that it is a natural transformation boils down to the following commutative diagram

$$\begin{array}{ccc} FG(V) = k^n & \longrightarrow & \mathrm{id}(V) = V \\ {\scriptstyle FG(f) = (\text{matrix for } f)} \downarrow & & \downarrow {\scriptstyle \mathrm{id}(f) = f \in \mathrm{Hom}(V,W)} \\ FG(W) = k^m & \longrightarrow & \mathrm{id}(W) = W \end{array}$$

[2]"op" is the **opposite category**, so arrows (morphs) point in the opposite direction than the original category.

5.4. NO EQUIVALENCE FOR PROJECTIVE VARIETIES

By sending

$$(X \subset \mathbb{P}^n) \mapsto (\widehat{X} \subset \mathbb{A}^{n+1}) \mapsto (S(X) = k[\widehat{X}])$$

we obtain a map

$$\{\text{proj.vars}\} \to \left\{ \begin{array}{c} \text{f.g. reduced } \mathbb{N}\text{-graded algebras } A \text{ generated by} \\ \text{finitely many elts in degree 1, with } A_0 = k \end{array} \right\}$$

"Conversely", given such an algebra A, pick generators $\alpha_0, \ldots, \alpha_n$ of degree 1, this determines a hom $\varphi : R \to A, x_i \mapsto \alpha_i$, then $X = \mathbb{V}(\ker\varphi) \subset \mathbb{P}^n$ satisfies $S(X) = R/\ker\varphi \cong A$ (notice $\ker\varphi$ is a homogeneous ideal). There is no equivalence of categories in this case: not all algebra homomorphisms give rise to projective morphisms of the associated projective varieties (not all morphisms $\widehat{X} \to \widehat{Y}$ descend to $X \to Y$, because they may not preserve the rescaling k-action). If we require the k-algebra homs to be grading-preserving, it becomes too restrictive: then only restrictions of linear embeddings $\mathbb{P}^n \hookrightarrow \mathbb{P}^m$ can arise, so for $n = m$ only projective equivalences would be morphs.

As mentioned in Section 3.11, $S(X)$ is not an isomorphism-invariant, so there cannot be an equivalence of categories of projective varieties in terms of the homogeneous coordinate rings $S(X)$.

6. PRODUCTS AND FIBRE PRODUCTS

6.0. ALGEBRA BACKGROUND: TENSOR PRODUCTS

The **tensor product** of two k-vector spaces $V \otimes W$ is a vector space of dimension $\dim V \cdot \dim W$ with basis $v_i \otimes w_j$ where v_i, w_j are bases for V, W.
Example. $\mathbb{R}^n \otimes \mathbb{R}^m \cong \mathbb{R}^{nm}$.
You can extend the symbol $\otimes$ to all vectors by declaring that $(\sum \lambda_i v_i) \otimes (\sum \mu_j w_j) = \sum (\lambda_i \mu_j) v_i \otimes w_j$.
Example. $0 \otimes w = 0 = v \otimes 0$, $(e_1 + 2e_3) \otimes (7e_1 + e_2) = 7e_1 \otimes e_1 + 14e_3 \otimes e_1 + e_1 \otimes e_2 + 2e_3 \otimes e_2$.
Exercise. $V^* \otimes W \cong \mathrm{Hom}(V,W)$ for finite dimensional v.s. V, W, where V^* is the dual of V.

For k-algebras A and B, the tensor product $A \otimes B$ (or $A \otimes_k B$) is the vector space as above, and multiplication is done componentwise. Thus a general element is a finite sum $\sum a_i \otimes b_i$ with $a_i \in A$, $b_i \in B$, and the product is $(\sum a_i \otimes b_i) \cdot (\sum a'_j \otimes b'_j) = \sum (a_i a'_j) \otimes (b_i b'_j)$ summing over all pairs i, j.

The tensor product is determined up to unique k-algebra isomorphism by a universal property. Namely, $A \otimes B$ is a k-algebra together with a k-alg hom $\varphi : A \times B \to A \otimes B$ which is a **balanced bihomomorphism**. Bihomomorphism means $\varphi(\cdot, b) : A \to A \otimes B$ is a k-alg hom for all $b \in B$, and similarly for $\varphi(a, \cdot)$. Balanced means $\varphi(\lambda a, b) = \varphi(a, \lambda b)$ for all $\lambda \in k$, $a \in A$, $b \in B$. The universal property is that any k-alg hom $\varphi' : A \times B \to C$ which is a balanced bihomomorphism must factorise through a unique k-alg hom $\psi : A \otimes B \to C$ (so $\varphi' = \psi \circ \varphi$).

Recall k is an algebraically closed field (this is crucial for the next two results).

Lemma 6.1. *Let A be a finitely generated reduced k-algebra. If $a \in A$ lies in all maximal ideals $\mathfrak{m} \subset A$ (equivalently: $\overline{a} = 0 \in A/\mathfrak{m}$), then $a = 0$.*

Proof. Let $p \in X = \mathrm{Specm}(A)$ be a point. Recall from 2.3 that p defines a maximal ideal $\mathfrak{m} = \mathfrak{m}_p \subset A$ and an evaluation isomorphism:

$$\varphi : A/\mathfrak{m} \xrightarrow{\cong} k.$$

Notice $\varphi(\overline{a}) = a(p)$, thus $a \notin \mathfrak{m}$ is equivalent to the statement $a(p) \neq 0$. Finally, if $a \in k[X] = A$ is a non-zero function (so $a \notin \mathbb{I}(X)$), then $a(p) \neq 0$ at some $p \in X$. □

Theorem 6.2. *Let A, B be k-algebras. Assume A is finitely generated.*

1. *If A, B are reduced, then so is $A \otimes B$.*
2. *If A, B are integral domains, then so is $A \otimes B$.*

Proof. (Non-examinable.)
1) Say $c=\sum a_i\otimes b_i\in A\otimes B$ is nilpotent. By bilinearity, WLOG b_i are linearly independent/k. Any max ideal $\mathfrak{m}\subset A$ yields an iso φ as in Lemma 6.1. Consider the k-algebra hom

$$A\otimes B\to (A/\mathfrak{m})\otimes B\cong k\otimes B\cong B,\qquad c=\sum a_i\otimes b_i\mapsto \sum \overline{a}_i\otimes b_i\mapsto \sum\varphi(\overline{a}_i)\otimes b_i\mapsto \sum \varphi(\overline{a}_i)b_i.$$

As B is reduced, the nilpotent element $\sum\varphi(\overline{a}_i)b_i$ is zero, thus $\varphi(\overline{a}_i)=0$ by independence/k, so $\overline{a}_i=0$, thus $a_i=0$ by Lemma 6.1, so $c=0$.
2) Say $(\sum a_i\otimes b_i)(\sum a_j'\otimes b_j')=0\in A\otimes B$, again WLOG b_i lin.indep./k, and b_i' lin.indep./k. Applying the hom from (1), $(\sum\varphi(\overline{a}_i)b_i)(\sum\varphi(\overline{a}_i')b_i')=0\in B$. As B is an I.D., one of those two factors is zero. By linear independence, for each $\mathfrak{m}$, either all $\varphi(\overline{a}_i)=0$, or all $\varphi(\overline{a}_j')=0$ (or both). Thus, either all $a_i\in\mathfrak{m}$ or all $a_j'\in\mathfrak{m}$ (but we don't know if the same case among those two will apply for all $\mathfrak{m}$). Geometrically this implies $X=\mathrm{Specm}(A)=\mathbb{V}(a_i:\ \text{all } i)\cup\mathbb{V}(a_j':\ \text{all } j)$. But X is irreducible as A is an I.D., so WLOG $X=\mathbb{V}(a_i:\ \text{all } i)$, so $a_i=0\in A$, thus $\sum a_i\otimes b_i=0\in A\otimes B$. □

6.1. PRODUCTS OF AFFINE VARIETIES

For affine varieties,
$X=\mathbb{V}(f_1,\dots,f_N)\subset\mathbb{A}^n,\quad f_j=f_j(x_1,\dots,x_n)\in k[x_1,\dots,x_n]$,
$Y=\mathbb{V}(g_1,\dots,g_M)\subset\mathbb{A}^m,\quad g_i=g_i(y_1,\dots,y_m)\in k[y_1,\dots,y_m]$.
The product $X\times Y$ is the affine variety

$$X\times Y=\mathbb{V}(f_1,\dots,f_N,g_1,\dots,g_M)\subset\mathbb{A}^{n+m}$$

using the coordinate ring $k[\mathbb{A}^{n+m}]=k[x_1,\dots,x_n,y_1,\dots,y_m]$.
Abbreviate $I=\mathbb{I}(X)$, $J=\mathbb{I}(Y)$, viewed as subsets in $k[\mathbb{A}^{n+m}]=k[x_1,\dots,x_n,y_1,\dots,y_m]$.
Observe that:[1]

$$X\times Y=\mathbb{V}(I\cup J)=\mathbb{V}(\langle I+J\rangle)\subset\mathbb{A}^{n+m}$$

where $\langle I\cup J\rangle=\langle I+J\rangle\subset k[x_1,\dots,x_n,y_1,\dots,y_m]$.
Here $I+J=\{f(x)+g(y): f(x)\in I, g(y)\in J\}$ as written is not yet an ideal in $k[x_1,\dots,x_n,y_1,\dots,y_m]$. It generates the ideal $\langle I+J\rangle=k[x_1,\dots,x_n,y_1,\dots,y_m]\cdot(I+J)=k[y_1,\dots,y_m]\cdot I+k[x_1,\dots,x_n]\cdot J$.
At the coordinate ring level:[2]

$$\begin{aligned}k[X\times Y]&=k[x_1,\dots,x_n,y_1,\dots,y_m]/\langle I+J\rangle\\&\cong k[x_1,\dots,x_n]/I\otimes_k k[y_1,\dots,y_m]/J\\&=k[X]\otimes_k k[Y]\end{aligned}$$

by identifying $x_i\cong x_i\otimes 1$ and $y_j\cong 1\otimes y_j$. The isomorphism is explicitly given by

$$\begin{aligned}k[x_1,\dots,x_n,y_1,\dots,y_m]/\langle I+J\rangle&\to k[x_1,\dots,x_n]/I\otimes_k k[y_1,\dots,y_m]/J\\ \sum\alpha_i\beta_i&\mapsto\sum\overline{\alpha}_i\otimes\overline{\beta}_i,\end{aligned}$$

where $\alpha_i\in k[x_1,\dots,x_n]$, $\beta_i\in k[y_1,\dots,y_m]$. The inverse map is $\sum\overline{\alpha}_i\otimes\overline{\beta}_i\mapsto\sum\alpha_i\beta_i$.
Exercise. Check that the two maps are well-defined.[3]

Lemma 6.3. *$\langle I+J\rangle=k[y_1,\dots,y_m]\cdot I+k[x_1,\dots,x_n]\cdot J$ is a radical ideal in $k[x_1,\dots,x_n,y_1,\dots,y_m]$.*

Proof. By Theorem 6.2.(1), since I,J are radical we deduce that $k[x_1,\dots,x_n]/I\otimes_k k[y_1,\dots,y_m]/J$ is reduced. By the above isomorphism, it follows that $k[x_1,\dots,x_n,y_1,\dots,y_m]/\langle I+J\rangle$ is reduced. □

Remark. If X,Y are irreducible then so is $X\times Y$, by Theorem 6.2.(2) or by a geometrical argument.[4]

[1] For aff./proj. vars., $X\times Y$ as a set is the usual $\{(a,b):a\in X,b\in Y\}$. It's the Zariski topology which is subtle. High-tech: all elements in Specm $k[X]\otimes_k k[Y]$ have the form $\mathfrak{m}_a\otimes\mathfrak{m}_b$, but Spec $k[X]\otimes_k k[Y]$ also has elements which are not of the form $\wp_1\otimes\wp_2$: e.g. $X=Y=\mathbb{A}^1$, the diagonal $D=\{(a,a):a\in\mathbb{A}^1\}\subset X\times Y$ corresponds to $\wp=\langle x_1-y_1\rangle$.
[2] The isomorphism is justified later. **Exercise.** Prove is using the universal property from Sec.6.0.
[3] Example: if $f_i\in I$, then $f_i\beta\in\langle I+J\rangle$ and maps to $\overline{f}_i\otimes\overline{\beta}=0$ as $\overline{f}_i=0\in k[x_1,\dots,x_n]/I$. Similarly $I\beta\to\overline{I}\otimes\overline{\beta}=0$.
[4] *Hints.* By contradiction, if $X\times Y=C_1\cup C_2$ for closed sets C_i, using irreducibility of Y show that $X=X_1\cup X_2$ where $X_i=\{x\in X: x\times Y\subset C_i\}$. These X_i are closed (the map $X\to X\times Y$, $x\mapsto(x,y)$ is continuous so $\{x\in X:(x,y)\in Z_i\}$ is closed for each y, now intersect these over all $y\in Y$). Finally use irreducibility of X.

6.2. PRODUCTS OF PROJECTIVE VARIETIES

For proj.vars. X,Y one can use the above affine construction locally to define the Zariski topology on $X\times Y$. We now show that one can equivalently carry out a global construction by using the Segre embedding from Section 4.2. Recall from that Section the notation: $\sigma_{n,m}:\mathbb{P}^n\times\mathbb{P}^m\hookrightarrow\mathbb{P}^{(n+1)(m+1)-1}$, the Segre variety $\Sigma_{n,m}=\sigma_{n,m}(\mathbb{P}^n\times\mathbb{P}^m)\subset\mathbb{P}^{nm+n+m}$, and the projection maps π_{col}, π_{row}.

Definition (Zariski topology on Products)**.** *The Zariski topology on $\mathbb{P}^n\times\mathbb{P}^m$ is the subspace topology on $\Sigma_{n,m}\subset\mathbb{P}^{nm+n+m}$ (i.e. we declare that $\sigma_{n,m}$ and $\pi_{\mathrm{col}}\times\pi_{\mathrm{row}}$ are isomorphisms).*
The Zariski topology on $X\times Y$ is the subspace topology on $\sigma_{n,m}(X\times Y)\subset\Sigma_{n,m}\subset\mathbb{P}^{nm+n+m}$ (i.e. we declare that $\sigma_{n,m}:X\times Y\to\sigma_{n,m}(X\times Y)$ is a homeomorphism).

Theorem. *$X\subset\mathbb{P}^n$, $Y\subset\mathbb{P}^m$ proj.vars. $\Rightarrow X\times Y$ is a proj.var. isomorphic to $\sigma_{n,m}(X\times Y)\subset\mathbb{P}^{nm+n+m}$.*

Proof. It remains to show that $\sigma_{n,m}(X\times Y)$ is a projective variety. This is an exercise.
Hint: Say $X=\mathbb{V}(F_1,\dots,F_N)$, $Y=\mathbb{V}(G_1,\dots,G_M)$, then show that

$$\sigma_{n,m}(X\times Y)=\Sigma_{n,m}\cap\mathbb{V}(F_k(z_{0j},\dots,z_{nj}),G_\ell(z_{i0},\dots,z_{im}):\ \text{all } k,\ell,i,j).\quad\square$$

If we intersect with the open sets

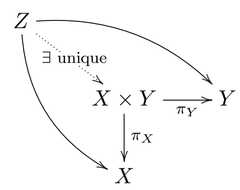
$$\begin{aligned}U_{0,\mathbb{P}^n}&=(x_0\neq 0)=\{[1:x_1:\cdots:x_n]\}\\U_{0,\mathbb{P}^m}&=(y_0\neq 0)=\{[1:y_1:\cdots:y_m]\}\end{aligned}$$

then $\sigma_{n,m}((X\times Y)\cap(U_{0,\mathbb{P}^n}\times U_{0,\mathbb{P}^m}))$ is described by the matrix $[x_iy_j]$ with first column $(1,x_1,\dots,x_n)$ (since $x_0=y_0=1$) and first row $(1,y_1,\dots,y_m)$. So Definition 6.2 above imposes precisely the vanishing of $f_k=F_k(1,x_1,\dots,x_n)$ and $g_\ell=G_\ell(1,y_1,\dots,y_m)$ (the other relations from $\Sigma_{n,m}$ tell us that the other cols/rows have no new information: they are rescalings of the first column/row). Thus the global construction with the Segre embedding agrees with the local affine construction.

6.3. CATEGORICAL PRODUCTS

Category Theory: let C be a category.
Examples. Category of Sets: Objects = sets, Morphisms = all maps between sets.
Category of Vector spaces: Obj = vector spaces, Morphs = linear maps.
Category of Topological spaces: Obj = top. spaces, Morphs = continuous maps.
Category of Affine varieties: Obj = aff.vars., Morphs = morphs of affine vars.

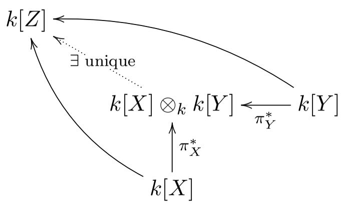
A **product** of $X,Y\in\mathrm{Ob}(C)$ (if it exists) is an object $X\times Y\in\mathrm{Ob}(C)$ with morphisms π_X, π_Y to X,Y s.t. for any $Z\in\mathrm{Ob}(C)$ with morphs to X,Y we have[1]



Example. For $C=$ Sets, $X\times Y=\{(x,y)\in X\times Y: x\in X, y\in Y\}$ is the usual product of sets.
Exercise. Show $X\times Y$ is unique up to canonical isomorphism, if it exists.

[1] Convention: if we write a diagram, we require that it commutes (unless we say otherwise).

Algebraically, we expect the “opposite” of the product, so the **coproduct** of $k[X], k[Y]$:



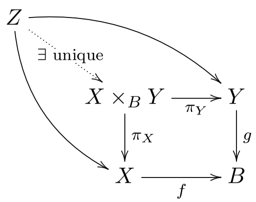
where $\pi_X^*(x_i) = x_i \otimes 1$, $\pi_Y^*(y_j) = 1 \otimes y_j$. Indeed, if the given maps into $k[Z]$ were φ, ψ, then the unique map is $\sum \alpha_i \otimes \beta_i \mapsto \sum \varphi(\alpha_i)\psi(\beta_i)$.
This, together with the equivalence of categories from Sec.5.3, is another proof of the result from Sec.6.1 that $k[X \times Y] \cong k[X] \otimes_k k[Y]$.
Example. $C =$ Sets: coproduct $X \sqcup Y$ is the disjoint union, with inclusions $X \to X \sqcup Y$, $Y \to X \sqcup Y$.
Exercise. For $C =$ Vector Spaces, the coproduct is the direct sum of vector spaces.

6.4. FIBRE PRODUCTS AND PUSHOUTS

This Section is non-examinable.
Motivation. In geometry, you study families of geometric objects labeled by a parameter space B. So $f : X \to B$ where $f^{-1}(b)$ is the geometric space in the family associated to the parameter b.
Example. $f : (\mathbb{V}(xy - t) \subset \mathbb{A}^2) \to \mathbb{A}^1$, $f(x, y, t) = t$, is a family of “hyperbolas” $xy = t$ in $\mathbb{A}^2$ depending on a parameter $t \in k$, which at $t = 0$ degenerates into a union of two lines (the two axes).
In set theory, the fibre product of two maps $f : X \to B$, $g : Y \to B$ (over the “base” B) is

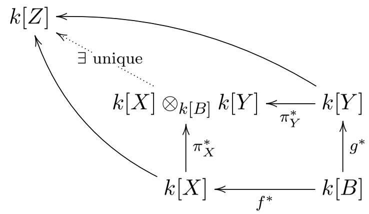
$$X \times_B Y = \{(x, y) \in X \times Y : f(x) = g(y) \in B\}.$$

Example. The fibre $f^{-1}(b)$ is the fibre product of $f : X \to B$ and $g =$ inclusion : $\{b\} \to B$.
Example. The intersection $X_1 \cap X_2$ in X is the fibre product of the inclusions $X_1 \to X$, $X_2 \to X$.
Category Theory: let C be a category.
The **fibre product** (or **pullback** or **Cartesian square**) of $f : X \to B$, $g : Y \to B$ (if it exists) is an object $X \times_B Y \in \mathrm{Ob}(C)$ with morphisms π_X, π_Y to X, Y s.t. for any $Z \in \mathrm{Ob}(C)$ with morphs to X, Y (commuting with f, g) we have



Exercise. $X \times_B Y$ is unique up to canonical isomorphism, if it exists.
Example. (If you have seen vector bundles.) Given a vector bundle $Y \to B$ over a manifold, and a map $f : X \to B$ of manifolds, then $X \times_B Y = \sqcup_{x \in X} Y_{f(x)}$ is the pullback vector bundle $f^*Y \to X$.

Algebraically, we expect the “opposite”, so the **pushout**[1]



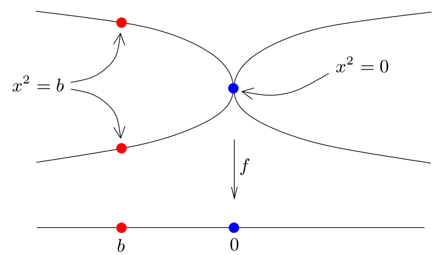
where $\pi_X^*(x_i) = x_i \otimes 1$, $\pi_Y^*(y_j) = 1 \otimes y_j$, and where[2]

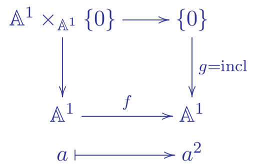
$$k[X] \otimes_{k[B]} k[Y] = k[X] \otimes_k k[Y] \,/\, \langle f^*(b) \otimes 1 - 1 \otimes g^*(b) : b \in k[B] \rangle.$$

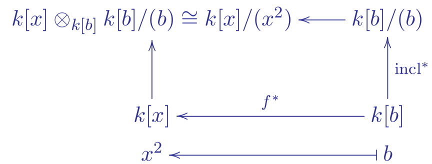
Example. For $C =$ Sets, the pushout of the inclusions $A \cap B \to A$, $A \cap B \to B$ is just the union $A \cup B$ (with obvious inclusions from A, B). The pushout of general maps $C \to A$, $C \to B$, is the disjoint union $A \sqcup B / \sim$ after identifying $a \sim b$ if a, b are images of some common $c \in C$.

Remark. $A = k[X] \otimes_{k[B]} k[Y]$ may have nilpotents (as in the next Example) in which case it does not correspond to the coordinate ring of an affine variety. However, we can **reduce** the algebra: $A_{\mathrm{red}} = A/\mathrm{nil}(A)$ where the nilradical $\mathrm{nil}(A)$ is the subalgebra of nilpotent elements. Then, as we want an affine variety, define $X \times_B Y$ to be “the” affine variety with coordinate ring A_{red}. It satisfies the pushout diagram for all *affine* varieties Z (note $\mathrm{nil}(A) \to \{0\}$ via $A \to k[Z]$ as $k[Z]$ is reduced). What has happened here is that even though $k[X] \otimes_{k[B]} k[Y]$ is the correct pushout in the category of rings (in particular, also in the category of k-algebras), it is not the correct pushout in the category of f.g. *reduced* k-algebras (equivalently, the category of affine varieties), so we had to reduce.

Example. Below is the most complicated way of solving the equation $x^2 = 0$ (!)
Observe the picture. We want to calculate the fibre product over 0 of $f : \mathbb{A}^1 \to \mathbb{A}^1$, $a \mapsto a^2$.







where $k[b]/(b)$ is the coordinate ring of the point $b = 0$ in $\mathbb{A}^1$. The above diagram proves that the fibre $f^{-1}(0)$ is Specm $(k[x]/(x))$ where we reduced $(k[x]/(x^2))_{\mathrm{red}} = k[x]/(x)$, so it is $\mathbb{V}(x) = \{0\} \subset \mathbb{A}^1$.

[1]In the Topology & Groups course, you have seen a pushout: in the Van Kampen theorem, when you take the free product with amalgamation of the first homotopy groups.
[2]we “identify” $f^*(b)$ and $g^*(b)$, in particular $(f^*(b)x) \otimes y \equiv x \otimes (g^*(b)y)$, but there are more relations as we take the ideal generated by those identifications.

6.5. GLUING VARIETIES

This Section is non-examinable.

The role of geometry/algebra above (pullback/pushout) can also be reversed, as in the case of gluing varieties. To glue varieties X, Y over a "common" open subset $U \hookrightarrow X$, $U \hookrightarrow Y$, we pushout:

$$\begin{array}{ccc} X \times_U Y & \longleftarrow & Y \\ \uparrow & & \uparrow \\ X & \longleftarrow & U \end{array}$$

which algebraically is the fibre product $k[X] \times_{k[U]} k[Y]$, namely the functions which agree on U.

Example. $\mathbb{P}^1 = \mathbb{A}^1 \times_{\mathbb{A}^1 \setminus \{0\}} \mathbb{A}^1$ is the gluing of two copies of $\mathbb{A}^1$ over $U = \mathbb{A}^1 \setminus \{0\}$ via the gluing maps $U \to \mathbb{A}^1, b \mapsto b$ and $U \to \mathbb{A}^1, b \mapsto b^{-1}$. Algebraically: $k[x] \times_{k[b,b^{-1}]} k[y]$, determined by the two homs $(x,0) \mapsto b$, $(0,y) \mapsto b^{-1}$. This corresponds to pairs of polynomial functions $f : \mathbb{A}^1 \to k$, $g : \mathbb{A}^1 \to k$ satisfying $f(b) = g(b^{-1})$, i.e. agreeing on the overlap U via the gluing maps.

Exercise. $k[x] \times_{k[b,b^{-1}]} k[y] \cong k$. Indeed the only global functions on $\mathbb{P}^1$ are the constant functions.

7. ALGEBRAIC GROUPS AND GROUP ACTIONS

7.1. ALGEBRAIC GROUPS

Definition. *G is an **algebraic group**[1] if G is an affine variety, and it has a group structure given by morphisms of affine varieties.*
Explicitly: multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are morphs of aff.vars.
*A **homomorphism** $G \to H$ of alg.groups is a hom of groups which is also a morph of aff.vars.*

EXAMPLES.
1) finite groups (viewed as a discrete set of points).
2) $SL(n,k) = \mathbb{V}(\det -1) \subset \mathbb{A}^{n^2}$.
3) $k^* = k \setminus \{0\} \cong \mathbb{V}(xy-1) \subset \mathbb{A}^2$ via $a \leftrightarrow (a, a^{-1})$, with $m =$ multiplication. Recall the coordinate ring is $k[k^*] = k[x,y]/(xy-1) \cong k[x, x^{-1}]$.
4) $k \cong \mathbb{A}^1$ with $m =$ addition.
5) $GL(n,k) =$ (non-singular $n \times n$ matrices$/k$) $\cong \mathbb{V}(y \cdot \det -1) \subset \mathbb{A}^{n^2+1}$, hence any Zariski closed subgroup will also be an algebraic group.
Examples of such subgroups: upper triangular matrices,[2] upper unipotent matrices,[3] and diagonal matrices. (Allowing only non-singular matrices)
6) If G, H alg.gps. then the product group $G \times H$ is an alg.gp.
Example: the **algebraic torus**[4] $\mathbb{G}_m = k^* \times \cdots \times k^*$ is an alg.gp.
7) For G algebraic group, define $G_0 =$(the[5] irreducible component containing 1). Exercise: Show that G_0 is an algebraic group. Show that the irreducible components of G are the cosets of G_0.
8) $H \subset G$ a subgroup of an algebraic group. Exercise: the closure $\overline{H}$ is an algebraic subgroup.
9) $\varphi : G \to H$ a morph of alg.gps. Exercise: $\ker \varphi \subset G$ is an algebraic subgp. Fact: $\mathrm{im}\, \varphi \subset H$ is an algebraic subgp.
10) Fact. Every alg.gp. is isomorphic to a closed subgp of some $GL(n,k)$.

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[1] Much of the theory is the algebraic analogue of the theory of Lie groups (groups which are also manifolds).
[2] $M_{ij} = 0$ for $i > j$.
[3] M upper triangular and all $M_{ii} = 1$.
[4] the "m" refers to the fact that we use multiplication.
[5] Non-examinable: there is only one irreducible component which contains 1. Indeed, suppose we had two such components X, Y. We need two facts: (1) the image of any irreducible variety under a continuous map is irreducible, and (2) if X, Y are irreducible then $X \times Y$ is irreducible. Thus the image under multiplication $m(X \times Y)$ is irreducible and contains both X, Y (since $X = m(X \times \{1\})$) hence $X = Y = m(X \times Y)$ by irreducibility.

7.2. GROUP ACTIONS BY ALGEBRAIC GROUPS ON AFFINE VARIETIES

Definition. *X aff.var., G alg.gp., then an **action** of G on X is a morphism $G \times X \to X$, $(g,x) \mapsto g \cdot x$ of aff.vars. such that $1 \cdot x = x$ and $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$.*

Example. $G = k^*$ acts on $X = \mathbb{A}^2$ by $t \cdot (a,b) = (t^{-1}a, tb)$. The orbits are:
$O_1 = \{(0,0)\}$.
$O_2 = k^* \cdot (1,0) = \{(a,0) : a \in k^*\}$.
$O_3 = k^* \cdot (0,1) = \{(0,b) : b \in k^*\}$.
$O(s) = k^* \cdot (1,s) = \mathbb{V}(xy - s) = \{(t^{-1}, ts) : t \in k^*\}$ where $s \in k^*$.
The partition by orbits is $\mathbb{A}^2 = O_1 \cup O_2 \cup O_3 \cup \cup_{s \in k^*} O(s)$.

Remark. In this Example, a function $f : X \to k$ which is G-invariant will be constant on each orbit. If f is continuous, then f takes the same value on O_1, O_2, O_3 because $O_1 \subset \overline{O}_2$, $O_1 \subset \overline{O}_3$. By Lemma 2.4, the topological quotient $\mathbb{A}^2/G$ (the space of orbits) cannot be an affine variety. Our goal is to define a better notion of quotient, which identifies the orbits O_1, O_2, O_3 so that this "good quotient" is an affine variety.

7.3. CATEGORICAL QUOTIENT and REDUCTIVE GROUPS

Definition. *The **categorical quotient** Y (if it exists) is an affine variety Y with a morphism $F : X \to Y$ such that F is constant on orbits, and F is "universal", meaning: for any other such data $Y', F' : X \to Y'$ we have*

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ & \searrow^{F'} & \downarrow \exists \text{ unique morph} \\ & & Y' \end{array}$$

Example. If you take $Y' =$ point, then $Y \to Y'$ maps everything to that point.

Exercise. Show that $Y, F : X \to Y$ are unique up to canonical isomorphism.

Remark. One does not require that $F : X \to Y$ is surjective (categorically: an epimorphism). It is not difficult to show[1] that for affine varieties F must be a **dominant** morphism (i.e. has dense image). At the end of the section we construct a non-surjective example.

The G-action on X also determines a G-action on the coordinate ring $k[X]$: $g \in G$ acts by

$$k[X] \to k[X], \ f \mapsto f^g \text{ where } f^g(a) = f(g^{-1}a).$$

This is a **linear action**, in the sense that G acts linearly on the coordinate ring:[2] $(f_1 + f_2)^g(a) = f_1(g^{-1}a) + f_2(g^{-1}a) = f_1^g(a) + f_2^g(a)$ and $(\lambda f)^g(a) = \lambda f(g^{-1}a) = \lambda f^g(a)$ for $\lambda \in k$, $a \in X \subset \mathbb{A}^n$.

Example. In the above Example,[3] k^* acts on $k[\mathbb{A}^2] = k[x,y]$ by[4] $t \cdot x = tx$ and $t \cdot y = t^{-1}y$.

The **G-invariant subalgebra of** $k[X]$ consists of the invariant functions

$$k[X]^G = \{f \in k[X] : f^g = f \text{ for all } g \in G\} \subset k[X].$$

Example. In the above Example, $k[x,y]^G = k[xy] \cong k[w] \cong k[\mathbb{A}^1]$ via $xy \leftarrow\!\shortmid w$.

Lemma 7.1. *If a morph $F : X \to Y$ is constant on orbits then $F^* : k[Y] \to k[X]^G$ lands in the invariant subalg.*

Proof. $(F^* f)^g(x) = (f \circ F)^g(x) = (f \circ F)(g^{-1}x) = f(F(x)) = (F^* f)(x)$. □

---

[1] Given a categorical quotient $Y \subset \mathbb{A}^N$, let Y' be the closure of $F(X) \subset \mathbb{A}^N$, then Y' also satisfies the universal property. By exercise sheet 2, being a dominant map is equivalent to having injective pull-back on coordinate rings, so $k[Y'] \to k[X]$ is injective. Hence $k[Y] \to k[X]$ is injective, since by the above universal property it is the composition of $k[Y] \to k[Y'] \to k[X]$ where the first map is an isomorphism by the previous exercise. So F is dominant (and $Y = Y'$).
[2] So $k[X]$ is a (typically infinite dimensional) representation of G.
[3] Notice that the action has "dualized" on the coordinate ring level.
[4] Explicitly: $x : \mathbb{A}^2 \to k$, $x(a,b) = a$, and $(t \cdot x)(a,b) = x(t^{-1} \cdot (a,b)) = x(ta, t^{-1}b) = ta = (tx)(a,b)$.

Assume[1] for the rest of this Section 7.3 that the characteristic $\operatorname{char} k = 0$.

Definition. *G is a (linearly)* ***reductive*** *group if every representation[2] of G is completely reducible,[3] i.e. isomorphic to a direct sum of irreducibles.[4]*

Examples of reductive groups. (Which we treat as facts)
1) Finite groups.
2) k^*.
3) $\mathbb{G}_m = k^* \times \cdots \times k^*$.
4) $SL(n,k)$.
5) $GL(n,k)$.
Non-example.
$G = k$ (with addition) is not reductive: consider the action[5] $k \ni a \mapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \in \operatorname{Aut}(k^2)$. This rep has the subrep $k \cdot \binom{1}{0}$ but we cannot find a complementary subrep (exercise).

Theorem (Nagata)**.** *Let G be a reductive alg.gp. acting on an aff.var. X. Then $k[X]^G$ is a f.g. reduced k-alg, i.e. $k[X]^G$ is isomorphic to the coordinate ring of an aff.var.*

Remark. $k[X]^G$ is obviously reduced as $k[X]$ is reduced. It is hard to show it is finitely generated.
Specm notation: if $A = k[X]^G$ is finitely generated, then by Section 5.3 there is an affine variety $\operatorname{Specm} A$ (unique up to isomorphism) whose coordinate ring is isomorphic to A.

Theorem. *Let G be a reductive alg.gp. acting on an aff.var. X. Then the inclusion*

$$j : k[X]^G \to k[X]$$

determines a categorical quotient given by

$$j^* : X \to X//G \equiv \operatorname{Specm} k[X]^G.$$

Explicitly: pick generators $f_1, \ldots, f_N$ for $k[X]^G$, then the image of

$$X \to \mathbb{A}^N,\ x \mapsto (f_1(x), \ldots, f_N(x))$$

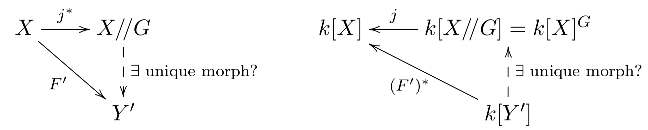
is an affine variety which is "the" categorical quotient of X by G.

Remark. Notice that $j^* : X \to X//G$ is surjective by construction, since $j^*(X) = \mathbb{V}(\ker \varphi) = X//G \subset \mathbb{A}^N$ where $\varphi : k[x_1, \ldots, x_N] \to k[X]^G$, $\varphi(x_i) = f_i$.

Proof.
Step 1. j^* is constant on orbits.
Proof. If $j^*(x) \neq j^*(gx)$, by Lemma 2.4 there is some $f \in k[X//G] = k[X]^G$ with $f(j^*x) \neq f(j^*(gx))$.
$\Rightarrow j(f)(x) = (j^{**}f)(x) = f(j^*x) \neq f(j^*(gx)) = (j^{**}f)(gx) = j(f)(gx)$.
$\Rightarrow$ Contradicts that $j(f) \in k[X]^G$ is G-invariant.

[1]The definitions of reductive and linearly reductive are different when $\operatorname{char} k \neq 0$. Linearly reductive (the definition above) implies reductive, but the converse can fail.
[2]A **representation** is a (finite dimensional) vector space V together with a homomorphism $\rho : G \to \operatorname{Aut}(V)$, where $\operatorname{Aut}(V)$ are the linear isos $V \to V$ (by picking a basis for V, you get $V \cong k^n$ and $\operatorname{Aut}(V) \cong GL(n,k)$, so ρ allows us to "represent" the action of G on V via a subgroup of the invertible $n \times n$ matrices). We usually just say "the representation V", and we write gv or $g \cdot v$ instead of $\rho(g)(v)$.
[3]Equivalently: (linearly) reductive means every G-stable vector subspace $W \subset V$ has some G-stable vector space complement W', i.e. $V = W \oplus W'$ and the action of G preserves the summands.
[4]Irreducible means not reducible. A rep V is **reducible** if there is a subrepresentation $0 \neq W \subsetneq V$. A **subrepresentation** $W \subset V$ is a G-stable vector subspace, meaning $G \cdot W \subset W$ (meaning $gw \in W$ for all $g \in G$, $w \in W$).
[5](Non-examinable) More generally, "unipotent elements are bad". The general definition of **reductive** excludes precisely these. An element r of a ring is **unipotent** if $r - 1$ is nilpotent. For example, any upper triangular matrix with 1 in each diagonal entry. More generally, a matrix is unipotent if and only if all of its eigenvalues are 1, since after conjugation it can be put into Jordan normal form, yielding such an upper triangular matrix.

Step 2. j^* is universal.



By Lemma 7.1, $(F')^*$ lands in $k[X]^G \subset k[X]$, and the diagram on the right commutes if the vertical map on the right is $(F')^* : k[Y'] \to k[X]^G$, and this is the unique map that works. □

EXAMPLES.
1) In the above Example (k^*-action on $\mathbb{A}^2$) $j : k[\mathbb{A}^1] \cong k[xy] = k[x,y]^G \to k[x,y] = k[\mathbb{A}^2]$, $j(xy) = xy$ determines the categorical quotient

$$j^* : \mathbb{A}^2 \to \mathbb{A}^1,\ j(a,b) = ab.$$

Notice, on orbits, j^* maps $O(s) \mapsto s$, whereas O_3, O_2, O_1 all map to $0 \in \mathbb{A}^1$.
Fact.[1] Let X be an affine variety with a linearly reductive group action by G. Given any two disjoint G-invariant closed subsets C_0, C_1 of X there is a function $f \in k[X]^G$ with $f(C_0) = 0$ and $f(C_1) = 1$.
Exercise. Two orbits map to the same point in the categorical quotient $\Leftrightarrow$ their closures intersect.
Corollary of the exercise. For finite groups G, the categorical quotient $X//G = X/G$ can be identified with the orbit space (since points are closed).
2) $G = \mathbb{Z}/2$ acting on $\mathbb{A}^2$ by $(-1) \cdot (a,b) = (-a,-b)$.
$\Rightarrow G$ acts on $k[\mathbb{A}^2] = k[x,y]$ by $(-1) \cdot x = -x$, $(-1) \cdot y = -y$.
$\Rightarrow k[x,y]^G = k[x^2, xy, y^2] \cong k[z_1, z_2, z_3]/(z_1 z_3 - z_2^2) = k[Y]$ where $Y = \mathbb{V}(z_1 z_3 - z_2^2) \subset \mathbb{A}^3$. So the categorical quotient is $\mathbb{A}^2 \to Y$, $(a,b) \mapsto (a^2, ab, b^2)$.
3) G alg.gp., $H \subset G$ any closed normal subgp.
Fact.[2] G/H is an algebraic group with coordinate ring[3] $k[G]^H$, so $G//H = G/H$.
4) The non-reductive group k, with addition, identified with $G = \{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$, acts on $X = SL(2,k)$ by left multiplication of matrices. We claim that $\mathbb{C}^2$ is a categorical quotient $X//G$, with $F : X \to \mathbb{C}^2$, $F(A) =$ (first column of A). Notice F is not surjective as $F(X) = \mathbb{C}^2 \setminus \{0\}$. Notice that $k[X]^G \subset k[X]$ is the k-algebra $k[x_{11}, x_{21}] \subset k[x_{ij}]$ generated by the entries of the first column. Then the proof of the previous theorem applies to this case, since $k[X]^G$ is finitely generated.

8. DIMENSION THEORY

8.1. GEOMETRIC DIMENSION

Let X be a variety (affine or projective). A **chain of length** m means a strict chain of inclusions

$$\emptyset \neq X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_m \tag{8.1}$$

where each $X_i \subset X$ is an irreducible subvariety.
One can start with $X_0 = \{p\}$ a point of X, and if X is irreducible then one can end with $X_m = X$.

Definition. *The* ***local dimension*** *$\dim_p X$ of X at a point $p \in X$ is the maximum over all lengths of chains starting with $X_0 = \{p\}$. The* ***dimension*** *of X is the maximum of the lengths of all chains,*

$$\dim X = \max_m (\exists \text{ chain } X_0 \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_m) = \max_{p \in X} \dim_p X.$$

[1]By *Urysohn's Lemma*, there is a continuous smooth function taking different values on C_0, C_1. Approximate this function by a polynomial to obtain $f \in k[X]$ taking different values on C_0, C_1. By adding/rescaling we can ensure $f(C_0) = 0, f(C_1) = 1$. To make the function G-invariant one simply applies the *Reynolds operator* $R : k[X] \to k[X]^G$ but we haven't constructed this in these notes. For finite groups G, it is easy to construct: $(Rf)(x) = \frac{1}{|G|} \sum_{g \in G} f(gx)$.
[2]It is not so easy to show that $\operatorname{Specm} k[G]^H \cong G/H$ are homeomorphic.
[3]H acts on $k[G]$ by $f^h = f \circ h^{-1}$, so $k[G]^H \subset k[G]$

*Say X has **pure dimension** if the $\dim_p X$ are equal for all $p \in X$.*
*The **codimension** of an irreducible subvariety $Y \subset X$ is*[1]

$$\operatorname{codim} Y = \max_m (\exists \text{ chain } Y \subsetneq X_1 \subsetneq X_2 \subsetneq \cdots \subsetneq X_{m-1} \subsetneq X_m).$$

EXAMPLES.
1. $\mathbb{A}^0 = \{0\} = \mathbb{V}(x_1,\ldots,x_n) \subset \mathbb{A}^1 = \mathbb{V}(x_2,\ldots,x_n) \subset \cdots \subset \mathbb{A}^{n-1} = \mathbb{V}(x_n) \subset \mathbb{A}^n$ so $\dim \mathbb{A}^n \geq n$.
2. $X = \mathbb{V}(xy, xz) = (yz-\text{plane}) \cup (x-\text{axis})$. Then $\dim_p X = 2$ at all points p in the plane, and $\dim_p X = 1$ at other points.
3. $X = (\text{point } p) \sqcup (\text{line}) \subset \mathbb{A}^2$ (disjoint union). Then $Y = \{p\} \subset X$ has codim $= 0$. Notice that $\dim X - \dim Y = 1 - 0 = 1 \neq \operatorname{codim} Y$, whereas $\dim_p X - \dim_p Y = 0 - 0 = 0 = \operatorname{codim} Y$.

Exercise. If $X = X_1 \cup \cdots \cup X_N$ is an irreducible decomposition, then $\dim X = \max \dim X_j$. If X has pure dimension, then $\dim X = \dim X_j$ for all j.
Exercise. An affine variety with $\dim X = 0$ is a finite collection of points.

FACT. $X = \mathbb{V}(I) \subset \mathbb{A}^n$ is a finite set of points $\Leftrightarrow$ $k[X]$ is a finite dimensional k-vector space. Indeed, the number of points is $d = \dim_k k[X]$, and $k[X] \cong k^d$ as k-algebras (exercise[2]).
So do not confuse $\dim k[X]$ and $\dim_k k[X]$.

Lemma 8.1. *If $X \subset Y$ then $\dim X \leq \dim Y$.*
If X, Y are irreducible and $X \subsetneq Y$ then $\dim X < \dim Y$.
(So for irreducibles $X \subset Y$, if $\dim X = \dim Y$ then $X = Y$.)

Proof. Any chain for X is a chain for Y. If $X \neq Y$ are irreds then can extend further: $X_{m+1} = Y$. □

FACT. $\dim \mathbb{P}^n = \dim \mathbb{A}^n = n$.

8.2. DIMENSION IN ALGEBRA

Let A be a ring (commutative with unit). A **chain of length** m means a strict chain of inclusions

$$\wp_0 \supsetneq \wp_1 \supsetneq \cdots \supsetneq \wp_{m-1} \supsetneq \wp_m \tag{8.2}$$

where each $\wp_i \subset A$ is a prime ideal.
One can start with a max ideal $\wp_0 = \mathfrak{m} \subset A$. If A is an integral domain one can end with $\wp_m = \{0\}$.

FACT. For A Noetherian, the descending chain condition holds for prime ideals, i.e. (8.2) eventually stops (however, this need not hold for general ideals).

Definition.
*The **height** $\operatorname{ht}(\wp)$ of a prime ideal is the maximal length of a chain with $\wp_0 = \wp$,*

$$\operatorname{ht}(\wp) = \max_m (\exists \text{ chain } \wp \supsetneq \wp_1 \supsetneq \cdots \supsetneq \wp_{m-1} \supsetneq \wp_m).$$

*The **Krull dimension** is*

$$\dim A = \max \operatorname{ht}(\mathfrak{m})$$

over max ideals $\mathfrak{m}$, i.e. the maximal length of chains.
*For an ideal $I \subset A$ the **height** is $\operatorname{ht}(I) = \min \operatorname{ht}(\wp)$ over all prime ideals $\wp$ containing I.*

EXAMPLES.
1. A field has dimension zero.
2. A PID has dimension 1 (unless it's a field), e.g. $\dim \mathbb{Z} = 1$.
3. Minimal prime ideals[3] are precisely those of height zero.
4. $(x_1,\ldots,x_n) \supset (x_1,\ldots,x_{n-1}) \supset \cdots \supset (x_1) \supset \{0\}$ shows $\dim k[x_1,\ldots,x_n] \geq n$.

[1] When X is irreducible, one can take $X_m = X$. One can define $\operatorname{codim} Y$ also for reducible Y as the minimum of all $\operatorname{codim} Y'$ for irreducible subvarieties $Y' \subset Y$. Example: the disjoint union $Y = (\text{point}) \sqcup (\text{line}) \subset \mathbb{A}^2$ has codim $= 1$.

[2] Consider the primary decomposition of $\mathbb{I}(X)$, and show that the minimal primes I_j are pairwise coprime, then use the Chinese remainder theorem: for any ring A, if I_j are coprime ideals (meaning $I_i + I_j = (1)$, which implies $I = \prod I_j = \cap I_j$) then $A/I \cong \prod A/I_j$ via the obvious map.

[3] **minimal prime ideal** means it does not contain any strictly smaller prime ideal.

EXERCISES.
1.[1] If you know about localisation (Sec.10), show that the **codimension** $\operatorname{codim}(\wp) = \dim A_\wp$ satisfies

$$\operatorname{codim}(\wp) = \dim A_\wp = \operatorname{ht}(\wp).$$

2.[2] If $\dim A = m$ and (8.2) holds, then $\dim A/\wp_j = m - j$.
3. Deduce that $\dim A \geq \dim(A/\wp) + \operatorname{codim}(\wp)$, with equality if $\wp = \wp_j$ as in (8.2) and $\dim A = m$.

We will assume the following two facts from algebra, which geometrically say that each equation we impose can cut down the dimension by at most one. Keep in mind (see Homework 2, ex.1) that it is not always possible to find exactly $\operatorname{ht}(\wp)$ generators for $\wp$.

Theorem 8.2 (Krull's principal ideal theorem, Hauptidealsatz)**.**
For any Noetherian ring A, if $f \in A$ is neither a zero divisor nor a unit, then

$$\operatorname{ht}((f)) = 1.$$

Exercise. By lifting a chain from $A/(f)$ to A, show that

$$\operatorname{ht}((f)) = 1 \Rightarrow \dim A/(f) \leq \dim A - 1.$$

Example. We check Krull's theorem in an easy case: for $f \in A$ **irreducible**[3] and A a UFD (e.g. $k[x_1,\ldots,x_n]$). In this case, $\wp_0 = (0) \subsetneq (f)$ is a chain, since (f) is prime.[4] So $\operatorname{ht}((f)) \geq 1$. We now show $0 \subsetneq \wp \subsetneq (f)$ is impossible. Suppose $0 \neq g \in \wp$ (want: $f \in \wp$ so $\wp = (f)$). As $\wp \subset (f)$, $g = f^m h$ for some $h \notin (f)$. As $h \notin (f)$ also $h \notin \wp$. As $\wp$ is prime, $f^m h \in \wp$ forces $f^m \in \wp$ and so forces $f \in \wp$.

Theorem (Krull's height theorem)**.** *For any Noetherian ring A, and $\langle f_1,\ldots,f_m\rangle \neq A$,*

$$\operatorname{ht}(\langle f_1,\ldots,f_m\rangle) \leq m.$$

So the height $\operatorname{ht}(\wp)$ is at most the number of generators of $\wp$. Conversely, if $\wp \subset A$ is a prime ideal of height m, then $\wp$ is a minimal prime ideal over an ideal generated by m elements.[5]

Corollary. $\dim k[x_1,\ldots,x_n] = n$.

Proof. We know the maximal ideals are $\langle x_1 - a_1,\ldots,x_n - a_n\rangle$, so they have height at most n by Krull's theorem, so $\dim k[x_1,\ldots,x_n] \leq n$. The above example showed $\dim k[x_1,\ldots,x_n] \geq n$. □

Remark. More generally, for A Noetherian, $\dim A[x] = \dim A + 1$. This also implies the Corollary.

The following two facts from algebra ensure that for k-algebras, dimension theory is not nasty:

Theorem. *Let A be a f.g. k-algebra.*[6] *Then*

$$\dim A = (\textit{maximal number of elements of } A \textit{ that are algebraically independent}/k).$$

If $\wp' \supset \wp$ are prime ideals in A, any two saturated[7] *chains from $\wp'$ to $\wp$ have the same length.*

[1] Hint: recall that prime ideals in the localization $A_\wp$ are in 1:1 correspondence with prime ideals of A inside $\wp$.

[2] Hint: recall that prime ideals of A/I are in 1:1 correspondence with prime ideals of A containing I.

[3] Recall an element $f \in A$ of a ring is **irreducible** if it is not zero or a unit, and it is not the product of two non-unit elements. Recall a **unit** f is an invertible element, i.e. $fg = 1$ for some $g \in A$.

[4] Recall, in any integral domain, prime implies irreducible, and in a Unique Factorization Domain the converse holds, so primes and irreducibles coincide. Recall $f \in A$ is **prime** if f is not zero and not a unit, and $f|gh$ implies $f|g$ or $f|h$ (equivalently: $A/(f)$ is an integral domain, i.e. $(f) \subset A$ is a non-zero prime ideal).

[5] Meaning $\wp$ corresponds to a minimal prime ideal of A/I where I is an ideal generated by m elements.

[6] For example, when A is reduced, the coordinate ring of an affine variety.

[7] i.e. a chain (8.2) that cannot be made longer by inserting more prime ideals.

Theorem 8.3. *Let A be a f.g. k-algebra and an integral domain.*[1] *Then*[2]

$$\dim A = \operatorname{trdeg}_k \operatorname{Frac}(A).$$

If $\dim A = m$, then all maximal ideals of A have height m, in fact every saturated chain from a maximal ideal to (0) has length m. Therefore

$$\operatorname{ht}(\wp) + \dim(A/\wp) = \dim A.$$

Thus the length of a saturated chain from $\wp'$ to $\wp$ is $\operatorname{ht}(\wp') - \operatorname{ht}(\wp) = \dim A/\wp - \dim A/\wp'$.

A simple application of this Theorem is (compare the Example after Theorem 8.2):

Corollary 8.4. *For irreducible $f \in R = k[x_1, \ldots, x_n]$ there is a maximal length chain*

$$\wp_0 \supsetneq \cdots \supsetneq \wp_{n-2} \supsetneq \wp_{n-1} = (f) \supsetneq \wp_n = (0).$$

Notice how $\dim R/(f) = n-1$ and $\operatorname{ht}((f)) = 1$ add up to $\dim R = n$.

Example. We prove the Corollary using transcendence degrees. As f cannot be constant, it involves at least one variable, say x_n. Then $\overline{x_1}, \ldots, \overline{x_{n-1}}$ in $R/(f)$ are algebraically independent over k (whereas $\overline{x}_n$ satisfies a polynomial relation over $k[x_1, \ldots, x_{n-1}]$, so $k(x_1, \ldots, x_{n-1}) \hookrightarrow \operatorname{Frac}(R/(f))$ is an algebraic extension). So $\dim R/(f) \geq n-1$, and by Krull $\dim R/(f) \leq n-1$. Hence equality.

8.3. GEOMETRIC DIMENSION = ALGEBRAIC DIMENSION

Theorem. *If $X \subset \mathbb{A}^n$ is an affine variety then*

$$\boxed{\dim X = \dim k[X]}$$

For a projective variety $X \subset \mathbb{P}^n$, $\dim X$ equals the maximal length of chains (8.2) of homogeneous prime ideals which do not contain the irrelevant ideal $(x_0, \ldots, x_n)$, in particular $\dim X = \dim \hat{X} - 1$.

Proof. Using Hilbert's Nullstellensatz, there is a bijection between chains in (8.1) and chains in (8.2): $\wp_j = \mathbb{I}(X_j)$ and $X_j = \mathbb{V}(\wp_j)$. The result for a projective variety follows by the projective Nullstellensatz (so, really, by the affine case applied to the affine cone $\hat{X}$). □

Exercise. For a maximal chain as above, $\operatorname{ht}(\wp_j) = \operatorname{codim} \mathbb{V}(\wp_j) = n - \dim \mathbb{V}(\wp_j)$.

Theorem. *For any irreducible affine variety $X \subset \mathbb{A}^n$,*

$$\dim X = n-1 \Leftrightarrow X = \mathbb{V}(f) \text{ for an irreducible } f \in R = k[x_1, \ldots, x_n].$$

The analogous holds for $X \subset \mathbb{P}^n$ an irreducible projective variety and f homogeneous in $k[x_0, \ldots, x_n]$.

Proof. ($\Rightarrow$): $\dim X = n-1 \Rightarrow \mathbb{I}(X) \neq (0) \Rightarrow \exists f \neq 0 \in \mathbb{I}(X)$. Since $\mathbb{I}(X)$ is prime, it must contain an irreducible factor of the factorization of f. So WLOG f is irreducible, hence prime (R is a UFD). Then $X \subset \mathbb{V}(f) \subsetneq \mathbb{A}^n$, so by Lemma 8.1, $\dim X \leq \dim \mathbb{V}(f) < \dim \mathbb{A}^n = n$ thus forcing $X = \mathbb{V}(f)$ since $\dim X = n-1$. ($\Leftarrow$): Follows by Corollary 8.4. □

Definition. *For an irreducible affine variety X, the* ***function field*** *is*

$$\boxed{k(X) = \operatorname{Frac}(k[X])}$$

[1] Thus, the coordinate ring of an *irreducible* affine variety.

[2] For an integral domain, one can construct the **fraction field** $\operatorname{Frac}(A)$ (mimicking the construction of $\operatorname{Frac}(\mathbb{Z}) = \mathbb{Q}$). Then $k \hookrightarrow \operatorname{Frac}(A)$ is a field extension. For any field extension $k \hookrightarrow K$ there exists a subset $B \subset K$, called **transcendence basis**, whose elements are algebraically independent over k (i.e. they do not satisfy a polynomial relation over k) and such that $k(B) \hookrightarrow K$ is an algebraic extension. Here $k(B)$ denotes the smallest subfield of K containing $k \cup B$. The **transcendence degree** $\operatorname{trdeg}_k K$ is the cardinality of B (FACT: it is independent of the choice of transcendence basis B).

Thus, by Theorem 8.3, for any irreducible affine variety X,

$$\boxed{\dim X = \operatorname{trdeg}_k k(X)}$$

Remark. Elements of $k(X)$ are ratios of polynomials, so they define functions $X \to k$ which are defined on an open subset of X (the locus where the denominator does not vanish).[1]

Example. $k(\mathbb{A}^n) = k(x_1, \ldots, x_n)$ has transcendence basis $x_1, \ldots, x_n$ so $\dim \mathbb{A}^n = n$.

Theorem. *For X, Y irreducible affine varieties, $\dim(X \times Y) = \dim X + \dim Y$.*

Proof. Exercise:[2] compare the trdeg_k for $k[X] = k[x_1, \ldots, x_n]/\mathbb{I}(X)$, $k[Y] = k[y_1, \ldots, y_m]/\mathbb{I}(Y)$ and $k[X \times Y] = k[x_1, \ldots, x_n, y_1, \ldots, y_m]/\langle \mathbb{I}(X) + \mathbb{I}(Y) \rangle \cong k[X] \otimes_k k[Y]$. □

Remark. Geometrically, $\operatorname{ht}(I)$ is the codimension of the subvariety $\mathbb{V}(I) \subset \operatorname{Spec}(A)$. For an irred affine subvar $Y \subset X$, $\dim X \geq \dim Y + \operatorname{codim}_X(Y)$ (which follows from $k[Y] \cong k[X]/\mathbb{I}(Y)$).

Remark. A proj.var. X is called a **complete intersection** if $\mathbb{I}(X)$ is generated by exactly $\operatorname{codim} X = \operatorname{ht} \mathbb{I}(X)$ elements. Recall the twisted cubic $X \subset \mathbb{P}^3$ has $\mathbb{I}(X) = \langle x^2 - wy, y^2 - xz, zw - xy \rangle \subset k[x, y, z, w]$, and it turns out that $\mathbb{I}(X)$ cannot[3] be generated by $2 = \operatorname{ht} \mathbb{I}(X) = \operatorname{codim} X$ elements.

8.4. NOETHER NORMALIZATION LEMMA

Theorem 8.5 (Algebraic version). *Let A be a f.g. k-algebra. Then there are injective k-alg homs*

$$k \hookrightarrow k[y_1, \ldots, y_d] \hookrightarrow A \tag{8.3}$$

where y_i are algebraically independent/k, and A is a finite module over $k[y_1, \ldots, y_d]$.
Moreover, if A is an integral domain, then

$$d = \operatorname{trdeg}_k \operatorname{Frac}(A).$$

A morph of aff vars $f : X \to Y$ is **finite** if $f^* : k[X] \leftarrow k[Y]$ is an **integral extension** (i.e. each element of $k[X]$ satisfies a monic polynomial with coefficients in $f^*k[Y]$).

Fact. If $f : X \to Y$ is a finite morph of irred.aff.vars. then
1) f is **quasi-finite**, meaning: each fibre $f^{-1}(p)$ is a finite collection of points;
2) f is a closed map (f(closed set) is closed);
3) f is surjective $\Leftrightarrow$ f^* is injective.

Example. $f : \mathbb{A}^1 \to \mathbb{A}^1$, $f(a) = a^2$: see the picture in Sec.6.4. So $f^* : k[b] \to k[x]$, $f^*(b) = x^2$. Notice x is integral over $k[b]$: the monic poly $p(x) = x^2 - b$ over $k[b]$ satisfies $p(x) = x^2 - f^*(b) = 0 \in k[x]$.

Remark. *(Non-examinable)* Quasi-finite does not imply finite. Let $f : \mathbb{V}(xy-1) \to \mathbb{A}^1$, $f(x, y) = x$ be the vertical projection from the hyperbola, it has finite fibres. Then $f^* : k[x] \to k[x, y]/(xy-1)$ is the inclusion, but y is not integral over $k[x]$ as $xy - 1$ is not monic. The algebra is not happy

[1] Think meromorphic functions.

[2] *Non-examinable Hints:* You want to show that the union of two transcendence bases $(\overline{f}_i), (\overline{g}_j)$ for $k[X]$, $k[Y]$ give a transcendence basis for $k[X \times Y]$, where $f_i \in k[x_1, \ldots, x_n]$, $g_j \in k[y_1, \ldots, y_m]$. Spanning is easy (hence $\dim X \times Y \leq \dim X + \dim Y$) but showing algebraic independence is harder. Suppose there was a dependency, then you would get $G_1 f^{I_1} + \cdots + G_\ell f^{I_\ell} \in \langle \mathbb{I}(X) + \mathbb{I}(Y) \rangle \subset k[x_1, \ldots, y_1, \ldots]$ where the G's are polynomials in the g_j, and the f^I are monomials $f_1^{i_1} \cdots f_a^{i_a}$ in the given $f_1, \ldots, f_a$. Now evaluate the y-variables at any $p \in Y$, to deduce $G_1(p), \ldots, G_\ell(p) = 0$ by algebraic independence of the f_i in $k[X]$. Deduce that $G_1, \ldots, G_\ell \in \mathbb{I}(Y)$, and from this conclude the result.
Another approach, is to use Noether's Normalization Lemma (Sec.8.4) to get finite surjective morphisms $X \to \mathbb{A}^a$, $Y \to \mathbb{A}^b$ and obtain a finite surjective morphism $\varphi : X \times Y \to \mathbb{A}^{a+b}$. The latter, implies that $k[X \times Y]$ is integral over $\varphi^*(k[\mathbb{A}^{a+b}]) = \varphi^*(k[f_1, \ldots, f_a, g_1, \ldots, g_b])$. The *Going Up (and Lying Over) Theorem* says that if a ring B is integral over a subring A, then any chain of prime ideals in A can be lifted to a chain of prime ideals in B (such that intersecting with A gives the original chain). Thus $\dim k[X \times Y] \geq a + b$, as required. That inequality can also be obtained more generally from the **fact** that if $\varphi : X \to Y$ is a surjective morphism of affine varieties, then $\dim X \geq \dim Y$.
That fact is proved using results from Sec.12.2 as follows. First replace Y by an irreducible component in Y of maximal dimension. Then replace X by an irreducible component in $\varphi^{-1}(Y)$ whose image is dense in Y (check it exists by using surjectivity and irreducibility of Y). Thus, φ is now a *dominant* morphism between irreducible affine varieties. This induces an extension on the function fields $\varphi^* : k(Y) \hookrightarrow k(X)$ which by basic field theory implies $\operatorname{trdeg}_k Y \leq \operatorname{trdeg}_k X$.

[3] $X = \mathbb{V}(I)$ for $I = \langle yw - x^2, z^2w - 2xyz + y^3 \rangle$, but this ideal is not radical.

about the "non-compactness" phenomenon that preimages are diverging near 0. Notice f is not a closed map. It turns out that an affine morphism $f : X \to Y$ is finite if and only if it is **universally closed** (meaning: for each morphism $Z \to Y$ the fibre product $X \times_Y Z \to Y$ is a closed map).

Theorem (Geometric version)**.** *Let $X \subset \mathbb{A}^n$ be an irreducible affine variety of dimension m. Then there is a finite surjective morphism $f : X \to \mathbb{A}^m$.*

Sketch proof. Take $A = k[X]$ in Theorem 8.5, and take Specm of (8.3) to obtain: $X \to \mathbb{A}^d \to$ point. The rest follows from the above Fact.[1] □

So any irreducible affine variety is a **branched covering** of affine space, meaning a morphism of affine varieties of the same dimension with dim("generic" fibers $f^{-1}(p)) = 0$ and which resembles the covering spaces we know from topology over the complement of a closed subset of "bad" points p called the **branch locus**. The **ramification locus** is the preimage f^{-1}(branch locus).[2]

One way to build $f : X \to \mathbb{A}^d$ is by linear projection, taking $y_1, \ldots, y_d$ to be generic linear polynomials in $x_1, \ldots, x_n$.

Theorem (Algebraic Version 2)**.** *When A is a f.g. k-algebra and an integral domain, one can in addition ensure that for the extensions of fields*

$$k \hookrightarrow K = k(y_1, \ldots, y_d) \hookrightarrow \mathrm{Frac}(A)$$

the first is a purely transcendental extension, the second is a primitive[3] *algebraic extension meaning*

$$\mathrm{Frac}\, A = \mathrm{Frac}\, K[z] \equiv K(z)$$

where $z \in A$ is algebraic over K. So only one polynomial relation is needed:

$$G(y_1, \ldots, y_d, z) = 0.$$

Theorem (Geometric Version 2)**.** *For X an irreducible aff var, $k[y_1, \ldots, y_d, z] \hookrightarrow A = k[X]$ induces a morphism $X \to \mathbb{A}^{d+1}$ which is a* birational equivalence[4]

$$X \dashrightarrow \mathbb{V}(G) \subset \mathbb{A}^{d+1}.$$

The conclusion is rather striking: every irreducible affine variety is birational to a hypersurface.

9. DEGREE THEORY

9.1. DEGREE

Recall (Sec.3.3) a **linear subvariety** of $\mathbb{P}^n$ is a projectivisation $L = \mathbb{P}$(a vector subspace $\widehat{L} \subset \mathbb{A}^{n+1}$).
$X \subset \mathbb{P}^n$ proj.var. $\Rightarrow$ the **degree** is

$\deg(X)$ = *# intersection points of X with a complementary linear subvariety in general position*
= *generic # $L \cap X$ for linear subvarieties $L \subset \mathbb{P}^n$ with* $\dim L + \dim X = n$.

We now explain the meaning of "general position" and "generic".

The Grassmannian which parametrizes all $\widehat{L} \subset \mathbb{A}^{n+1}$ above is $G = \mathrm{Gr}(n+1-\dim X, n+1)$.
Fact. There is a non-empty open subset $U \subset G$ such that the number of intersection points $\# L \cap X$ for $\hat{L} \in U$ is finite and independent of $\hat{L}$, and we call that number **deg**(X).

[1]**Exercise.** Show directly that the fibres are finite by using that each $x_i \in k[X]$ satisfies a monic poly over $k[y_1, \ldots, y_d]$. To show the fibre $f^{-1}(p)$ is non-empty, consider $f^*\langle y_1 - p_1, \ldots, y_d - p_d\rangle \subset k[X]$. (You may need Nakayama's lemma: for any rings $A \subset B$, if B is a finite A-module then $\mathfrak{a}B \neq B$ for any maximal ideal $\mathfrak{a} \subset A$).

[2]Compare B3.2 Geometry of Surfaces: non-constant holomorphic maps between Riemann surfaces are locally of the form $z \mapsto z^n$ which has ramification locus $\{0\}$ if $n > 1$. So near most points it is a local biholomorphism.

[3]In fact, one proves that one can choose $y_1, \ldots, y_d$ so that $k(y_1, \ldots, y_d) \hookrightarrow \mathrm{Frac}(A)$ is a finite separable extension. Then the primitive element theorem from Galois theory applies.

[4]We will see these later in the course. A rational map $X \dashrightarrow Y$ is a map defined on an open subset of X defined using rational functions in $k(X)$ rather than polynomial functions in $k[X]$. It is birational if there is a rational map $Y \dashrightarrow X$ such that the two composites are the identity where they are defined. Think of a birational map as being "an isomorphism between open dense subsets".

Corollary 9.1. *If $U' \subset G$ is any non-empty open subset such that $\# L \cap X$ is finite and independent of $\hat{L} \in U'$, then this number equals* $\deg(X)$.

Proof. G is irreducible by Lemma 4.14, so by Sec.2.6 we know $U \cap U'$ is non-empty (and dense). □

Thus the "bad" L (yielding a different finite or infinite number) must lie inside some proper closed subset $V \subset G$, which is thought of as "small" since $G \setminus V$ is open and dense. The "good" $\widehat{L} \in G \setminus V$ are called "in general position", and that finite number $\deg(X)$ is often called the "generic" number or the "expected" number of intersection points. When X is irreducible, $\deg(X)$ is in fact the maximal possible finite number of intersection points of $L \cap X$ for all L (compare Example 3 below).

Examples.
1) $X = H$ hyperplane $\Rightarrow \deg X = 1$, for example $\mathbb{V}(x_0) \cap \mathbb{V}(x_2, \cdots, x_n) = \{[0 : 1 : 0 : \cdots : 0]\}$.
2) $X = \mathbb{P}^n \subset \mathbb{P}^n$, $L =$ any point $\Rightarrow \deg \mathbb{P}^n = 1$.
3) The reducible variety $X = H_0 \cup \{[1:0:1]\} = \{[0:y:1] : y \in k\} \cup \{[0:1:0], [1:0:1]\} \subset \mathbb{P}^2$ generically intersects a line in one point, but $L = \mathbb{P}(\mathrm{span}_k(e_0, e_2)) = \{[x:0:1] : x \in k\} \cup \{[1:0:0]\}$ intersects X twice. On the affine patch $z = 1$, $X = (y$-axis $\cup$ a point on the x-axis), and $L = x$-axis.
4) $X = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$.
$L = \mathbb{V}(ax + by + cz) \overset{1:1}{\longleftrightarrow}$ (plane $\widehat{L} \subset \mathbb{A}^3) \in \mathrm{Gr}(2,3) \overset{1:1}{\longleftrightarrow}$ (normal to the plane) $= [a:b:c] \in \mathbb{P}^2$.
We now calculate $L \cap X$. We want to go to an affine patch $x \neq 0$, but must not forget intersection points outside of that. If $x = 0$, then $y = 0$, and if $c \neq 0$ then also $z = 0$, but $[0]$ is not allowed in $\mathbb{P}^2$. Thus assume $c \neq 0$. Then $x \neq 0$, WLOG $x = 1$. Solving: $y = \frac{-cz-a}{b}$ if $b \neq 0$ and $z = y^2 = (\frac{-cz-a}{b})^2$ gives two solutions z if the discriminant of the quadratic equation is non-zero (check the discriminant is $b^2(b^2 - 4ac)$). Thus $\deg X = 2$, and the set of "bad" $L \equiv [a:b:c] \in \mathbb{P}^2$ forms a subset of $\mathbb{V}(c) \cup \mathbb{V}(b) \cup \mathbb{V}(b^2(b^2 - 4ac))$, hence a subset of $\mathbb{V}(bc(b^2 - 4ac))$.

Remark. $\mathbb{P}^1 \cong \mathbb{V}(xz - y^2)$ (Veronese map), yet $\deg \mathbb{P}^1 = 1$, $\deg \mathbb{V}(xz - y^2) = 2$. Thus the degree depends (unsurprisingly) on the embedding into projective space.

Definition. $X \subset \mathbb{A}^n \equiv U_0 \subset \mathbb{P}^n$ aff.var. $\Rightarrow \deg X = \deg(\overline{X} \subset \mathbb{P}^n)$.

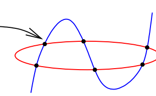
Theorem. $F \in R = k[x_0, \ldots, x_n]$ *homogeneous of degree d with no repeated factors* $\Rightarrow \deg \mathbb{V}(F) = d$.

Proof. $L =$ any line, $X = \mathbb{V}(F)$.
$\Rightarrow X \cap L = \mathbb{V}(F|_L) \subset L \cong \mathbb{P}^1$.
After a linear change of coordinates, WLOG $L = \mathbb{V}(x_2, \ldots, x_n)$.
$\Rightarrow F|_L =$ degree d homog.poly[1] in x_0, x_1 (if $\deg F|_L < d$ then L is not generic enough).
$\Rightarrow \#$(zeros of poly) $\leq d$, and generically[2] it has d zeros. □

Fact. (Weak Bézout's Theorem)[3]
Let $X, Y \subset \mathbb{P}^n$ be proj.vars. of pure dimension with[4] $\dim X \cap Y = \dim X + \dim Y - n$, then

$$\deg X \cap Y \leq \deg X \cdot \deg Y.$$

$2 \cdot 3 = 6$
intersections



[1]Put $t = x_1/x_0$ to get a (non-homogeneous) poly in one variable, and you find all roots (explicitly, if $t = a$ is a root then the original homog.poly had a root for $[x_0 : x_1] = [1 : a]$, and it remains to check whether $[0 : 1]$ was a root).

[2]There is a general notion of discriminant (essentially the **resultant polynomial** or the square of the **Vandermonde polynomial**), and genericity is ensured if the discriminant is non-zero.

[3]**Remark.** For n projective hypersurfaces $X_1, \ldots, X_n \subset \mathbb{P}^n$ of degrees $d_1, \ldots, d_n$ then $\#(X_1 \cap \cdots \cap X_n) = d_1 d_2 \cdots d_n$ generically (it is also $d_1 d_2 \cdots d_n$ if it is not infinite, provided that one counts intersections with multiplicities). The key trick is: $Y \subset \mathbb{P}^n$ proj.var., $\dim X = \delta$, $\deg X = d_1$, $H \subset \mathbb{P}^n$ hypersurf of $\deg H = d_2$ not containing irred components of X, then $X \cap H$ has $\dim = \delta - 1$ and $\deg = d_1 d_2$.

[4]This dimension condition is what you would get for vector subspaces $X, Y \subset k^n$ with $X + Y = k^n$.

9.2. HILBERT POLYNOMIAL

$X \subset \mathbb{P}^n$ proj.var. We now relate the degree to Sections 3.10 and 3.11.
$S(X) = k[\widehat{X}] = \oplus_{m\geq 0} S(X)_m$, where $S(X)_m$ is the vector space $k[x_0,\ldots,x_n]_m/\mathbb{I}(X)_m$. Define

$$\boxed{h_X : \mathbb{N}\to\mathbb{N},\ \ h_X(m) = \dim_k S(X)_m = \binom{m+n}{m} - \dim_k \mathbb{I}(X)_m}$$

EXAMPLES.
1) $h_{\mathbb{P}^n}(m) = \binom{m+n}{m} = \frac{(m+n)!}{m!n!} = \frac{1}{n!}(m+n)\cdots(m+1) = \frac{1}{n!}m^n +$ lower order.
2) $X = \mathbb{V}(F) \subset \mathbb{P}^2$, for F irred.homog. of deg d. Then $\mathbb{I}(X)_m = \{\alpha F : \deg\alpha = m-d\}$. Thus

$$\begin{aligned} h_X(m) &= \binom{m+2}{m} - \binom{m-d+2}{m-d} = \tfrac{(m+2)(m+1)}{2} - \tfrac{(m-d+2)(m-d+1)}{2} \\ &= \tfrac{1}{2}(m^2+3m+2-(m^2-2md+3m)-(d-2)(d-1)) = dm - \tfrac{(d-1)(d-2)}{2} + 1. \end{aligned}$$

Fact. (Degree-genus formula for algebraic curves). $g = \mathrm{genus}(X) = \frac{(d-1)(d-2)}{2}$.
Thus $h_X(m) = dm - g + 1$.

FACT.
$X \subset \mathbb{P}^n$ proj.var.
$\Rightarrow$ there exists $p_X \in k[x]$ and there exists m_0 such that for all[1] $m \geq m_0$,

$$h_X(m) = p_X(m).$$

p_X is called the **Hilbert polynomial** of $X \subset \mathbb{P}^n$. Moreover, the leading term of p_X is

$$\boxed{\frac{\deg X}{(\dim X)!}\cdot m^{\dim X}}$$

Remark. p_X depends on the embedding $X \subset \mathbb{P}^n$.
Remark. Other coefficients of p_X are also "discrete invariants" of X. So we only "care" to compare varieties with equal Hilbert polynomial.
Remark. $X, Y \subset \mathbb{P}^n$, if $X \equiv Y$ are linearly equivalent[2] then $p_X = p_Y$.

9.3. FLAT FAMILIES

A **flat family** of varieties is[3] a proj.var. $X \subset \mathbb{P}^n$ together with a surjective morphism

$$\pi : X \to B$$

where B is an irred.proj.var. (or quasi-proj.var.) and the fibers $X_b = \pi^{-1}(b)$ have the same Hilb.poly.
Example. $\phi : \mathbb{P}^1 \to \mathbb{P}^1$, $[x] \mapsto [f_0(x) : f_1(x)]$ where f_0, f_1 are homogeneous of the same degree. Assume f_0, f_1 are linearly independent/k (so $af_0 - bf_1 \neq 0$ for all $(a,b) \in k^2 \setminus \{(0,0)\}$). Then $\phi^{-1}[a:b] = \mathbb{V}(bf_0 - af_1) \subset \mathbb{P}^1$ is a hypersurf of degree d, hence (by Homework 3, ex.2) they have the same Hilbert polynomial for all a, b (in fact the Hilb.poly is the constant d).
Non-example. The **blow-up** of $\mathbb{A}^2$ at the origin is

$$B_0\mathbb{A}^2 = \{\text{any line through } 0 \text{ in } \mathbb{A}^2 \text{ together with any choice of point on the line}\}$$

together with the map $\pi : B_0\mathbb{A}^2 \to \mathbb{A}^2$ which projects to the chosen point on the line. Explicitly:

$$\mathbb{P}^1 \times \mathbb{A}^2 \supset \mathbb{V}(xw - yz) = B_0\mathbb{A}^2 \to \mathbb{A}^2,\ ([x:y],(z,w)) \mapsto (z,w).$$

If $(z,w) \neq 0$, $\pi^{-1}(z,w) = ([z:w],(z,w)) =$ one point[4] (so $B_0\mathbb{A}^2$ is the same as $\mathbb{A}^2$ except over the point 0). Whereas over[5] 0: $\pi^{-1}(0,0) = \{([x:y],(0,0))\} \cong \mathbb{P}^1$. Notice the dimension of the fibers

[1] Think: "for large m, h_X *really* is a polynomial".
[2] i.e. $X \cong Y$ is induced by a (linear) isomorphism $\mathbb{P}^n \cong \mathbb{P}^n$.
[3] This definition is equivalent to the usual definition of flat family (see Hartshorne III.9).
[4] Given a non-zero point in $\mathbb{A}^2$, there is a unique line through the point and 0.
[5] Think of $\pi^{-1}(0) \cong \mathbb{P}^1$ as parametrizing the tangential directions along which lines in $\mathbb{A}^2$ approach the origin.

jumps at 0. Compactifying the above[1] we obtain the blow-up $\pi : B_p\mathbb{P}^2 \to \mathbb{P}^2$ of $\mathbb{P}^2$ at $p = [0:0:1]$, which is not a flat family (the degree of the Hilbert poly of the fibers jumps at p).

10. LOCALISATION THEORY

10.1. LOCALISATION IN ALGEBRA

Let A be a ring (commutative with 1).
Definition 10.1. *$S \subset A$ is a* ***multiplicative set*** *if*[2]

$$1 \in S \ \ \textit{and}\ \ S\cdot S \subset S.$$

EXAMPLES.
1). $S = A \setminus \{0\}$ for any integral domain A.
2). $S = A \setminus \wp$ for any prime ideal $\wp \subset A$.
3). $S = \{1, f, f^2, \ldots\}$ for any $f \in A$.
The definition of localisation of A at S mimics the construction of the fraction field $\mathrm{Frac}(A)$ for an integral domain A, so mimicking $\mathrm{Frac}(\mathbb{Z}) = \mathbb{Q}$. Recall $\mathrm{Frac}(A)$ consists of fractions $\frac{r}{s}$, which formally are thought of as pairs $(r,s) \in A \times (A \setminus \{0\})$, subject to identifying fractions $\frac{r}{s} \sim \frac{r'}{s'}$ if $rs' = r's$.
Definition 10.2. *The* ***localisation*** *of A at S is*

$$S^{-1}A = (A \times S)/\sim$$

where we abbreviate the pairs (r,s) by $\frac{r}{s}$, and the equivalence relation is:

$$\frac{r}{s} \sim \frac{r'}{s'} \iff t(rs' - r's) = 0 \textit{ for some } t \in S. \tag{10.1}$$

We should explain why t appears in (10.1). Algebraically t ensures that $\sim$ is an equivalence relation.
Exercise. Check that $\sim$ is a transitive relation (notice you need to use a clever t).
In many examples, t is not necessary: if A is an integral domain and $0 \notin S$, then (10.1) forces $rs' - r's = 0$ (since there are no zero divisors $t \neq 0$ in S).
Geometric Motivation. The t plays a crucial role in ensuring that localisation identifies the functions that ought to be thought of as equal. Consider $X = \mathbb{V}(xy) = (x\text{-axis}) \cup (y\text{-axis}) \subset \mathbb{A}^2$ and $A = k[X] = k[x,y]/(xy)$. What are the "local functions" near the point $p = (1,0)$? We want to formally invert all those functions $f \in A$ which do not vanish at p:

$$\begin{aligned} S &= \{f \in A : f(p) \neq 0\} \\ &= A \setminus \mathbb{I}(p) \\ &= \{f \in A : f \notin \langle x-1, y\rangle\}. \end{aligned}$$

For example, $x \in S$ since it does not vanish at $p = (1,0)$. Consider the global functions 0 and y: these are different in A. However, once we localise near p, by restricting 0 and y to a neighbourhood of p such as $(x\text{-axis}) \setminus 0 = X \setminus \mathbb{V}(x)$, then the local functions 0 and y become equal. So we want $y = \frac{y}{1} = \frac{0}{1} = 0$ in $S^{-1}A$. Indeed, $t \cdot (y \cdot 1 - 0 \cdot 1) = 0 \in A$ using $t = x \in S$. Without t in (10.1) this would have failed. Moreover, we want the local functions of X near p to agree with the local functions of the irreducible component $\mathbb{V}(y) = (x\text{-axis})$ near p, so we expect (and we prove later) that $S^{-1}A$ is isomorphic to the k-algebra $k[x]$ after inverting all $h \in k[x] \setminus \mathbb{I}(p)$:

$$k[x]_{\mathbb{I}(p)} = k[x][\tfrac{1}{h} : h(p) \neq 0] \subset \mathrm{Frac}(k[x]) = k(x).$$

Exercise.[3] $S^{-1}A = 0 \Leftrightarrow 0 \in S$.
Exercise. Show that

$$\frac{r}{s} = 0 \in S^{-1}A \Leftrightarrow (tr = 0 \text{ for some } t \in S) \Leftrightarrow r \in \bigcup_{t\in S} \mathrm{Ann}(t).$$

[1] $\pi : B_p\mathbb{P}^2 \to \mathbb{P}^2$ is an isomorphism on the complement of $\pi^{-1}(p)$.
[2] $S \cdot S \subset S$ means $st \in S$ for all $s, t \in S$. Some books require that $0 \notin S$, but we do not.
[3] *Hint.* Consider 1.

In particular, for an integral domain A, $\frac{r}{s} = 0 \Leftrightarrow r = 0$ (assuming $0 \notin S$).

EXAMPLES.

1). $A_f = S^{-1}A$ is the localisation of A at $S = \{1, f, f^2, \ldots\}$. So

$$A_f = \{\tfrac{r}{f^m} : r \in A, m \geq 0\}/\sim$$

where for example $\frac{r}{f^m} = \frac{rf}{f^{m+1}}$, and more generally $\frac{r}{f^m} = \frac{r'}{f^n} \Leftrightarrow f^N(rf^n - r'f^m) = 0$ for some $N \geq 0$.

- if f is nilpotent, so $f^N = 0 \in S$ for some N, so $A_f = \{0\}$. Indeed: $A_f = 0 \Leftrightarrow f$ is nilpotent.
- if A is an integral domain,

$$A_f = A[\tfrac{1}{f}] \subset \mathrm{Frac}(A).$$

2). $A = k[x, y]/(xy)$, $S = \{1, x, x^2, \ldots\}$ then $y = \frac{y}{1}$ is zero since y is annihilated by $x \in S$. Thus

$$S^{-1}A \cong k[x]_x = k[x, x^{-1}] \subset \mathrm{Frac}(k[x]) = k(x).$$

Exercise. In general, $A_f \cong A[z]/(zf - 1)$ (we have seen this trick before).

$S^{-1}A$ is a ring in a natural way:

$$\frac{r}{s} + \frac{r'}{s'} = \frac{rs' + r's}{ss'} \qquad \frac{r}{s} \cdot \frac{r'}{s'} = \frac{rr'}{ss'}$$

with zero $0 = \frac{0}{1}$ and identity $1 = \frac{1}{1}$, and it comes with a canonical ring homomorphism

$$\pi : A \to S^{-1}A, \quad a \mapsto \frac{a}{1}$$

which has kernel

$$\ker \pi = \{a \in A : ta = 0 \text{ for some } t \in S\} = \bigcup_{t \in S} \mathrm{Ann}(t).$$

If A is an integral domain then $\pi : A \hookrightarrow S^{-1}A$ is injective (assuming $0 \notin S$).

Exercise. Check the above statements (in particular, that the operations are well-defined).

EXAMPLES.

1). $S = A \setminus \wp$, then the **localisation of A at the prime ideal** $\wp$ is[1]

$$A_\wp = \{\tfrac{r}{s} : r \in A, s \notin \wp\}/\sim.$$

2). For an integral domain A, let $S = A \setminus \{0\}$, then the localisation at $\wp = (0)$ is:

$$S^{-1}A = A_{(0)} = \mathrm{Frac}(A).$$

Definition 10.3. *A is a **local ring** if it has a unique maximal ideal $\mathfrak{m} \subset A$. The field $A/\mathfrak{m}$ is called **residue field**.*

Exercise.[2] A is local $\Leftrightarrow$ there exists an ideal $\mathfrak{m} \subsetneq A$ such that all elements in $A \setminus \mathfrak{m}$ are units.

Lemma 10.4. *$A_\wp$ is a local ring with maximal ideal $\wp A_\wp = \{\frac{r}{s} : r \in \wp, s \notin \wp\}/\sim$.*

Proof. Notice $\wp \cdot A_\wp$ is an ideal. Suppose $\frac{r}{s} \notin \wp A_\wp$. Then $r \notin \wp$. So $\frac{r}{s}$ is a unit since $\frac{s}{r} \in A_\wp$. □

Key Exercise. For A an integral domain,

$$A = \bigcap_{\max \mathfrak{m} \subset A} A_\mathfrak{m} = \bigcap_{\text{prime } \wp \subset A} A_\wp \subset \mathrm{Frac}(A).$$

[1] Don't get confused with A_f. For A_f we invert f. For $A_\wp$ we invert everything *except* what's in $\wp$!

[2] *Hints.* To show $\mathfrak{m}$ is maximal: $\mathfrak{m} \subsetneq I \subset A$ implies I contains a unit, so $I = A$. Conversely, if $u \in A \setminus \mathfrak{m}$ were not a unit, then there is a maximal ideal containing the ideal $\langle u \rangle$, and this cannot equal $\mathfrak{m}$.

Exercise.[1] Let $\varphi : A \to B$ be a ring hom, and $\wp \subset B$ a prime ideal. Abbreviate $\varphi^*\wp = \varphi^{-1}(\wp)$. Show there is a natural local ring hom[2]

$$A_{\varphi^*\wp} \to B_\wp. \tag{10.2}$$

Example. Localising $\mathbb{Z}$ at a prime (p): $\mathbb{Z}_{(p)} = \{\frac{a}{b} : p \nmid b\}$ has max ideal $\mathfrak{m}_p = p\mathbb{Z}_{(p)} = \{\frac{a}{b} : p | a, \, p \nmid b\}$.

Exercise. The residue field is $\mathbb{Z}_{(p)}/\mathfrak{m}_p \cong \mathbb{Z}/(p)$, $\frac{a}{b} \mapsto ab^{-1}$.

As an exercise in algebra, try proving the following:

FACT. There is a 1:1 correspondence

$$\begin{array}{rcl} \{\text{prime ideals } I \subset A \text{ with } I \cap S = \emptyset\} & \leftrightarrow & \{\text{prime ideals } J \subset S^{-1}A\} \\ I & \mapsto & J = I \cdot S^{-1}A = \{\frac{i}{s} : i \in I, s \in S\} \\ I = \pi^{-1}(J) = \{i \in A : \frac{i}{1} \in J\} & \leftarrow\!\shortmid & J. \end{array}$$

In particular, for a prime ideal $\wp \subset A$,

$$\begin{array}{rcl} \{\text{prime ideals } I \subset \wp \subset A\} & \leftrightarrow & \{\text{prime ideals } J \subset A_\wp\} \\ I = \pi^{-1}(J) & \leftrightarrow & J = IA_\wp. \end{array}$$

Exercise. If A is Noetherian, then $S^{-1}A$ is Noetherian.

Exercise. $S^{-1}(A/I) \cong (S^{-1}A)/(IS^{-1}A)$, in particular

$$(A/I)_\wp \cong A_\wp/IA_\wp.$$

Example. Consider again $A = k[x, y]/(xy) = k[X]$, so $X = X_1 \cup X_2$ where $X_1 = \mathbb{V}(y) = (x\text{-axis})$ and $X_2 = \mathbb{V}(x) = (y\text{-axis})$. Consider $p = (1, 0) \in X_1 \setminus X_2$ and $\mathfrak{m}_p = \mathbb{I}(p)$. Recall any $f \in (y) \subset k[X_2] = k[y]$ becomes zero in $A_{\mathfrak{m}_p}$ because $xf = 0 \in A$, where $x \in S = k[X] \setminus \mathfrak{m}_p$. So let $I = yA \subset A$, then $IA_{\mathfrak{m}_p} = 0 \subset A_{\mathfrak{m}_p}$. Thus, since $A/I \cong k[x] = k[X_1]$:

$$k[X]_{\mathfrak{m}_p} = A_{\mathfrak{m}_p} \cong A_{\mathfrak{m}_p}/IA_{\mathfrak{m}_p} \cong (A/I)_{\mathfrak{m}_p} \cong k[X_1]_{\mathfrak{m}_p} = k[x][\tfrac{1}{h} : h(p) \neq 0] \subset k(x)$$

as promised. In general, if you localize at a point p which only belongs to one irreducible component, then the local ring at p agrees with the local ring of the irreducible component at p.

Exercise. $S^{-1}\sqrt{I} = \sqrt{S^{-1}I}$, in particular localising radical ideals gives radical ideals.

10.2. LOCALISATION FOR AFFINE VARIETIES: regular functions and stalks

Motivation. *We now want to consider the k-algebra of functions that are naturally defined near a point p, and we expect that any function which doesn't vanish at p should be invertible near p.*

For any topological space X, a **germ of a function** near a point $p \in X$ means a function $f : U \to k$ defined on a neighbourhood $U \subset X$ of p, where we identify two such functions $U \to k$, $U' \to k$ if they agree on a smaller neighbourhood of p. So a germ is an equivalence class $[(U, f)]$.

Let X be an affine variety, and $p \in X$. A function $f : U \to k$ defined on a neighbourhood of p is called **regular at** p if on some open $p \in W \subset U$, the following functions $W \to k$ are equal,

$$f = \frac{g}{h} \qquad \text{some } g, h \in k[X] \text{ and } h(w) \neq 0 \text{ for all } w \in W.$$

We write $\mathcal{O}_X(U)$ for the k-algebra of functions $f : U \to k$ regular at all points in an open $U \subset X$. The **stalk** $\mathcal{O}_{X,p}$ is the k-algebra of germs of regular functions at p, so equivalence classes of pairs (U, f) with $p \in U \subset X$ open and $f : U \to k$ a regular function, where we identify $(U, f) \sim (V, g)$ if $f|_W = g|_W$ on an open $p \in W \subset U \cap V$. **Exercise.** Check this is a k-algebra in the obvious sense.

EXAMPLES.

1) For any $f \in k[X]$, $f : X \to k$ is regular at each point (consider $U = X$ and $f = \frac{f}{1}$). We will show in Theorem 11.2 that functions regular at each point of X always arise in this way. So

$$\mathcal{O}_X(X) \cong k[X].$$

[1] *Hint.* If $S \subset A$ is multiplicative such that $\varphi(S) \subset B$ consists of units, there's an obvious hom $S^{-1}A \to B$, $\frac{r}{s} \mapsto \frac{\varphi(r)}{\varphi(s)}$.

[2] a hom of local rings $R_1 \to R_2$ sending the max ideal $\mathfrak{m}_1$ to (a subset of) the max ideal $\mathfrak{m}_2$.

2) For $X = \mathbb{A}^1$, $m \in \mathbb{N}$, $f : U = D_x = \mathbb{A}^1 \setminus \{0\} \to k$, $f(x) = \frac{1}{x^m}$ is regular at any $p \in U$, so $f \in \mathcal{O}(U)$.
3) More generally, for any $f \in k[X]$, recall $D_f = X \setminus \mathbb{V}(f)$. Corollary 11.3 will show that

$$\mathcal{O}_X(D_f) \cong k[X]_f.$$

4) Let $X = \mathbb{V}(xy) \subset \mathbb{A}^2$ (the union of the two axes). Let $U = X \setminus \mathbb{V}(y) = (x\text{-axis}) \setminus \{0\}$. Then $f : U \to k$, $f(x,y) = \frac{y}{x} \in \mathcal{O}(U)$, but $(U, f) \sim (U, 0)$ as $y = 0 : U \to k$, so $[(U, f)] = 0$.

Lemma 10.5. *At $p \in X$, the stalk of the structure sheaf $\mathcal{O}_X$ is:*

$$\boxed{\mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p}}$$

where $\mathfrak{m}_p = \mathbb{I}(p) = \{f \in k[X] : f(p) = 0\}$ is the maximal ideal corresponding to p.

Proof. The isomorphism is defined by

$$(U, f) \mapsto \frac{g}{h}$$

where $f|_U = \frac{g}{h}$ for $g, h \in k[X]$, $h(p) \neq 0$. The map is well-defined: $h(p) \neq 0 \Rightarrow h \notin \mathfrak{m}_p \Rightarrow \frac{g}{h} \in k[X]_{\mathfrak{m}_p}$. Moreover, if $(U, f) \sim (U', f')$, so $\frac{g}{h} = \frac{g'}{h'}$ on a basic open $p \in D_s \subset U \cap U'$, where $s \in k[X]$, then $gh' - g'h = 0$ on D_s. Since $s(p) \neq 0$, we have $s \notin \mathfrak{m}_p$. Thus $s \cdot (gh' - g'h) = 0$ everywhere on X, so $s \cdot (gh' - g'h) = 0$ in $k[X]$. Thus $\frac{g}{h} = \frac{g'}{h'}$ in $k[X]_{\mathfrak{m}_p}$.
We build the inverse map: for $h \notin \mathfrak{m}_p$, let $U = D_h$, then send $\frac{g}{h} \mapsto (U, \frac{g}{h})$. Moreover, if $\frac{g}{h} = \frac{g'}{h'}$ in $k[X]_{\mathfrak{m}_p}$, then $s \cdot (gh' - g'h) = 0$ for some $s \in k[X] \setminus \mathfrak{m}_p$. Then $s(p) \neq 0$ so $p \in D_s$, and $gh' - g'h = 0$ on D_s. Thus $\frac{g}{h} = \frac{g'}{h'}$ as functions $D_s \to k$, as required.
By construction, the two maps are inverse to each other, so we have an isomorphism. □

Example. For an irreducible variety X, we get an integral domain A, so Lemma 10.5 becomes:

$$\mathcal{O}_{X,p} = k[X]_{\mathfrak{m}_p} = k[X][\tfrac{1}{h} : h(p) \neq 0] \subset \mathrm{Frac}(k[X]) = k(X)$$

and the Key Exercise, from Section 10.1, implies[1]

$$k[X] = \bigcap_{p \in X} \mathcal{O}_{X,p} \subset \mathcal{O}_{X,p} \subset k(X).$$

The FACT, from Section 10.1, translates into geometry as the 1:1 correspondence:

$$\begin{array}{rcl} \{\text{irreducible subvarieties } Y \subset X \text{ passing through } p\} & \leftrightarrow & \{\text{prime ideals in } \mathcal{O}_{X,p}\} \\ Y = \mathbb{V}(J) & \leftrightarrow & J \cdot \mathcal{O}_{X,p} = \{f \in \mathcal{O}_{X,p} : f(Y) = 0\}. \end{array}$$

where $J = \mathbb{I}(Y)$. In particular, the point $Y = \{p\}$ corresponds to the maximal ideal $\mathfrak{m}_p \mathcal{O}_{X,p} \subset \mathcal{O}_{X,p}$.
By Lemma 10.4, $\mathfrak{m}_p \mathcal{O}_{X,p} \subset \mathcal{O}_{X,p}$ is the unique maximal ideal. The quotient recovers our field k:

$$\mathbb{K}(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p \mathcal{O}_{X,p} \cong k, \ \frac{g}{h} \mapsto \frac{g(p)}{h(p)}. \tag{10.3}$$

Warning. Not all function spaces arise as a localisation of $k[X]$. For example $f = \frac{x}{y} = \frac{z}{w} \in k(X)$ where $X = \mathbb{V}(xw - yz) \subset \mathbb{A}^4$ defines a regular function $f \in \mathcal{O}_X(D_y \cup D_w)$. But it turns out that one cannot write $f = \frac{g}{h}$ on all of $D_y \cup D_w$ for $g, h \in k[X]$ (this is caused by the fact that $k[X] = k[x, y, z, w]/(xw - yz)$ is not a UFD). So $\mathcal{O}_X(D_y \cup D_w)$ is not a localisation of $k[X]$, unlike $\mathcal{O}_X(D_y) = k[X]_y$, $\mathcal{O}_X(D_w) = k[X]_w$, $\mathcal{O}_X(D_y \cap D_w) = \mathcal{O}_X(D_{yw}) = k[X]_{yw}$ which are all localisations.

[1] Recall the first equality implies the theorem "regular at all points implies polynomial" for an irred.aff.var. (Theorem 11.2). That k is algebraically closed comes into play: all max ideals arise as $\mathfrak{m}_p = \mathbb{I}(p)$ for $p \in X$.

10.3. HOMOGENEOUS LOCALISATION: projective varieties

Let $A = \oplus_{m \geq 0} A_m$ be an $\mathbb{N}$-graded ring. Let $S \subset A$ be a multiplicative set consisting only of homogeneous elements. Then $S^{-1}A = \oplus_{m \in \mathbb{Z}} (S^{-1}A)_m$ has a $\mathbb{Z}$-grading: if $r \in A, s \in S$ are homogeneous elements then $m = \deg \frac{r}{s} = \deg(r) - \deg(s) \in \mathbb{Z}$.

Exercise. Show that $(S^{-1}A)_0 \subset S^{-1}A$ is a subring.

Example. For $A = k[x_0, \ldots, x_n]$, $(S^{-1}A)_0$ is important: they are the rational functions $\frac{F(x_0,\ldots,x_n)}{G(x_0,\ldots,x_n)}$ for F, G homogeneous polys of equal degree, so $\frac{F(p)}{G(p)} \in k$ is well-defined[1] for $p \in \mathbb{P}^n$ with $G(p) \neq 0$.

Definition 10.6. *The **homogeneous localisation** is the subring $(S^{-1}A)_0$ of $S^{-1}A$. Abbreviate by $A_{(f)} = (A_f)_0$ the h.localisation at $\{1, f, f^2, \ldots\}$ for a homogeneous element $f \in A$; and $A_{(\wp)} = (A_\wp)_0$ for the h.localisation at all homogeneous elements in $A \setminus \wp$ for a homogeneous prime ideal $\wp \subset A$.*

Let $X \subset \mathbb{P}^n$ be a q.p.v. Recall $S(\overline{X})$ is the homogeneous coordinate ring of the projective variety $\overline{X} \subset \mathbb{P}^n$, the projective closure of $X \subset \mathbb{P}^n$. Abbreviate $X_0 = \overline{X} \cap U_0$. Then:

Lemma 10.7. *For any $p \in X_0$,*

$$\mathcal{O}_{X,p} \cong k[X_0]_{\mathfrak{m}_{p,0}} \cong S(\overline{X})_{(\mathfrak{m}_p)}$$

where $\mathfrak{m}_{p,0} = \{f \in k[X_0] : f(p) = 0\}$ and $\mathfrak{m}_p = \{F \in S(\overline{X}) : F(p) = 0\}$.

Proof. The mutually inverse morphisms are given by homogenising and dehomogenising. Explicitly, where $d = \max(\deg(f), \deg(g))$,

$$\frac{f(x_1, \ldots, x_n)}{g(x_1, \ldots, x_n)} \mapsto \frac{x_0^d f(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})}{x_0^d g(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})} \qquad \frac{F(x_0, x_1, \ldots, x_n)}{G(x_0, x_1, \ldots, x_n)} \mapsto \frac{F(1, x_1, \ldots, x_n)}{G(1, x_1, \ldots, x_n)}.$$ □

Exercise. [See Hwk sheet 1, ex.5.] Show that the projectivisation $\overline{X} \subset \mathbb{P}^2$ of $X = \mathbb{V}(y - x^3) \subset \mathbb{A}^2$ is not iso to $\mathbb{P}^1$ by computing the local ring $\mathcal{O}_{\overline{X},p}$ at $p = [0 : 1 : 0]$ (compare with local rings of $\mathbb{P}^1$). Show $\mathbb{V}(y - x^3) \cong \mathbb{V}(y - x^2)$ as affine varieties in $\mathbb{A}^2$, but their projectivisations in $\mathbb{P}^2$ are not iso.

11. QUASI-PROJECTIVE VARIETIES

11.1. QUASI-PROJECTIVE VARIETY

Aim: Define a large class of varieties which contains both affine vars, projective vars, and open sets e.g. $k^* \subset k$, such that any open subset of a variety in this class is also in this class.

Definition. *A **quasi-projective variety** $X \subset \mathbb{P}^n$ is any open subset of a projective variety, so*

$$X = U_J \cap \mathbb{V}(I)$$

where $U_J = \mathbb{P}^n \setminus \mathbb{V}(J)$, so X is an intersection[2] of an open and a closed subset of $\mathbb{P}^n$.

EXAMPLES.
1) Affine $X \subset \mathbb{A}^n$: then $X = \mathbb{A}^n \cap \overline{X}$ (exercise[3]).
2) Projective $X \subset \mathbb{P}^n$: then $X = \mathbb{P}^n \cap X$.
3) $\mathbb{A}^2 \setminus \{0\} = (U_0 \cap (U_1 \cup U_2)) \cap \mathbb{P}^2$ (viewing[4] $\mathbb{A}^2 \equiv U_0 \subset \mathbb{P}^2$).
4) Any open subset of a q.p.var. is also a q.p.var., since $U_{J'} \cap (U_J \cap \mathbb{V}(I)) = (U_{J'} \cap U_J) \cap \mathbb{V}(I)$.

Definition. A **morphism of q.p.vars.** $X \to Y$ is defined just as for proj.vars., so locally

$$p \mapsto [F_0(p) : \cdots : F_m(p)]$$

[1] i.e. unchanged under the k^*-rescaling action which defines $\mathbb{P}^n$.
[2] such sets are called **locally closed subsets**.
[3] Recall $\overline{X} \subset \mathbb{P}^n$ is the projective closure of $X \subset \mathbb{A}^n \equiv U_0 \subset \mathbb{P}^n$, and recall Theorem 3.3.
[4] $\{[1 : * : *]\} = \{[x_0 : x_1 : x_2] : x_0 \neq 0\}$ and we exclude the case $x_1 = x_2 = 0$ by taking $U_1 \cup U_2 = \mathbb{P}^2 \setminus \mathbb{V}(x_1, x_2)$.

for homogeneous polys $F_0, \dots, F_m$ of the same degree (where $X \subset \mathbb{P}^n$, $Y \subset \mathbb{P}^m$).
Remark. For X, Y affine, this agrees with the definition of morph of aff.vars.:

$$\begin{array}{ccc} [x_0 : \cdots : x_n] & & [F_0(x) : \cdots : F_m(x)] \\ \| & & \| \\ [1 : y_1 : \cdots : y_n] & \longmapsto & [1 : f_1(y) : \cdots : f_m(y)] = [x_0^d : x_0^d f_1(y) : \cdots : x_0^d f_m(y)] \end{array}$$

where $y_i = x_i/x_0$ ($x_0 \neq 0$), $d = \max \deg f_i$, and $F_0(x) = x_0^d$, $F_i(x) = x_0^d f_i(y)$ (notice $\deg F_i = d$).

Corollary. *$X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^n$ q.p.vars. If there are mutually inverse polynomial maps $X \to Y$ and $Y \to X$, then $X \cong Y$ as q.p.vars.*

Warning. The converse is false: $\mathbb{A}^2 \supset \mathbb{V}(xy-1) \cong \mathbb{A}^1 \setminus 0$ q.p.vars, but not via a polynomial map:

$$\begin{array}{ccc} (x,y) & & x \\ \| & & \| \\ \mathbb{P}^2 \supset [x:y:1] = [x : x^{-1} : 1] & \longmapsto & [x:1] \in \mathbb{P}^1 \\ \| & & \| \\ [x^2 : 1 : x] & \longleftarrow & [x:1] \\ [x^2 : y^2 : xy] & \longleftarrow & [x:y] \\ [x:y:z] & \longmapsto & [x:z] \end{array}$$

Definition. *A q.p.var. X is **affine** if it is isomorphic (as q.p.vars) to an aff.var. $Y = \mathbb{V}(I) \subset \mathbb{A}^n$. We will often write $k[X]$ when we mean $k[Y] = k[x_1, \dots, x_n]/\mathbb{I}(Y)$.*

Example. $k^* \subset k$ is affine.

11.2. QUASI-PROJECTIVE VARIETIES ARE LOCALLY AFFINE

Lemma 11.1. *X aff.var., $f \in k[X]$. Then $D_f = X \setminus \mathbb{V}(f)$ is an affine q.p.var. with*[1]

$$k[D_f] = \{\tfrac{g}{f^m} : D_f \to k \text{ where } g \in k[X], m \geq 0\} \cong k[X][\tfrac{1}{f}] \cong k[X]_f$$

defining[2] *$k[X][\frac{1}{f}] \equiv k[X][x_{n+1}]/(f x_{n+1} - 1)$, so we introduced a formal inverse "$x_{n+1} = \frac{1}{f}$". It will become clear in Sec.10 that this is one equivalent*[3] *definition of the localisation $k[X]_f$ at f.*

Proof. Define $\widetilde{I} = \langle \mathbb{I}(X), x_{n+1} f - 1 \rangle \subset k[x_1, \dots, x_n, x_{n+1}]$
$\Rightarrow \mathbb{V}(\widetilde{I}) \subset \mathbb{A}^{n+1}$ is affine with a new coordinate function x_{n+1} which is reciprocal to f,

$$k[\mathbb{V}(\widetilde{I})] = k[X][x_{n+1}]/(f x_{n+1} - 1) \equiv k[X][\tfrac{1}{f}].$$

Subclaim. $\varphi : D_f \to \mathbb{V}(\widetilde{I})$ is an iso of q.p.vars, via

$$a = (a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, \tfrac{1}{f(a)})$$

with inverse $(b_1, \dots, b_n) \leftarrow\!\shortmid (b_1, \dots, b_n, b_{n+1})$.
Pf of Subclaim. View $D_f \subset \mathbb{A}^n \equiv U_0 \subset \mathbb{P}^n$ via $(a_1, \dots, a_n) \leftrightarrow [1 : a_1 : \cdots : a_n]$ and $\mathbb{V}(\widetilde{I}) \subset \mathbb{A}^{n+1} \equiv U_0 \subset \mathbb{P}^{n+1}$ via $(a_1, \dots, a_{n+1}) \leftrightarrow [1 : a_1 : \cdots : a_{n+1}]$. Then φ is the restriction of $F : \mathbb{P}^n \to \mathbb{P}^{n+1}$,

$$\begin{array}{ccc} [1 : \frac{a_1}{a_0} : \cdots : \frac{a_n}{a_0}] & \overset{\text{“}\varphi\text{”}}{\longmapsto} & [1 : \frac{a_1}{a_0} : \cdots : \frac{a_n}{a_0} : \frac{1}{f(\frac{a_1}{a_0}, \cdots, \frac{a_n}{a_0})}] \\ \| & & \| \\ [a_0 : a_1 : \cdots : a_n] & \underset{F}{\longmapsto} & [a_0 \widetilde{f}(a) : a_1 \widetilde{f}(a) : \cdots : a_{n-1} \widetilde{f}(a) : a_0^{\deg f + 1}] \end{array}$$

[1] For X not irreducible, we may worry about the definition of localisation: $\frac{g}{f^a} = \frac{h}{f^b} \in k[X]_f \Leftrightarrow f^\ell(f^b g - f^a h) = 0$ for some $\ell \geq 0$. But evaluating at $p \in D_f$ (thus $f(p) \neq 0$) implies $f(p)^b g(p) - f(p)^a h(p) = 0 \in k$, so $\frac{g(p)}{f(p)^a} = \frac{h(p)}{f(p)^b}$. So also the functions $\frac{g}{f^a} = \frac{h}{f^b} : D_f \to k$ agree.

[2] For X irreducible, we can view $k[X][\frac{1}{f}]$ as the subalgebra $\{\frac{g}{f^m} : g \in k[X], m \geq 0\} \subset k(X) = \operatorname{Frac} k[X]$.

[3] The k-alg.hom $k[X][x_{n+1}]/(f x_{n+1} - 1) \to k[X]_f$ sending $x_{n+1} \mapsto \frac{1}{f}$ (and $g \in k[X]$ to $\frac{g}{1}$) has inverse $\frac{g}{f^a} \mapsto g x_{n+1}^a$.

where we homogenised: $\widetilde{f}(a) = \widetilde{f}(a_0, \dots, a_n) = a_0^{\deg f} f(\frac{a_1}{a_0}, \cdots, \frac{a_n}{a_0})$, and in the second vertical identification we rescaled by $a_0 \widetilde{f}(a)$. The local inverse is $[a_0 : \cdots : a_n] \leftarrow\!\shortmid [a_0 : \cdots : a_{n+1}] \in U_0$ (the composites give the identity, using that $\widetilde{f}(a) \neq 0$ on D_f, so we may rescale by $\frac{1}{\widetilde{f}(a)}$). □

Theorem. *Every q.p.var. has a finite open cover by affine q.p.subvars. In particular, affine open subsets form a basis for the topology.*

Proof. $\mathbb{P}^n \supset X = U_J \cap \mathbb{V}(I) = \mathbb{V}(F_1, \dots, F_N) \setminus \mathbb{V}(G_1, \dots, G_M)$ (where we picked generators for J, I). WLOG[1] it suffices to check the claim on the open $U_0 \cap X$. Then $U_0 \cap X$ is $\mathbb{V}(f_1, \dots, f_N) \setminus \mathbb{V}(g_1, \dots, g_M) = \cup_j \mathbb{V}(f_1, \dots, f_N) \setminus \mathbb{V}(g_j) = \cup_j D_{g_j}$ where D_{g_j} is the basic open subset $(g_j \neq 0) \subset \mathbb{V}(f_1, \dots, f_N)$, and where $f_1 = F_1|_{x_0=1} \in k[x_1, \dots, x_n]$ so $f_1(a) = F_1(1, a)$ etc. Now apply Lemma 11.1. □

11.3. REGULAR FUNCTIONS

Motivation. $\mathbb{A}^1 \setminus \{0\} \cong \mathbb{V}(xy-1) \subset \mathbb{A}^2$. We want to allow the function $\frac{1}{x^m} = y^m$.

Definition. *X aff.var., $U \subset X$ open.*

$$\begin{aligned} \mathcal{O}_X(U) &= \{\text{regular functions } f : U \to k\} \\ &= \{f : U \to k : f \text{ is regular at each } p \in U\} \end{aligned}$$

*Recall, f **regular at** p means: on some open $p \in W \subset U$, the following functions $W \to k$ are equal,*

$$f = \frac{g}{h} \qquad \text{some } g, h \in k[X] \text{ and } h(w) \neq 0 \text{ for all } w \in W.$$

Example 1. $U = D_x = \mathbb{A}^2 \setminus \mathbb{V}(x) \subset \mathbb{A}^2$, $f : D_x \to k$, $f(x,y) = \frac{y}{x} \in \mathcal{O}_X(D_x)$.
2. For any $g, h \in k[X]$, with $h \neq 0$, we have $\frac{g}{h} \in \mathcal{O}_X(D_h)$.

REMARKS.
1) Some books just say $h(p) \neq 0$, and this is enough[2] since we can always replace W by $W \cap D_h$.
2) We are *not* saying that $f = \frac{g}{h}$ holds on all of U, only *locally*.
3) We are *not* saying that g, h are unique (e.g. in $\mathbb{Q}$, $\frac{2}{3} = \frac{4}{6}$).
4) Later we will prove that if $U \cong Y \subset \mathbb{A}^n$ is affine, then $\mathcal{O}_X(U)$ is isomorphic to the classical $k[Y]$. By making W smaller, we can always assume W is a basic open set D_β for some polynomial function $\beta : X \to k$ (and $\beta(p) \neq 0$). As D_β is affine, $\mathcal{O}_X(D_\beta) = k[D_\beta] = k[X]_\beta$, therefore $f = \frac{\alpha}{\beta^N}$ as functions $D_\beta \to k$, for some $\alpha, \beta \in k[X]$, $N \in \mathbb{N}$. By replacing β by β^N, we can assume $f = \frac{\alpha}{\beta}$ (so $N = 1$).
5) Some books always abbreviate $k[U] = \mathcal{O}_X(U)$, but we will try to avoid this to prevent confusion.

Definition. *X q.p.var., $U \subset X$ open.*

$$\mathcal{O}_X(U) = \{F : U \to k : F \text{ is regular at each } p \in U\}.$$

*F **regular at** p means: on some* affine *open $p \in W \subset U$, $F|_W$ is regular at p as previously defined.*

REMARKS.
1) Recall the affine open covering $U_i = (x_i \neq 0) \subset \mathbb{P}^n$. Suppose $p \in X \cap U_i$. Note that $X \cap U_i$ is an open set in $U_i \cong \mathbb{A}^n$. Then near p, F is equal to a ratio of two polynomials in the variables $x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n$ whose denominator does not vanish at p. Following Remark 4 above, we can also pick an affine open $D_\beta \subset U_i \cong \mathbb{A}^n$ so that $F = \frac{\alpha}{\beta}$ as a function $D_\beta \to k$ or equivalently as an element of the localisation $k[D_\beta] \cong k[U_i]_\beta = k[x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n]_\beta$. If you want to view F as a function $\mathbb{P}^n \to k$ defined near[3] p, you need to homogenise by replacing each x_j by x_j/x_i. Clearing denominators will give a ratio of homogeneous polynomials of the same degree.
2) Gluing regular functions. Given open sets U_1, U_2 in a q.p.var. X, and regular functions

[1] because $\mathbb{P}^n$ has an open cover by U_i.

[2] Although I find the meaning of the equality $f = \frac{g}{h}$ unclear, on the larger W.

[3] The function is not defined on all of $\mathbb{P}^n$ as the denominator may vanish (recall global morphs $\mathbb{P}^n \to k$ are constant).

$f_1 \in \mathcal{O}_X(U_1)$ and $f_2 \in \mathcal{O}_X(U_2)$, observe that the necessary and sufficient condition to be able to find a glued regular function $f \in \mathcal{O}_X(U_1 \cup U_2)$ (meaning, it restricts to f_i on U_i) is that $f_1|_{U_1\cap U_2} = f_2|_{U_1 \cap U_2}$. Indeed, define $f = f_i$ on U_i, then $f : U_1 \cup U_2 \to k$ is well-defined, and regularity follows because regularity is a local condition and we already know it is satisfied by f_1, f_2 on U_1, U_2.

Exercise. (Non-examinable) Using Remark 2 and Sec.15.5, show $\mathcal{O}_X$ is a sheaf (of k-algs) on X.

3) Let $\varphi : X \cong Y$ be isomorphic q.p.vars, and $U \subset X$ an open set, so $V = \varphi(U) \subset Y$ is an open set. Then we have an iso $\varphi^* : \mathcal{O}_Y(V) \cong \mathcal{O}_X(U)$, $F \mapsto F \circ \varphi$. (*Hint:* first read Sec.11.4).

Warning. For $f \in \mathcal{O}_X(U)$, it may not be possible to find a fraction $f = \frac{g}{h}$ that works on all of U.

Example. For the affine variety $X = \mathbb{V}(xw - yz) \subset \mathbb{A}^4$, $f = \frac{x}{y} = \frac{z}{w} \in k(X) = \operatorname{Frac} k[X]$ defines a rational function $f \in \mathcal{O}_X(D_y \cup D_w)$ on the q.p.var. $U = D_y \cup D_w$ since $\frac{x}{y} \in \mathcal{O}_X(D_y)$ and $\frac{z}{w} \in \mathcal{O}_X(D_w)$, but one cannot[1] find a global expression $f = \frac{g}{h}$ defined on all of U.

Theorem 11.2. *X affine variety $\Rightarrow \mathcal{O}_X(X) = k[X]$.*

Proof.[2] **Claim 1.** $k[X] \subset \mathcal{O}_X(X)$. *Proof 1.* $f \in k[X] \Rightarrow f = \frac{f}{1}$ on X, so it is regular everywhere. ✓
Claim 2. $\mathcal{O}_X(X) \subset k[X]$. *Proof 2.* $\forall p \in X$, $\exists$ open $p \in U_p \subset X$:

$$\mathcal{O}_X(X) \ni f = \tfrac{g_p}{h_p} \text{ as maps } U_p \to k,$$

where $g_p, h_p \in k[X]$, and $h_p \neq 0$ at all points of U_p. Since basic open sets are a basis for the Zariski topology, we may assume $U_p = D_{\ell_p}$ for some $\ell_p \in k[X]$ (possibly making U_p smaller). We now need:[3]
Trick. $\frac{g_p}{h_p} = \frac{g_p \ell_p}{h_p \ell_p}$ *on D_{ℓ_p}. Replacing g_p, h_p by $g_p\ell_p, h_p\ell_p$, we may assume $g_p = h_p = 0$ on $\mathbb{V}(\ell_p)$. As $h_p \neq 0$ at points of $U_p = D_{\ell_p}$, we deduce $D_{h_p} = D_{\ell_p}$. So $f = \frac{g_p}{h_p}$ on $U_p = D_{h_p}$, and $g_p = 0$ on $\mathbb{V}(h_p)$.*

Now consider the ideal $J = \langle h_p : p \in X \rangle \subset k[X]$.
Then $\mathbb{V}(J) = \emptyset$ since $h_p(p) \neq 0$. By Hilbert's Nullstellensatz, $J = k[X] = \langle 1 \rangle$ so $1 = \sum \alpha_i h_{p_i} \in k[X]$ for some finite collection of $p_i \in X$, and $\alpha_i \in k[X]$. Abbreviate $h_i = h_{p_i}$, $g_i = g_{p_i}$, $D_i = U_{p_i} = D_{h_{p_i}}$. Note that $1 = \sum \alpha_i h_i$ implies[4] that the D_i are an open cover of X. On the overlap $D_i \cap D_j$, we know $\frac{g_j}{h_j} = f = \frac{g_i}{h_i}$, so $h_i g_j = h_j g_i$ on $D_i \cap D_j$. By the above Trick, $h_i g_j = h_j g_i$ also holds on $\mathbb{V}(h_i) = X \setminus D_i$ since $g_i = h_i = 0$ there, and also on $\mathbb{V}(h_j) = X \setminus D_j$ since $g_j = h_j = 0$ there. Thus $h_i g_j = h_j g_i$ holds everywhere on X as $X = (D_i \cap D_j) \cup \mathbb{V}(h_i) \cup \mathbb{V}(h_j)$. Thus, on X, we deduce

$$f = \frac{g_j}{h_j} = 1 \cdot \frac{g_j}{h_j} = \sum_i \alpha_i h_i \cdot \frac{g_j}{h_j} = \sum_i \alpha_i \frac{h_i g_j}{h_j} = \sum_i \alpha_i \frac{h_j g_i}{h_j} = \sum_i \alpha_i g_i \in k[X]. \quad \square$$

Corollary 11.3. *$D_h \subset X$ for an aff.var. $X \subset \mathbb{A}^n$, then*

$$\mathcal{O}_X(D_h) = \{\tfrac{g}{h^m} : D_h \to k, \text{ where } m \geq 0,\ g \in k[X]\} \cong k[X][\tfrac{1}{h}] \cong k[X]_h.$$

Proof. Follows from Lemma 11.1 and Theorem 11.2. One can also prove it directly, by mimicking the previous proof: $f = \frac{g_p}{h_p}$ on $D_h \cap U_p$, then $\mathbb{V}(\langle h_p \rangle) \subset \mathbb{V}(h)$, so by Nullstellensatz $h^m \in \langle h_p \rangle$, and arguing as above one deduces $h^m = \sum \alpha_i h_i$, then $h^m f = \sum \alpha_i g_i$ and finally $f = \frac{\sum \alpha_i g_i}{h^m} \in k[D_h]$. $\square$

[1] this is essentially caused by the fact that $k[X]$ is not a UFD.

[2] In the previous version of these notes, we had assumed that X was irreducible, and instead of using the ideal J and the cover $D_i \cap D_j$, we argued that $g_i h_j = h_i g_j$ on $D_i \cap D_j$ forces $X = \overline{D_i \cap D_j} \subset \mathbb{V}(g_i h_j - h_i g_j)$ since $D_i \cap D_j$ is an open dense set for irreducible X, and thus $g_i h_j = h_i g_j$ holds on all of X.

[3] We cannot use Remark 4 above, otherwise we have a circular argument. Also, we need the trick, because otherwise later in the proof $g_j h_i = g_i h_j$ will only hold on $D_i \cap D_j$, so $f|_{D_j} = \frac{g_j}{h_j} = \sum_i \alpha_i h_i \frac{g_j}{h_j} = \sum_i \alpha_i g_i$ will only hold on $\cap_i D_i$.

[4] $\emptyset = \mathbb{V}(\langle h_i \rangle) = \cap_i \mathbb{V}(h_i)$ so $X = X \setminus \cap_i \mathbb{V}(h_i) = \cup_i X \setminus \mathbb{V}(h_i) = \cup_i D_i$. Equivalently, if $x \in X \setminus \cup D_i$ then $h_i(x) = 0$ for all i, contradicting the equation $\sum \alpha_i h_i = 1$.

Example.[1] Let $X = \mathbb{A}^2 \setminus \{0\}$. Then $\mathcal{O}_X(X) = k[x, y]$ (which implies that X is not affine[2]). Indeed, $\mathbb{A}^2 \setminus \{0\} = D_x \cup D_y$, so $f \in k[X]$ defines regular functions $f_1 = f|_{D_x} \in k[D_x]$, $f_2 = f|_{D_y} \in k[D_y]$ which agree on the overlap: $f_1|_{D_x \cap D_y} = f|_{D_x \cap D_y} = f_2|_{D_x \cap D_y} \in k[D_x \cap D_y]$ (conversely such compatible regular f_1, f_2 determine a unique glued $f \in k[D_x \cup D_y]$). Compare $k[D_x], k[D_y]$ inside $\operatorname{Frac} k[\mathbb{A}^2] = k(x, y)$, so $k[X] = k[D_x] \cap k[D_y] \subset k(x, y)$, and[3] $k[D_x] \cap k[D_y] = k[x, y]_x \cap k[x, y]_y = k[x, y]$.

Exercise. $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$, i.e. the constant functions.

11.4. REGULAR MAPS ARE MORPHISMS OF Q.P.VARIETIES

Definition. *X, Y q.p.vars., $F : X \to Y$ is a **regular map** if $\forall p \in X$, $\exists$ open affines $p \in U \subset X$, $F(p) \in V \subset Y$ (in particular $U \cong Z_U \subset \mathbb{A}^n$ and $V \cong Z_V \subset \mathbb{A}^m$ are affine) such that*

$$F(U) \subset V \qquad \text{and} \qquad Z_U \cong U \xrightarrow{F|_U} V \cong Z_V \subset \mathbb{A}^m \text{ is defined by } m \text{ regular functions}^4.$$

Lemma. *F is a regular map $\Leftrightarrow$ F is a morph of q.p.vars.*

Proof. Exercise.[5] $\square$

11.5. THE STALK OF GERMS OF REGULAR FUNCTIONS

Definition. *The ring of **germs of regular functions at** p (or the **stalk of** $\mathcal{O}_X$ **at** p) is*

$$\mathcal{O}_{X,p} = \{\text{pairs } (f, U) : \text{any open } p \in U \subset X,\ \text{any function } f : U \to k \text{ regular at } p\}/\sim$$

where $(f, U) \sim (f', U') \Leftrightarrow f|_W = f'|_W$ some open $p \in W \subset U \cap U'$.

For $F : X \to Y$ a morph of q.p.vars. we get a ring hom on stalks,

$$F_p^* : \mathcal{O}_{Y,F(p)} \to \mathcal{O}_{X,p}, \ \ F_p^*(U, g) = (F^{-1}(U), F^* g)$$

where $F^* : \mathcal{O}_Y(U) \to \mathcal{O}_X(F^{-1}(U))$, $F^* g = g \circ F$.

Lemma. *"Knowing F_p^* for all $p \in X$ determines F".*
More precisely: if $F, G : X \to Y$ satisfies $F_p^* = G_p^*$ $\forall p \in X$ then $F = G$.

Proof. Exercise (compare Homework 3, ex.4). $\square$

Remark. All the above are steps towards the proof that $\mathcal{O}_X$ is a sheaf on X, called **structure sheaf**, and $(X, \mathcal{O}_X)$ is a locally ringed space, indeed a scheme (since it is locally affine), see Sec.15.

12. THE FUNCTION FIELD AND RATIONAL MAPS

12.1. FUNCTION FIELD

For an irred.aff.var. X, $k[X]$ is an integral domain, so we can[6] define the **function field**

$$k(X) = \operatorname{Frac} k[X] = \{f = \tfrac{g}{h} : g, h \in k[X]\} \,/\, (\tfrac{g}{h} = \tfrac{\widetilde{g}}{\widetilde{h}} \Leftrightarrow g\widetilde{h} = \widetilde{g}h)$$

[1] Notice, this says: if you are regular on $\mathbb{A}^2 \setminus \{0\}$ then you must be regular also at 0. The analogous statement holds for holomorphic functions of 2 (or more) variables (*Hartogs' extension theorem*), unlike the 1-dimensional case $\mathbb{A}^1 \setminus \{0\}$ where poles and essential singularities can arise.

[2] If X were affine, it would be isomorphic to $\mathbb{A}^2$, as it has the same coordinate ring. At the coordinate ring level, we obtain some isomorphism $\varphi : k[\mathbb{A}^2] = \mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2) \to \mathcal{O}_X(X)$. The preimage of the prime ideal $I = \langle x, y \rangle \subset \mathcal{O}_X(X)$ yields a prime ideal $J = \varphi^{-1}(I) \subset k[\mathbb{A}^2]$. But $\mathbb{V}(I) = \emptyset \subset X$, so $\mathbb{V}(J) = \varphi^*(\mathbb{V}(I)) = \emptyset \subset \mathbb{A}^2$, so $J = k[x, y]$ by the affine Nullstellensatz. But φ is an isomorphism, so $I = \varphi(J) = k[x, y]$, contradiction.

[3] $f/x^k = g/y^\ell \Leftrightarrow y^\ell f = x^k g \in k[x, y] \Leftrightarrow x^k | f, y^\ell | g$, so $f/x^k \in k[x, y]$.

[4] In other words, $Z_U \to Z_V$ is defined by polynomials using the $\mathbb{A}^n, \mathbb{A}^m$ coordinates.

[5] Hint. for an affine open $U \subset X$, there is an aff.var. Z such that $U \cong Z \subset \mathbb{A}^n$. Check that $\mathcal{O}_X(U) \cong \mathcal{O}_Z(Z) \cong k[Z]$, using Theorem 11.2 for the last iso. Therefore a map defined by regular functions is locally a polynomial map.

[6] **Remark.** For X reducible ($k[X]$ not an integral domain) the analogue of $\operatorname{Frac} k[X]$ is the total ring of fractions: localize $k[X]$ at $S = \{\text{all } f \in k[X] \text{ which are not zero divisors}\}$. For $k[X]$ (or any Noetherian reduced ring), $S^{-1}k[X] \cong \prod \operatorname{Frac}(k[X]/\wp_i)$ where $\wp_i$ are the minimal prime ideals (geometrically, the irred components X_i of X). This is not a field: it is a product of fields $k(X_i)$. An element in $S^{-1}k[X]$ is one rational function on each X_i compatibly on $X_i \cap X_j$.

Example. $\frac{g}{h} \in k(X) \Rightarrow \frac{g}{h} \in \mathcal{O}_X(D_h)$ is a regular function on the open $D_h = X \setminus \mathbb{V}(h) \subset X$.
Example. Let $X = \mathbb{V}(xw - yz) \subset \mathbb{A}^4$. Then $f = \frac{x}{y} = \frac{z}{w} \in k(X)$. Notice $f \in \mathcal{O}_X(D_y \cup D_w)$.

Lemma 12.1. *$U, U' \neq \emptyset$ affine opens in an irred aff var $X \Rightarrow \forall$ basic open $\emptyset \neq D_h \subset U \cap U'$,*

$$k(U) \cong k(D_h) \cong k(U').$$

Proof. $U \cong Z = \mathbb{V}(I) \subset \mathbb{A}^n$, so[1] $k[D_h] \cong k[Z]_h$, so $k(D_h) \cong \mathrm{Frac}(k[Z]_h) \cong \mathrm{Frac}(k[Z]) \cong k(Z) = k(U)$. □

Remark. There is an obvious restriction map $\varphi : k(U) \to k(D_h)$, $\frac{f}{g} \mapsto \frac{\pi(f)}{\pi(g)}$ using the canonical map $\pi : k[U] = k[Z] \hookrightarrow k[Z]_h = k[D_h]$. The above proves φ is bijective. These restrictions are compatible: the composite $k(U) \to k(D_h) \to k(D_{hh'})$ equals $k(U) \to k(D_{h'}) \to k(D_{hh'})$ (note: $D_{hh'} = D_h \cap D_{h'}$).

Exercise. For irreducible affine X, we can compare various rings inside the function field:[2]

$$\boxed{k[U] = \mathcal{O}_X(U) = \bigcap_{D_h \subset U} \mathcal{O}_X(D_h) = \bigcap_{p \in U} \mathcal{O}_{X,p} \quad \subset \quad \mathcal{O}_{X,p} = k[X]_{\mathfrak{m}_p} \quad \subset \quad \mathrm{Frac}(k[X]) = k(X)}$$

Definition 12.2. *For X an irred q.p.v. and $\emptyset \neq U \subset X$ an affine open, define $k(X) = k(U)$.*
Exercise. Show that this field is independent (up to iso) on the choice of U. (*Hint.* above Lemma.)

12.2. RATIONAL MAPS AND RATIONAL FUNCTIONS

Motivation. Let X, Y be irred aff vars. Recall k-alg homs $k[X] \to k[Y]$ are in 1:1 correspondence with polynomial maps $X \leftarrow Y$. Do k-alg homs $k(X) \to k(Y)$ correspond to maps geometrically?
Example. $k(t) \to k(t)$, $t \mapsto \frac{1}{t}$ corresponds to $\mathbb{A}^1 \leftarrow \mathbb{A}^1$ given by $a \mapsto \frac{1}{a}$, defined on the open $\mathbb{A}^1 \setminus \{0\}$.

Definition 12.3. *For X an irred q.p.v., a **rational map** $f : X \dashrightarrow Y$ is a regular map defined on a non-empty open subset of X, and we identify rational maps which agree on a non-empty open subset.*

Remark. So a rational map is an equivalence class $[(U, F)]$ where $\emptyset \neq U \subset X$ is open, $F : U \to Y$ is a morph of q.p.v.'s. We identify $(U, F) \sim (U', F')$ if $F|_{U \cap U'} = F'|_{U \cap U'}$. By definition of regular map, we can always assume that $F : U \to V \subset Y$ is a polynomial map between affine opens $U \subset X, V \subset Y$.
Remark. Since X is irred, $U \subset X$ is dense, so f is "defined almost everywhere". X irreducible ensures that intersections of finitely many non-empty open subsets are non-empty, open and dense.

EXAMPLES.
1). $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$, $[x_0 : \cdots : x_n] \to [x_0 : \cdots : x_{n-1}]$ is defined on $U = \mathbb{P}^n \setminus \{[0 : \cdots : 0 : 1]\}$.
2). $f : X \dashrightarrow \mathbb{A}^n$ determines regular $f_1, \ldots, f_n : U \to \mathbb{A}^1$ in $\mathcal{O}_X(U)$ on some open $\emptyset \neq U \subset X$.
3). $f_i \in \mathcal{O}_X(U_i)$ for opens $\emptyset \neq U_i \subset X$ yield $f = (f_1, \ldots, f_n) : X \dashrightarrow \mathbb{A}^n$, defined on $U = \cap U_i$.
4). An example of (2)/(3): $\frac{g}{h} \in k(X)$ determines $X \dashrightarrow \mathbb{A}^1$, $a \mapsto \frac{g(a)}{h(a)}$, defined on $U = D_h \subset X$.

Definition 12.4. *A **rational function**[3] is a rational map $f : X \dashrightarrow \mathbb{A}^1$.*

[1]To clarify: $h : X \to k$ is a polynomial map, defining $D_h = (h \neq 0) \subset X$. Since $D_h \subset U$, we also have $D_h = (h|_U \neq 0) \subset U$ for the restricted function $h|_U : U \to k$. Also h defines a polynomial function h' on Z via $Z \cong U \subset X \to k$ (above we abusively called h' again h) defining $D_{h'} = (h' \neq 0) \subset Z$. Now $D_h, D_{h'}$ are isomorphic, so their coordinate rings are also iso. Explicitly: $k[D_h] \cong k[X]_h \cong \mathcal{O}_X(D_h) \cong \mathcal{O}_U(D_{h|_U}) \cong \mathcal{O}_Z(D_{h'}) \cong k[Z]_{h'}$.

[2]Sec.10 defines localisation, and Lemma 10.5 shows $\mathcal{O}_{X,p} \cong k[X]_{\mathfrak{m}_p} \subset k(X)$ consists of fractions $\frac{f}{g}$ with $g \in k[X] \setminus \mathfrak{m}_p$.

[3]**Cultural Remark. Chow's theorem**: every compact complex manifold $X \subset \mathbb{P}^n$ (holomorphically embedded) is a smooth proj var; every meromorphic function is a rational function; holo maps between such mfds are regular maps.
Example (Courses B3.2/B3.3): for X a compact connected Riemann surface, $k(X) = \{$meromorphic functions $X \dashrightarrow \mathbb{A}^1_{\mathbb{C}} = \mathbb{C}\} = \{$holomorphic maps $X \to \mathbb{P}^1\} \setminus \{$constant function $\infty\}$. The following categories are equivalent:
(1) non-singular irred **projective algebraic curves** (i.e. dim = 1) over $\mathbb{C}$ with morphs the non-constant regular maps,
(2) compact connected Riemann surfaces with morphs the non-constant holomorphic maps,
(3) the opposite of the category of **algebraic function fields** in one variable/$\mathbb{C}$ (meaning: a f.g. field extension $\mathbb{C} \hookrightarrow K$ with $\mathrm{trdeg}_{\mathbb{C}} K = 1$, so a finite field extension $\mathbb{C}(t) \hookrightarrow K$) with morphs the field homs fixing $\mathbb{C}$.
So any two meromorphic functions are algebraically dependent/k, and compact connected Riemann surfaces are iso iff their function fields are iso (this may fail for singular curves, and compactness is crucial to ensure X is algebraic). The "non-constant" condition ensures the maps are dominant.

Lemma 12.5. *For X an irred q.p.v.,*

$$k(X) \cong \{\text{rational functions } f : X \dashrightarrow \mathbb{A}^1\}, \ \tfrac{g}{h} \mapsto [(D_h, \tfrac{g}{h})].$$

Remark. Analogous to: for X aff var, $k[X] \cong \{$polynomial functions $f : X \to \mathbb{A}^1\}$, $g \mapsto (X \xrightarrow{g} \mathbb{A}^1)$.

Proof. WLOG (by restricting to a non-empty open affine in X) we may assume X is an irreducible affine variety. By definition, a rational function is determined by a representative on any non-empty open subset, so we can pick an (arbitrarily small) basic open subset $D_h \subset X$ with[1] $f = [(D_h, \frac{g}{h})]$ for some $\frac{g}{h} \in k(D_h)$. By Lemma 12.1, this corresponds to a unique element in $k(D_h) \cong k(X)$, and the element constructed is independent of the choice of D_h by the Remark under Lemma 12.1. □

Warning. Rational maps may not compose: $\mathbb{A}^1 \dashrightarrow \mathbb{A}^1$, $a \mapsto 0$ and $\mathbb{A}^1 \dashrightarrow \mathbb{A}^1$, $a \mapsto \frac{1}{a}$.
$f = [(U, F)] : X \dashrightarrow Y$, $g = [(V, G)] : Y \dashrightarrow Z$ have a well-defined composite $g \circ f : X \dashrightarrow Z$ if $F(U) \cap V \neq \emptyset$: then $g \circ f$ is defined on the open $F^{-1}(F(U) \cap V)$. To ensure composites with f are always defined, independently of g, we want $F(U)$ to hit every open in Y, i.e. $F(U) \subset Y$ is dense.

Definition. *$f = [(U, F)] : X \dashrightarrow Y$ is **dominant** if the image $F(U) \subset Y$ is dense.*

Exercise. The definition is independent of the choice of representative (U, F).
Exercise. Let $f : X \dashrightarrow Y$ be dominant, and $g : Y \dashrightarrow X$ a rational map satisfying $g \circ f = \mathrm{id}_X$ (an equality of rational maps, i.e. $g \circ f = \mathrm{id}_X$ on some non-empty open set). Show g is dominant.

Definition. *A **birational equivalence** $f : X \dashrightarrow Y$ is a dominant rational map between q.p.v.'s which has a rational inverse, i.e. there exists a rational map $g : Y \dashrightarrow X$ with $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$ (equalities of rational maps). We say $X \simeq Y$ are **birational**.*

EXAMPLES.
1). $\mathbb{A}^n \simeq \mathbb{P}^n$ are birational via the inclusion $\mathbb{A}^n \cong U_0 \subset \mathbb{P}^n$, which has rational inverse $\mathbb{P}^n \to \mathbb{A}^n$, $[x_0 : \cdots : x_n] \to (\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0})$ defined on U_0.
2). For an irred q.p.v. $X \subset \mathbb{P}^n$, $X \simeq \overline{X}$ via the inclusion $X \hookrightarrow \overline{X}$.
3). For an irred q.p.v. $X \subset \mathbb{P}^n$, $X \cap U_i \simeq X$ via the inclusion, assuming $X \cap U_i \neq \emptyset$ (i.e. $X \not\subset \mathbb{V}(x_i)$).
4). The **Cremona transformation** $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, $[x : y : z] \mapsto [yz : xz : xy]$, defined on the open where at least two coords are non-zero. Dividing by xyz, this rational map is equivalent to $[x : y : z] \mapsto [\frac{1}{x} : \frac{1}{y} : \frac{1}{z}]$, defined on the open where all coords are non-zero. This map is its own inverse, so birational.

Lemma 12.6. *For X, Y irred aff vars, $f : X \dashrightarrow Y$ determines a k-alg hom $f^* : k[Y] \to k(X)$ via*

$$(y : Y \to \mathbb{A}^1) \mapsto (f^* y = y \circ f : X \dashrightarrow \mathbb{A}^1).$$

Moreover $\boxed{f^* \text{ injective} \Leftrightarrow f \text{ dominant}}$ *in which case we get a k-alg hom $f^* : k(Y) \to k(X)$, $\frac{g}{h} \mapsto \frac{f^* g}{f^* h}$.*

Proof. $f = [(U, F)]$ defines $f^* y = [(U, F^* y)] = [(U, y \circ F)]$. The lack of injectivity of the linear map F^* depends on its kernel. For $y \neq 0$,

$$F^* y = 0 \Leftrightarrow y(F(u)) = 0 \ \forall u \in U \Leftrightarrow F(u) \in \mathbb{V}(y) \ \forall u \in U \Leftrightarrow F(U) \subset \mathbb{V}(y) \subset Y.$$

$F(U)$ not dense $\Leftrightarrow F(U) \subset$ (some proper closed subset say $\mathbb{V}(J) \neq X) \subset \mathbb{V}(y)$, any $y \neq 0 \in J$.
For the final claim: $f^* h \neq 0$ if $h \neq 0$ (since f^* inj). □

12.3. Equivalence: IRREDUCIBLE Q.P.VARS. AND F.G. FIELD EXTENSIONS

Theorem 12.7. *There is an equivalence of categories*[2]

$$\begin{array}{ccc} \{\text{irred q.p.v. } X, \text{ with rational dominant maps}\} & \to & \{\text{f.g. field extensions } k \hookrightarrow K, \text{ with } k\text{-alg homs}\}^{\mathrm{op}} \\ X & \mapsto & k(X) \\ (f = \varphi^* : X \dashrightarrow Y) & \mapsto & (\varphi = f^* : k(X) \leftarrow k(Y)) \end{array}$$

[1]If $f = \frac{g}{h^N}$ we can always replace h by h^N to assume $N = 1$.

[2]A field extension $k \hookrightarrow K$ is **finitely generated** if there are elements $\alpha_1, \ldots, \alpha_n$ such that the homomorphism $k(x_1, \ldots, x_n) = \mathrm{Frac}\, k[x_1, \ldots, x_n] \to K$, $x_i \mapsto \alpha_i$ is surjective. Notice we allow fractions, unlike finitely generated k-algebras where you only allow polynomials in the generators.

In particular, the following properties hold:

(1) $f^{**} = f$ *and* $\varphi^{**} = \varphi$;
(2) $X \overset{f}{\dashrightarrow} Y \overset{g}{\dashrightarrow} Z \Rightarrow (g \circ f)^* = f^* \circ g^* : k(X) \overset{f^*}{\longleftarrow} k(Y) \overset{g^*}{\longleftarrow} k(Z)$;
(3) $k(X) \overset{\varphi}{\longleftarrow} k(Y) \overset{\psi}{\longleftarrow} k(Z) \Rightarrow (\varphi \circ \psi)^* = \psi^* \circ \varphi^* : X \overset{\varphi^*}{\dashrightarrow} Y \overset{\psi^*}{\dashrightarrow} Z$.
(4) $X \simeq Y$ *birational q.p.v.'s* $\Leftrightarrow k(X) \cong k(Y)$ *iso* k*-algs.*

Remark. Recall the equiv {affine vars, aff morphs} $\to$ {f.g. reduced k-algs, k-alg homs}, $X \mapsto k[X]$. This was not an iso of cats: to build X from the k-alg A, one chooses generators $g_1, \ldots, g_n \in A$ to get $\varphi : k[x_1, \ldots, x_n] \twoheadrightarrow A$, $x_i \mapsto g_i$, so $\overline{\varphi} : k[x_1, \ldots, x_n]/\ker\varphi \cong A$. Then $X = \mathbb{V}(\ker\varphi) \subset \mathbb{A}^n$.

Proof.

Claim 1. f induces $\varphi = f^*$.
Pf. WLOG X, Y are affine (since f is represented by an affine map $F : U \to V$ on open affines and $k(U) = k(X)$, $k(V) = k(Y)$ by definition). By Lemma 12.6, $f : X \dashrightarrow Y$ determines $\frac{g}{h} \mapsto \frac{f^*g}{f^*h}$. ✓

Claim 2. For field extensions $k \hookrightarrow A$, $k \hookrightarrow B$, any k-alg hom $A \to B$ is a field extension (i.e. inj).
Pf. Let $J = \ker(A \to B)$. As J is an ideal in a field A, it is either 0 (done) or A (false: $1 \mapsto 1$). ✓

Claim 3. For X, Y irred affine, a k-alg hom $\varphi : k(Y) \to k(X)$ determines a birational $f : X \dashrightarrow Y$.
Pf. By Claim 2, φ is injective (in particular an injection $k[Y] \hookrightarrow k(Y) \to k(X)$). Let $y_1, \ldots, y_n$ be generators of $k[Y]$ (if $Y \subset \mathbb{A}^n$ then $k[Y]$ is generated by the coordinate functions $\overline{y_j}$). Then

$$\varphi(y_j) = \frac{g_j}{h_j} \in k(X).$$

Let $U = \cap D_{h_j}$, then $\varphi(y_j) \in \mathcal{O}_X(U)$. Since $k[Y]$ is generated by the y_j, also $\varphi(k[Y]) \subset \mathcal{O}_X(U)$. WLOG U is affine (replace U by a smaller basic open). Then[1] $\mathcal{O}_X(U) = k[U]$. The inclusion $\varphi : k[Y] \hookrightarrow k[U]$ corresponds to a morph $\varphi^* : U \to Y$ of aff vars (see above Remark), and φ^* is dominant since φ is injective (Lemma 12.6), so it represents a dominant $\varphi^* : X \dashrightarrow Y$.
Remark. Explicitly, for $u \in U \subset X$,

$$u \mapsto (\varphi(y_1)(u), \ldots, \varphi(y_n)(u)) = \left(\tfrac{g_1(u)}{h_1(u)}, \ldots, \tfrac{g_n(u)}{h_n(u)}\right) \in Y \subset \mathbb{A}^n.$$

Claim 4. For X, Y q.p.v.'s, a k-alg hom $\varphi : k(Y) \to k(X)$ determines a birational $f : X \dashrightarrow Y$.
Pf. $k(X) = k(U)$, $k(Y) = k(V)$ for affine opens U, V. By Claim 3, $k(V) = k(Y) \to k(X) = k(U)$ defines $U \dashrightarrow V$, which represents $X \dashrightarrow Y$. ✓

Claim 5. For any f.g. $k \hookrightarrow K$, there is an irred q.p.v. X with $K \cong k(X)$.
Pf. Pick generators $k_1, \ldots, k_n$ of K, let $R = k[x_1, \ldots, x_n]$, define $\varphi : R \to K$, $x_j \mapsto k_j$. Let $J = \ker\varphi$, then $R/J \hookrightarrow K$, so J is a prime ideal as K is an integral domain. Let $X = \mathbb{V}(J) \subset \mathbb{A}^n$ be the irreducible affine variety corresponding to R/J. Then $k(X) \cong K$ since $k(X) = \operatorname{Frac} R/J \hookrightarrow K$ contains the generators k_j in the image. ✓

Exercise. Prove properties (1)-(4) (these follow from analogous known claims for affine morphs).

Claim 6. The functor in the claim is an equivalence of categories.
Pf. It's fully faithful by $f^{**} = f$, $\varphi^{**} = \varphi$ (Property (1)).
It's essentially surjective by Claim 5. ✓ □

Corollary 12.8. *Any irreducible affine variety is birational to a hypersurface in some affine space.*

Proof. WLOG X is affine (restrict to an affine open). By Noether normalisation (Section 8.4), for an irred.aff.var. X,

$$k \hookrightarrow k(y_1, \ldots, y_d) \hookrightarrow k(X) \cong k(y_1, \ldots, y_d, z) = \operatorname{Frac} k[y_1, \ldots, y_d, z]/(G)$$

where $y_1, \ldots, y_d$ are algebraically independent$/k$, $d = \dim X = \operatorname{trdeg}_k k[X]$, and $z \in k[X]$ satisfies an irreducible poly $G(y_1, \ldots, y_d, z) = 0$. Since $\mathbb{V}(G) \subset \mathbb{A}^{n+1}$ has $k[\mathbb{V}(G)] = k[y_1, \ldots, y_d, z]/(G)$, the above iso $k(X) \cong k(\mathbb{V}(G))$ implies via Theorem 12.7 that $X \dashrightarrow \mathbb{V}(G)$ are birational. □

[1] Recall the Theorem: X an affine variety $\Rightarrow \mathcal{O}_X(X) = k[X]$.

Definition 12.9. *A q.p.v. X is* ***rational*** *if it is birational to $\mathbb{A}^n$ for some n.*
Remark. *By the Thm, X rational $\Leftrightarrow k(X) \cong k(x_1, \ldots, x_n)$ is a purely transcendental extension of k.*

13. TANGENT SPACES

13.1. TANGENT SPACE OF AN AFFINE VARIETY

For a more detailed discussion of the tangent space, we refer to the Appendix Section 17.

$F \in k[x_1, \ldots, x_n]$.
$p = (p_1, \ldots, p_n) \in \mathbb{A}^n$.
The linear polynomial $d_pF \in k[x_1, \ldots, x_n]$ is defined by

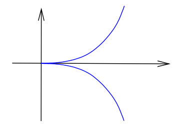
$$\boxed{d_pF = dF|_{x=p} \cdot (x - p) = \sum \tfrac{\partial F}{\partial x_j}(p) \cdot (x_j - p_j)}$$

Example. $p = 0$, $F(x) = F(0) + a_0x_0 + \cdots + a_nx_n +$ quadratic $+$ higher. The linear part of this Taylor expansion is $d_0F = \sum a_jx_j$.

Definition. *The* ***tangent space*** *to an aff.var. $X \subset \mathbb{A}^n$, with $\mathbb{I}(X) = \langle F_1, \ldots, F_n \rangle$, is*

$$\boxed{T_pX = \mathbb{V}(d_pF_1, \ldots, d_pF_N) = \cap \ker dF_i \subset \mathbb{A}^n}$$

REMARKS.
1) T_pX is an intersection of hyperplanes $\mathbb{V}(d_pF_i)$, so it is a linear subvariety.
2) T_pX is the plane which "best" approximates X near p. Notice $p \in T_pX$.
3) By translating, $-p + T_pX$, we obtain the *vector space* which "best" approximates X near p (with 0 "corresponding" to p). This is also often called the tangent space.
Silly example. $X = \mathbb{A}^n$, $\mathbb{I}(\mathbb{A}^n) = \{0\}$ so $T_p\mathbb{A}^n = \mathbb{A}^n$.
Example. The cusp $X = \mathbb{V}(y^2 - x^3) = \{(t^2, t^3) : t \in k\}$ is determined by $F = y^2 - x^3$.



At $p = (t^2, t^3)$,

$$dF = -3x^2\,dx + 2y\,dy = \begin{pmatrix} -3x^2 \\ 2y \end{pmatrix}$$
$$d_pF = -3t^4(x - t^2) + 2t^3(y - t^3).$$

For $t \neq 0$, $T_pV = \ker d_pF$ is the (1-dimensional) straight line perpendicular to $(-3t^4, 2t^3)$. But at $t = 0$, $d_pF = 0$ so $T_pX = \mathbb{V}(0) = k^2$ is 2-dimensional.

Exercise. Recall a line through p has the form $\ell(t) = p + tv$ for some $v \in k^n$. A line is called **tangent** to X at p if $F_i(\ell(t))$ has a repeated[1] root at $t = 0$. Show that

$$T_pX = \cup(\text{lines tangent to } X \text{ at } p).$$

Definition. *$p \in X$ is a* ***smooth point*** *if*[2]

$$\dim_k T_pX = \dim_p X.$$

$p \in X$ is a ***singular point***[3] *if* $\dim_k T_pX > \dim_p X$. *Abbreviate* $\operatorname{Sing}(X) = \{$*all singular points*$\} \subset X$.

[1] We know $t = 0$ is a root, since the F_i vanish at $p \in X$.
[2] Recall: $\dim_p X =$(the dimension of the irreducible component of X containing p), Section 8.1.
[3] (Non-examinable) Fact: $\dim T_pX \geq \dim_p X$ always holds. Intuitively: if the d_pF_i are linearly independent then the F_i are also "independent near p", so each equation $F_i = 0$ cuts down by one the dimension of X at p. Over complex numbers, this is a consequence of the implicit function theorem. More generally, one way to prove this is via the Noether Normalization Lemma (Geometric Version 2) from Sec.8.4 and applying the following fact to the projection from the tangent "bundle" $TX = \{(p, v) \in X \times \mathbb{A}^n : v \in T_pX\} \to X$, $(p, v) \mapsto p$. **Fact.** Given any regular surjective map $f : X \to Y$ of irreducible q.p.vars, then $\dim F \geq \dim X - \dim Y$ for any component F of $f^{-1}(y)$, and any $y \in Y$. Moreover, $\dim f^{-1}(y) = \dim X - \dim Y$ holds on a non-empty open (hence dense) subset of $y \in Y$.

Theorem. *Let X be an irreducible aff.var. of dimension d with $\mathbb{I}(X) = \langle F_1, \ldots, F_N \rangle$.*
$\Rightarrow$ $\operatorname{Sing}(X) \subset X$ *is a closed subvariety given by the vanishing in X of all $(n-d) \times (n-d)$ minors of the Jacobian matrix*

$$\operatorname{Jac} = \left(\frac{\partial F_i}{\partial x_j} \right).$$

Proof. T_pX is the zero set of

$$\varphi_p : \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial F_1}{\partial x_1}\big|_p & \cdots & \frac{\partial F_1}{\partial x_n}\big|_p \\ \vdots & & \\ \frac{\partial F_N}{\partial x_1}\big|_p & \cdots & \frac{\partial F_N}{\partial x_n}\big|_p \end{pmatrix} \cdot \begin{pmatrix} x_1 - p_1 \\ \vdots \\ x_n - p_n \end{pmatrix}$$

Hence $p \in \operatorname{Sing} X \Leftrightarrow \dim \varphi_p^{-1}(0) > d \Leftrightarrow \dim \ker \operatorname{Jac}_p > d \Leftrightarrow$ all $(n-d) \times (n-d)$ minors vanish.[1] □

Example. For the cusp: $F = y^2 - x^3$, $\operatorname{Jac} = (-3x^2 \quad 2y)$, $n = 2$, $d = 1$. So 1×1 minors all vanish precisely when $(x, y) = (0, 0)$.

13.2. INTRINSIC DEFINITION OF THE TANGENT SPACE OF A VARIETY

Theorem. *X aff.var., $p \in X$, and recall $\mathfrak{m}_p = \{\frac{f}{g} \in \mathcal{O}_{X,p} : f(p) = 0\} \subset \mathcal{O}_{X,p}$. Then, canonically,*

$$\boxed{T_pX \cong \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*}$$

*(the vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$, before dualization, is called the **cotangent space**).*

Proof. WLOG (after a linear iso of coords) assume $p = 0 \in \mathbb{A}^n$.
Notation. To avoid confusion, we first list below the maximal ideals that will arise in the proof:

$$\begin{array}{rcccl} k[\mathbb{A}^n] & \supset & \mathfrak{m} & = & \{f : \mathbb{A}^n \to k : f(0) = 0\} = \langle x_1, \ldots, x_n \rangle \\ k[X] & \supset & \overline{\mathfrak{m}} & = & \{f : X \to k : f(0) = 0\} = \mathfrak{m} \cdot k[X] = \mathfrak{m} + \mathbb{I}(X) \\ \mathcal{O}_{X,0} & \supset & \mathfrak{m}_0 & = & \{\frac{f}{g} : f, g \in k[X], g(0) \neq 0, f(0) = 0\} = \mathfrak{m} \cdot \mathcal{O}_{X,0}. \end{array}$$

Step 1. We prove it for $X = \mathbb{A}^n$.
$d_0F = \sum \frac{\partial F}{\partial x_i}\Big|_0 \cdot x_i$ is a linear functional $\mathbb{A}^n \equiv T_0\mathbb{A}^n \to k$, so $d_0F \in (T_0\mathbb{A}^n)^*$. Thus

$$d_0 : k[x_1, \ldots, x_n] \to (T_0\mathbb{A}^n)^*,\ F \mapsto d_0F$$

and d_0 is linear.[2] Restricting to the maximal ideal $\mathfrak{m} = (x_1, \ldots, x_n)$ of those F with $F(0) = 0$,

$$d_0|_{\mathfrak{m}} : \mathfrak{m} \to (T_0\mathbb{A}^n)^*.$$

$d_0|_{\mathfrak{m}}$ is linear and surjective.[3]
Subclaim. $\ker d_0|_{\mathfrak{m}} = \mathfrak{m}^2$, hence $d_0|_{\mathfrak{m}} : \mathfrak{m}/\mathfrak{m}^2 \to (T_0\mathbb{A}^n)^*$ is an iso.
Proof. $d_0F = 0 \Leftrightarrow \frac{\partial F}{\partial x_i}(0) = 0\, \forall i \Leftrightarrow$ (F only has monomials of degrees ≥ 2) $\Leftrightarrow F \in \mathfrak{m}^2$. ✓
Step 2. We prove it for general X.
The inclusion $j : T_0X \hookrightarrow T_0\mathbb{A}^n$ is injective, so the dual map[4] is surjective,

$$j^* : \mathfrak{m}/\mathfrak{m}^2 \cong (T_0\mathbb{A}^n)^* \to (T_0X)^*$$

$\Rightarrow j^* \circ d_0 : \mathfrak{m} \to (T_0X)^*$ surjective.
Subclaim.[5] $\ker j^* \circ d_0 = \mathfrak{m}^2 + \mathbb{I}(X) = \overline{\mathfrak{m}}^2 \subset k[X]$, hence $\mathfrak{m}/(\mathfrak{m}^2 + \mathbb{I}(X)) \cong (T_0X)^*$.
Proof. $F \in \ker(j^* \circ d_0) \Leftrightarrow j^*d_0F = d_0F|_{T_0X} = 0 \Leftrightarrow d_0F \in \mathbb{I}(T_0X)$
$\Leftrightarrow d_0F \in \langle d_0F_1, \ldots, d_0F_N \rangle$ where $\mathbb{I}(X) = \langle F_1, \ldots, F_N \rangle$.

[1] Otherwise we would find $n-d$ linearly independent columns (the columns involved in that minor), and hence the rank would be at least $\dim = n-d$, so the kernel would be at most $\dim = d$.
[2] $d_0(\lambda F + \mu G) = \lambda\, d_0F + \mu\, d_0G$.
[3] $d_0x_i = x_i$ are a basis for $(T_0\mathbb{A}^n)^*$.
[4] explicitly, j^* is just the restriction map: $j^*F = F \circ j = F|_{T_0X} : T_0X \xrightarrow{j} T_0\mathbb{A}^n \xrightarrow{F} k$.
[5] $\overline{\mathfrak{m}}$ denotes the image of $\mathfrak{m}$ in the quotient $k[X] = R/\mathbb{I}(X)$.

$\Leftrightarrow d_0F = \sum a_i\, d_0F_i$ where $a_i \in k[x_1, \ldots, x_n]$.
$\Leftrightarrow d_0(F - \sum a_iF_i) = -\sum(d_0a_i) \cdot F_i(0) = 0$ (since $0 = p \in \mathbb{V}(F_1, \ldots, F_N)$).
$\Leftrightarrow F - \sum a_iF_i \in \ker d_0|_{\mathfrak{m}} = \mathfrak{m}^2$.
$\Leftrightarrow F \in \mathbb{I}(X) + \mathfrak{m}^2$. ✓
Finally[1]

$$(T_0X)^* \cong \mathfrak{m}/(\mathfrak{m}^2 + \mathbb{I}(X)) \cong \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$$

where the last iso is one of the "isomorphism theorems".[2] Now localise:
Claim. $\varphi : \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2 \cong \mathfrak{m}_0/\mathfrak{m}_0^2, f \mapsto \frac{f}{1}$ (the theorem then follows).
Proof. **Subclaim 1.** φ is surjective.
Proof. For $\frac{f}{g} \in \mathfrak{m}_0$, let $c = g(0) \neq 0$.
$\Rightarrow \varphi(\frac{f}{c}) - \frac{f}{g} = \frac{f}{c} - \frac{f}{g} = \frac{f}{1} \cdot (\frac{1}{c} - \frac{1}{g}) \in \mathfrak{m}_0^2$ (since $\frac{f}{1} \in \mathfrak{m}_0$ and $(\frac{1}{c} - \frac{1}{g}) \in \mathfrak{m}_0$).
$\Rightarrow \varphi(\frac{f}{c}) = \frac{f}{g}$ modulo $\mathfrak{m}_0^2$. ✓
Subclaim 2. φ is injective.
Proof. Need to show $\ker \varphi = 0$. Suppose $\frac{f}{1} \in \mathfrak{m}_0^2$. Thus[3] $\frac{f}{1} = \sum \frac{g_i}{h_i} \cdot \frac{g_i'}{h_i'}$ where $g_i, g_i' \in \overline{\mathfrak{m}}$ and $h_i, h_i' \in k[X] \backslash \overline{\mathfrak{m}}$. Take common denominators (and redefine g_i) to get $\frac{f}{1} = \frac{\sum g_ig_i'}{h}$ for some $h \in k[X] \backslash \overline{\mathfrak{m}}$. Then $s \cdot (fh - \sum g_ig_i') = 0 \in k[X]$ for some $s \in k[X] \setminus \overline{\mathfrak{m}}$. Thus $sfh \in \overline{\mathfrak{m}}^2$. Since $f \in \overline{\mathfrak{m}}$, also[4] $(sh - s(0)h(0))f \in \overline{\mathfrak{m}}^2$. Thus $s(0)h(0)f \in \overline{\mathfrak{m}}^2$, forcing[5] $f \in \overline{\mathfrak{m}}^2$. So $\frac{f}{1} = 0 \in \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ as required. □

Remark. That also proved $T_pX \cong (\mathcal{I}_p/\mathcal{I}_p^2)^*$ where $\mathcal{I}_p = \langle x_1 - p_1, \ldots, x_n - p_n \rangle \subset k[X]$.

Corollary. *T_pX only*[6] *depends on an open neighbourhood of $p \in X$.*

Proof. By the Theorem, it only depends on the local ring $\mathcal{O}_{X,p}$ (and its unique maximal ideal $\mathfrak{m}_p$). □

Definition. *For X a q.p.var. we define the tangent space at $p \in X$ by $T_pX = \left(\mathfrak{m}_p/\mathfrak{m}_p^2\right)^*$.*

Remark. In practice, you pick an affine neighbourhood of $p \in X$, then calculate the affine tangent space using the Jacobian.

13.3. DERIVATIVE MAP

Lemma. *For $F : X \to Y$ a morph of q.p.vars., on stalks $F^* : \mathcal{O}_{Y,F(p)} \to \mathcal{O}_{X,p}$ is a local*[7] *ring hom $\mathfrak{m}_{F(p)} \to \mathfrak{m}_p$, $g \mapsto F^*g = g \circ F$.*

Proof. $g(F(p)) = 0$ implies $(F^*g)(p) = 0$. □

$F : X \to Y$ morph of q.p.vars. We want to construct the **derivative map**

$$D_pF : T_pX \to T_{F(p)}Y.$$

By the Lemma, $F^*(\mathfrak{m}_{F(p)}) \subset \mathfrak{m}_p$, so $F^*(\mathfrak{m}_{F(p)}^2) \subset \mathfrak{m}_p^2$, and thus[8]

$$F^* : \mathfrak{m}_{F(p)}/\mathfrak{m}_{F(p)}^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2.$$

Its dual defines the derivative map:

$$D_pF = (F^*)^* : (\mathfrak{m}_p/\mathfrak{m}_p^2)^* \to (\mathfrak{m}_{F(p)}/\mathfrak{m}_{F(p)}^2)^*.$$

[1] using that $\mathbb{I}(X) \subset \mathfrak{m}$, since $f|_X = 0 \Rightarrow f|_p = 0$.
[2] we quotient numerator and denominator by a common submodule, $\mathbb{I}(X)$. Explicitly: $\mathfrak{m} \to \overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ is surjective and the kernel is easily seen to be $\mathfrak{m}^2 + \mathbb{I}(X)$.
[3] By definition $\mathfrak{m}_0^2$ is generated by products of any two elements from $\mathfrak{m}_0$, so it involves a sum and not just one $\frac{gg'}{hh'}$.
[4] since $sh - s(0)h(0)$ and f both vanish at 0.
[5] since s, h do not vanish at 0.
[6] So it is independent of the choice of F_i with $\mathbb{I}(X) = \langle F_1, \ldots, F_N \rangle$, and it is independent of the choice of embedding $X \subset \mathbb{A}^n$, i.e. it is an isomorphism invariant.
[7] meaning: max ideal $\to$ max ideal.
[8] This F^* is called the pullback map on cotangent spaces.

Exercise. Show that locally, on affine opens around $p, F(p)$, you can identify D_pF with the Jacobian matrix of F. More precisely: locally $F : \mathbb{A}^n \to \mathbb{A}^m$, $p = 0$ and $F(p) = 0$, and $\mathrm{Jac}\, F = (\frac{\partial F_i}{\partial x_j})$ acts by left multiplication $\mathbb{A}^n \equiv T_0\mathbb{A}^n \to \mathbb{A}^m \equiv T_0\mathbb{A}^m$.

Example. $F : \mathbb{A}^1 \to \mathbb{V}(y - x^2) \subset \mathbb{A}^2$, $F(t) = (t, t^2)$, $F(0) = (0, 0)$.
For $\mathbb{A}^1$: $\mathfrak{m}_0 = t \cdot k[t]_{(t)} \subset k[t]_{(t)}$ (we invert anything which is not a multiple of t).
For $\mathbb{A}^2$: $\mathfrak{m}_{F(0)} = (x, y) \cdot (k[x, y]/(y - x^2))_{(x,y)} \subset (k[x, y]/(y - x^2))_{(x,y)}$.
$F^* : \overline{ax + by + \text{higher}} \in \mathfrak{m}_{F(0)}/\mathfrak{m}^2_{F(0)} \mapsto \overline{at + bt^2} = a\bar{t} \in \mathfrak{m}_0/\mathfrak{m}_0^2$.
$\Rightarrow D_0F = (F^*)^* : t^* \mapsto x^*$, where $t^*(a\bar{t}) = a$ and $x^*(\overline{ax + by}) = a$.
$\Rightarrow D_0F = \binom{1}{0}$ in the basis x^*, y^* on the target (and basis t^* on the source).
This agrees with the Jacobian matrix of partial derivatives:
$D_0F = \binom{\partial_t F_1|_0}{\partial_t F_2|_0} = \binom{1}{2t}\big|_{t=0} = \binom{1}{0}$.

14. BLOW-UPS

14.1. BLOW-UPS

The blow-up of $\mathbb{A}^n$ at the origin is the set of lines in $\mathbb{A}^n$ with a given choice of point:

$$B_0\mathbb{A}^n = \{(x, \ell) : \mathbb{A}^n \times \mathbb{P}^{n-1} : x \in \ell\} = \mathbb{V}(x_iy_j - x_jy_i) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$$

using coords $(x_1, \dots, x_n)$ on $\mathbb{A}^n$, $[y_1 : \cdots : y_n]$ on $\mathbb{P}^{n-1}$. That $x \in \ell$ means $(x_1, \dots, x_n)$ and $(y_1, \cdots, y_n)$ are proportional, equivalently the matrix with those rows has rank 1 so 2×2 minors vanish.
Exercise. Via the linear iso $x \mapsto x - p$, describe the blow-up $B_p\mathbb{A}^n$ at p.

The morphism

$$\pi : B_0\mathbb{A}^n \to \mathbb{A}^n,\ \pi(x, [y]) = x,$$

is birational with inverse[1] $\mathbb{A}^n \to B_0\mathbb{A}^n$, $x \mapsto (x, [x])$ defined on $x \neq 0$. The fibre $\pi^{-1}(x)$ is a point with the exception of the **exceptional divisor**[2]

$$E_0 = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}.$$

Thus $\pi : B_0\mathbb{A}^n \setminus E_0 \to \mathbb{A}^n \setminus 0$ is an iso, and π collapses E_0 to the point 0.
In fact $E_0 \cong \mathbb{P}^{n-1} \cong \mathbb{P}(T_0\mathbb{A}^n)$ is the projectivisation of the tangent space:[3] the closure of the preimage $\{(vt, [vt]) : t \neq 0\}$ of the punctured line $t \mapsto tv$, $t \neq 0$, contains the new point $(0, [v])$ (using that $[vt] = [v] \in \mathbb{P}^{n-1}$ by rescaling).
Definition. For $X \subset \mathbb{A}^n$ an affine variety with $0 \in X$, the **proper transform** is

$$B_0X = \textbf{closure}(\,\pi^{-1}(X \setminus \{0\})\,) \subset B_0\mathbb{A}^n.$$

Again $\pi : B_0X \to X$ is birational, and

$$E = \pi^{-1}(0) \cap B_0X$$

is the exceptional divisor. B_0X only keeps track of directions $E \subset E_0$ at which X approaches 0, unlike the **total transform**

$$\pi^{-1}(X) = B_0X \cup E_0.$$

Example. $X = \mathbb{V}(xy) = (x\text{-axis}) \cup (y\text{-axis}) \subset \mathbb{A}^2$. Then

$$\pi^{-1}(X \setminus 0) = \{((x, y), [a : b]) \in \mathbb{A}^2 \times \mathbb{P}^1 : xb - ya = 0, xy = 0, (x, y) \neq (0, 0)\}.$$

Solving: $((x, 0), [1 : 0])$ for $x \neq 0$, $((0, y), [0 : 1])$ for $y \neq 0$.
Then B_0X is the closure: $(\mathbb{A}^1 \times 0, [1 : 0]) \sqcup (0 \times \mathbb{A}^1, [0 : 1])$, a disjoint union of lines! The exceptional divisor E consists of two points: $((0, 0), [1 : 0])$, $((0, 0), [0 : 1])$, the 2 directions of the lines in X.

[1] a non-zero x determines the line uniquely: $\ell = [x_1 : \cdots : x_n]$.
[2] **Divisor** here just means codimension 1 subvariety, although more generally divisor refers to formal $\mathbb{Z}$-linear combinations of such (these are called Weil divisors).
[3] more accurately, of the normal space to $\{0\} = T_00 \subset T_0\mathbb{A}^n$: we keep track of how x converges normally into 0.

14.2. RESOLUTION OF SINGULARITIES

Blow-ups are important because they provide a way to desingularise a variety X, i.e. finding a smooth variety X' which is birational to the original variety X. Of course, X' is not unique.
Example. The cuspidal curve $X = \mathbb{V}(y^2 - x^3) \subset \mathbb{A}^2$ is singular at 0. Use coords $((x, y), [a : b])$ on $B_0\mathbb{A}^2$, $xb - ya = 0$. Notice $D_a = X \cap (a \neq 0)$ can be viewed as a subset of $\mathbb{A}^2$ using coords (x, b), since WLOG $a = 1$, then $y = xb$. Substitute into our equation: $0 = y^2 - x^3 = x^2b^2 - x^3$. The proper transform is obtained by dropping the x^2 factor: $b^2 - x = 0$ (check this). Thus B_0X is a smooth curve, a parabola, birational to X.

Hironaka's Theorem (Hard!). *Assume* $\mathrm{char}\, k = 0$. *For any p.v./q.p.v.* X, *there is a smooth p.v./q.p.v.* X' *and a morph* $\pi : X' \to X$ *which is birational, such that*

$$\pi : X' \setminus \pi^{-1}(\mathrm{Sing}(X)) \to X_{\text{smooth}} = X \setminus \mathrm{Sing}(X)$$

is an iso. If X *is affine, then* $X' = B_I(X)$ *can be constructed as the blow-up of* X *along a (possibly non-radical) ideal* $I \subset k[X]$ *(see Section 14.3), with*

$$\mathbb{V}(I) = \mathrm{Sing}(X).$$

14.3. BLOW-UPS ALONG SUBVARIETIES AND ALONG IDEALS

Definition. For affine X, and $I = \langle f_1, \dots, f_N \rangle \subset k[X]$, define $B_I(X)$ to be the **graph** of $f : X \dashrightarrow \mathbb{P}^{N-1}$, $f(x) = [f_1(x) : \cdots : f_N(x)]$, meaning:

$$B_IX = \textbf{closure}(\{(x, f(x)) : x \in X \setminus \mathbb{V}(I)\}) \subset X \times \mathbb{P}^{N-1}.$$

The morph

$$\pi : B_I(X) \to X,\ \pi(x, [v]) = x$$

is birational with inverse $x \mapsto (x, f(x))$ (defined on $X \setminus \mathbb{V}(I)$). The **exceptional divisor** is

$$E = \pi^{-1}(\mathbb{V}(I)).$$

Definition. The **blow-up along a subvariety** Y is

$$B_YX = B_{\mathbb{I}(Y)}X.$$

Exercise. For $Y = \{0\}$ (so $I = \mathbb{I}(0) = (x_1, \dots, x_n)$), show B_YX is the proper transform B_0X.
Remark. B_IX is independent of the choice of generators f_j, but it depends[1] on I and not just $\mathbb{V}(I)$.
Definition. For q.p.v. $X \subset \mathbb{P}^n$, and $I \subset S(\overline{X})$ homog., pick homog. gens $f_1, \dots, f_N$ of the same degree[2]. Thus $f : \overline{X} \dashrightarrow \mathbb{P}^{N-1}$ determines $B_I\overline{X} \subset \overline{X} \times \mathbb{P}^{N-1}$ as before, and define

$$B_IX = B_I\overline{X} \cap (X \times \mathbb{P}^{N-1}).$$

15. SCHEMES

Section 15 is an introduction to modern algebraic geometry. It is conceptually central to the subject. However, for the purposes of exams, almost all of section 15 is non-examinable. The only topics you need to know are: (1) the definition of Spec, Specm *in 15.1; (2) the Zariski topology on spectra in 15.2; (3) morphisms between spectra in 15.3.*

[1] e.g. $B_{(x^2,y)}\mathbb{A}^2$ is singular but $B_{(x,y)}\mathbb{A}^2$ is smooth, although $\mathbb{V}(x^2, y) = \mathbb{V}(x, y)$.
[2] Recall the trick: $\mathbb{V}(f) = \mathbb{V}(z_0f, z_1f, \dots, z_nf)$. So we can get f_j of equal degree.

15.1. Spec OF A RING and THE "VALUE" OF FUNCTIONS ON Spec

A = any ring (commutative with 1).
The **affine scheme**[1] for A is the spectrum $\operatorname{Spec} A$, where

$$\boxed{\operatorname{Spec} A = \{\text{prime ideals } \wp \subset A\} \supset \{\text{max ideals } \mathfrak{m} \subset A\} = \operatorname{Specm} A}$$

Here A plays the role of the coordinate ring

$$A = \mathcal{O}(\operatorname{Spec} A) = \text{"ring of global regular functions"}$$

where $\mathcal{O}$ is called the **structure sheaf** (more on this later).
Remark. Notice $\mathcal{O}(\operatorname{Spec} k[x]/x^2) = k[x]/x^2$ remembers that 0 is a double root of x^2, whereas the affine coordinate ring $k[\mathbb{V}(x^2)] = k[x]/x$ does not.
Question: In what sense are elements of A "functions" on $\operatorname{Spec} A$?

$$\begin{array}{rcl} f \in A \Rightarrow \text{"function" } \operatorname{Spec} A & \to & ?? \\ \wp & \mapsto & f(\wp) \in \mathbb{K}(\wp) = \operatorname{Frac}(A/\wp) \end{array}$$

where we need to explain[2] what $f(\wp)$ is, inside the fraction field of the integral domain $A/\wp$:

$$\begin{array}{ccccc} A & \to & A/\wp & \hookrightarrow & \mathbb{K}(\wp) \\ f & \mapsto & \overline{f} & \mapsto & f(\wp) = \dfrac{\overline{f}}{1} \in \mathbb{K}(\wp) \end{array}$$

Remark. It is not actually a function: the target $\mathbb{K}(\wp)$ is a field which depends on the given $\wp$!
Example. $A = \mathbb{Z}$.
$\operatorname{Spec} A = \{(0)\} \cup \{(p) : p \text{ prime}\}$.
$\mathbb{K}(0) = \operatorname{Frac}(\mathbb{Z}/0) = \mathbb{Q}$, $\mathbb{K}(p) = \operatorname{Frac}(\mathbb{Z}/p) = \mathbb{Z}/p$.
Consider $f = 4$.
$f((0)) = 4 \in \mathbb{Q}$.
$f((3)) = (4 \bmod 3) = 1 \in \mathbb{Z}/3$.
$f((2)) = 0 \in \mathbb{Z}/2$, since $4 \in (2)$.
Exercise. $\boxed{f(\wp) = 0 \Leftrightarrow f \in \wp}$

When $\wp = \mathfrak{m}$ is a maximal ideal, $A/\mathfrak{m}$ is already a field, so $\mathbb{K}(\mathfrak{m}) = A/\mathfrak{m}$, thus:

$$f(\mathfrak{m}) = (f \text{ modulo } \mathfrak{m}) \in A/\mathfrak{m}.$$

Example. $A = k[x]$ corresponds to the affine variety $\operatorname{Specm} A = \mathbb{A}^1$. Consider a polynomial $f(x) \in A$, and the ideal $\mathfrak{m} = (x-2)$. Then $f(\mathfrak{m}) = (f \bmod x-2) \in k[x]/(x-2)$ corresponds to the value $f(2) \in k$ via the identification $\mathbb{K}(\mathfrak{m}) = k[x]/(x-2) \cong k$, $x \mapsto 2$.
Remark. For an affine variety $X \subset \mathbb{A}^n$, so taking $A = k[X]$, the maximal ideals $\mathfrak{m}_a = \langle x_1 - a_1, \ldots, x_n - a_n\rangle$ correspond to the points $a \in X \subset \mathbb{A}^n$, and the "function" f at $\mathfrak{m}_a$ just means reducing f modulo $\mathfrak{m}_a$. But $k[X]/\mathfrak{m}_a \cong k$ via the evaluation map $g(x) \mapsto g(a)$, so we get an actual function on the maximal ideals:

$$f : \operatorname{Specm} A \to k,\ \mathfrak{m}_a \mapsto f(\mathfrak{m}_a) = f(a)$$

in other words, this is the polynomial function $\mathbb{V}(I) \to k$ defined by the polynomial $f \in k[x_1, \ldots, x_n]/I$, so the value $f(a)$ is obtained by plugging in the values $x_i = a_i$ in f.
Example. $A = k[X] = R/I$ for an affine variety $X \subset \mathbb{A}^n$, where $R = k[x_1, \ldots, x_n]$.
For $f \in A$, we obtain $f : X = \operatorname{Specm} A \to k$ as remarked above, and this is the polynomial function obtained via $k[X] \cong \operatorname{Hom}(X, k)$. Example: $x_i \in A$ defines the i-th coordinate function $\overline{x}_i : X \to k$.
For $\wp \subset A$ a prime ideal, we obtain a subvariety $Y = \mathbb{V}(\wp) \subset X$, and you should think of $f(\wp)$ as the restriction to Y of the polynomial function $f : X \to k$, so $f(\wp) : Y \to k$. Indeed, let $\overline{A} = k[Y] = A/\wp$ and $\overline{f} = (f \bmod \wp) \in \overline{A}$. Then the restriction $f|_Y : Y \to k$ equals the function $\overline{f} : \operatorname{Specm} \overline{A} \to k$

[1]These will be the local models for general schemes.
[2]here we write $\overline{f}$ to mean f modulo $\wp$, so the coset $f + \wp \in A/\wp$.

which corresponds to the "function" $f(\wp)$ arguing as before.[1]
Remark. The values $\overline{f} \in \mathbb{K}(\wp)$ "determine" the image of f in any field $\mathbb{F}$ under any homomorphism $\varphi : A \to \mathbb{F}$. Indeed (assuming φ is not the zero map), $\wp = \ker \varphi$ is a prime ideal since $A/\wp \hookrightarrow \mathbb{F}$ is an integral domain, so φ factorises as $A \to A/\wp \hookrightarrow \mathbb{K}(\wp) \hookrightarrow \mathbb{F}$ since $\mathbb{K}(\wp)$ is the smallest field containing $A/\wp$, so $\varphi(f)$ is determined by $\overline{f} \in \mathbb{K}(\wp)$ and the field extension $\mathbb{K}(\wp) \hookrightarrow \mathbb{F}$.

15.2. THE ZARISKI TOPOLOGY ON Spec

Define the **Zariski topology** on $\operatorname{Spec} A$ and $\operatorname{Specm} A$ by declaring as closed sets:

$$\mathbb{V}(I) = \{\wp \in \operatorname{Spec} A : \wp \supset I\} \subset \operatorname{Spec} A$$
$$\mathbb{V}(I) = \{\mathfrak{m} \in \operatorname{Specm} A : \mathfrak{m} \supset I\} \subset \operatorname{Specm} A$$

for any ideal $I \subset A$. More generally, for a subset $S \subset A$, we write $\mathbb{V}(S)$ to mean $\mathbb{V}(\langle S\rangle)$. For example,

$$\mathbb{V}(f) = \{\wp \in \operatorname{Spec} A : \wp \ni f\} = \{\wp \in \operatorname{Spec} A : f(\wp) = 0\}.$$

Again we have basic open sets

$$D_f = \{\wp : f(\wp) \neq 0\} = \{\wp : f \notin \wp\} \subset \operatorname{Spec} A$$
$$D_f = \{\mathfrak{m} : f(\mathfrak{m}) \neq 0\} = \{\mathfrak{m} : f \notin \mathfrak{m}\} \subset \operatorname{Specm} A$$

for each $f \in A$, which define a basis for the topology.
Exercise. $\operatorname{Spec} A \setminus \mathbb{V}(\wp) = \{\text{prime ideals not containing } \wp\} = \cup_{f \in \wp} D_f$.
The elements of $\operatorname{Specm} A$ are called the **closed points**[2] of $\operatorname{Spec} A$. A point of a topological space is called **generic** if it is dense.[3] So a **generic point** $\wp \in \operatorname{Spec} A$ is a point satisfying $\mathbb{V}(\wp) = \operatorname{Spec} A$.
Examples.
1. For $A = R = k[x_1, \ldots, x_n]$, then $\operatorname{Specm} A \equiv k^n$ via

$$\begin{array}{rcl} \mathfrak{m}_a = \langle x_1 - a_1, \ldots, x_n - a_n\rangle & \stackrel{1:1}{\longleftrightarrow} & a \\ \mathbb{V}(I) = \{\mathfrak{m}_a : \mathfrak{m}_a \supset I\} \subset \operatorname{Specm} A & \stackrel{1:1}{\longleftrightarrow} & \{a \in k^n : \{a\} = \mathbb{V}_{\text{classical}}(\mathfrak{m}_a) \subset \mathbb{V}_{\text{classical}}(I)\} \\ & & = \mathbb{V}_{\text{classical}}(I) \subset \mathbb{A}^n. \end{array}$$

So $\operatorname{Specm} R \cong \mathbb{A}^n$ are homeomorphic, and $\mathcal{O}(\mathbb{A}^m) = R$.
$\operatorname{Spec} R$ contains all irreducible subvarieties $Y = \mathbb{V}(\wp) \subset \mathbb{A}^n$:

$$\begin{array}{rcl} \operatorname{Spec} A & \stackrel{1:1}{\longleftrightarrow} & \operatorname{Specm} A \cup \{\text{prime ideals } \wp \subset R \text{ which are not maximal}\} \\ & \stackrel{1:1}{\longleftrightarrow} & \mathbb{A}^n \cup \{\text{irred subvars } Y \subset \mathbb{A}^n \text{ which are not points}\} \\ & \stackrel{1:1}{\longleftrightarrow} & \{\text{all irred subvars } Y \subset \mathbb{A}^n\} \end{array}$$

This is unlike the Euclidean topology (for $k = \mathbb{R}$ or $\mathbb{C}$) where the only non-empty irreducible sets are single points, so we don't notice interesting "points" apart from $\mathbb{A}^n$.
2. For $X \subset \mathbb{A}^n$ aff.var., let $I = \mathbb{I}(X)$, so $k[X] = R/I$ where $R = k[x_1, \ldots, x_n]$.

$$X \cong \operatorname{Specm}(R/I) \text{ are homeomorphic, and } \mathcal{O}(X) = k[X] = R/I$$
$$a = \mathbb{V}(\overline{\mathfrak{m}}_a) = \{\overline{f} \in k[X] : \overline{f}(a) = 0\} \text{ where } \overline{\mathfrak{m}}_a = \mathfrak{m}_a + I \subset R/I = k[X].$$

3. For $A = \mathbb{Z}$,

$$\operatorname{Spec} \mathbb{Z} = \{\text{the closed points } \{p\} \text{ for } p \text{ prime}\} \cup \{\text{the generic point } (0)\}$$

Note: (p) is maximal, $\mathbb{V}(p) = \{(p)\}$, and (0) is generic since $\mathbb{V}((0)) = \operatorname{Spec} \mathbb{Z}$ as $(0) \subset (p)$ for all p.
4. For $A = k[x]$,

$$\operatorname{Spec} k[x] = \{(x-a) : a \in k\} \cup \{(0)\} \leftrightarrow \mathbb{A}^n \cup (\text{generic point}).$$

[1]identifying $\mathbb{K}(\overline{\mathfrak{m}}) \cong k$ via evaluation, for any max ideal $\overline{\mathfrak{m}} \subset \overline{A}$, i.e. a max ideal $\mathfrak{m} \subset A$ which contains $\wp$.
[2]"closed" because $\mathbb{V}(\mathfrak{m}) = \{\mathfrak{m}\}$.
[3]i.e. its closure is everything.

Note: 0 is generic as $\mathbb{V}((0)) = \operatorname{Spec} k[x]$ as $(0) \subset \langle x-a\rangle$.
5. For $A = k[x]/x^2$,

$$\begin{aligned}&\operatorname{Specm} A = \operatorname{Spec} A = \{(x)\} = \text{one point}\\ &\mathcal{O}(\operatorname{Spec} A) = A = k[x]/x^2\\ &A \ni f = a + bx : \operatorname{Spec} A \to k,\ (x) \mapsto a = (f \bmod (x) \in \mathbb{K}((x)) \cong A/x \cong k).\end{aligned}$$

So we have a two-dimensional space of functions (two parameters: $a, b \in k$), even though when we consider the values of the functions we only see one parameter worth of functions. So the ring of functions $\mathcal{O}(\operatorname{Spec} A)$ also remembers tangential information:[1] the tangent vector $\frac{\partial}{\partial x}\big|_{x=0}$, namely the operator acting on functions as follows,

$$\frac{\partial}{\partial x}\Big|_{x=0} f = b.$$

Why is this a reasonable definition? The "ringed space" $\operatorname{Spec} A$ is not the same as $\operatorname{Spec} k[x]/x$: it remembers that it arose as a deformation of $\operatorname{Spec} B = \{\text{two points } \alpha, \beta \in \mathbb{A}^1\}$ as $\alpha, \beta \to 0$ where

$$B = k[x]/(x-\alpha)(x-\beta) \cong k \oplus k$$

where $\alpha, \beta \in k$ are non-zero distinct deformation parameters, and the second isomorphism[2] is evaluation at α, β respectively. So $f = a + bx \mapsto (a + b\alpha) \oplus (a + b\beta)$, so we can independently pick the two values of f at the two points $\{\alpha, \beta\} = \operatorname{Specm} B$, giving a two-dimensional family of functions. The derivative $\partial_x f|_{x=0} = b = \lim \frac{f(\alpha)-f(\beta)}{\alpha-\beta}$ as we let α, β converge to 0.

Exercise. An affine variety $X \subset \mathbb{A}^n$ is irreducible if and only if $\operatorname{Spec} k[X]$ has a generic point.
Exercise. Knowing the value of $f \in A$ at a generic point determines the value of f at all points.
Example. $f \in \mathbb{Z}$, then $f((0)) = \frac{f}{1} \in \mathbb{K}((0)) = \mathbb{Q}$ determines $f((p)) \in \mathbb{K}((p)) = \mathbb{Z}/p$ (reduce mod p).

15.3. MORPHISMS BETWEEN Specs

Apart from the motivation coming from deformation theory, another convincing reason for preferring $\operatorname{Spec} A$ over $\operatorname{Specm} A$, is that we get a category of affine schemes because we have morphisms:

Definition. *The morphisms*[3]

$$\operatorname{Hom}(\operatorname{Spec} A, \operatorname{Spec} B) = \{\varphi^* : \operatorname{Spec} A \to \operatorname{Spec} B \textit{ induced by ring homs } \varphi : B \to A\}$$

where $\varphi^*(\wp) = \varphi^{-1}(\wp) \subset A$, *for any prime ideal* $\wp \subset B$.

Exercise. The preimage of a prime ideal under a ring hom is always prime.
Warning. This exercise fails for maximal ideals. **Example.** For the inclusion $\varphi : \mathbb{Z} \to \mathbb{Q}$, $\varphi^{-1}(0) = (0) \subset \mathbb{Z}$ is not maximal even though $(0) \subset \mathbb{Q}$ is maximal. Similarly, for the inclusion $\varphi : k[x] \to k(x) = \operatorname{Frac} k[x]$, $\varphi^{-1}(0) = (0)$ is not maximal since $(0) \subset (x)$.
Remark. We did not notice this issue when dealing with affine varieties, which was the study of Specm of f.g. reduced k-algs, because in that case morphisms exist between the Specm.
Exercise.[4] More generally: for any f.g. k-algebras A, B, and $\varphi : A \to B$ a k-alg hom, prove that $\operatorname{Specm} A \leftarrow \operatorname{Specm} B : \varphi^*$ is well-defined, namely $\varphi^*(\mathfrak{m}) = \varphi^{-1}(\mathfrak{m})$ is always maximal.

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[1] More generally: a closed point $\mathfrak{m} \in \operatorname{Spec} A$ corresponds to a k-alg hom $A \to k$ (with kernel $\mathfrak{m}$), which corresponds to a map $\{\text{point}\} = \operatorname{Spec} k \to \operatorname{Spec} A$, and the same holds if we replace $k \cong k[x]/x$. Whereas a map $\operatorname{Spec} k[x]/x^2 \to \operatorname{Spec} A$ corresponds to a k-alg hom $A \to k[x]/x^2$ which defines a closed point together with a "tangent vector".

[2] This is the Chinese Remainder Theorem. Explicitly: $1 = \frac{x-\beta}{\alpha-\beta} + \frac{\alpha-x}{\alpha-\beta}$, so $k[x]/(x-\alpha)(x-\beta) \cong k[x]/(x-\alpha) \oplus k[x]/(x-\beta)$ via $g \mapsto \frac{x-\beta}{\alpha-\beta} g \oplus \frac{\alpha-x}{\alpha-\beta} g$. Finally, $k[x]/(x-\gamma) \cong k$ via $f \mapsto f(\gamma)$.

[3] Categorically: Spec is a functor Rings $\to \mathrm{Top}^{op}$ from the category of rings (commutative) to the opposite of the category of topological spaces and continuous maps.

[4] *Hints.* $k \subset A/\varphi^{-1}(\mathfrak{m}) \subset B/\mathfrak{m} \cong$ some field. When k is algebraically closed, we know $B/\mathfrak{m} \cong k$, so we are done. For general k, we already know $\varphi^{-1}(\mathfrak{m})$ is prime so $A/\varphi^{-1}(\mathfrak{m})$ is a domain. Finally use: (1) f.g. k-alg + field $\Rightarrow$ algebraic/k $\Rightarrow$ finite field extension/k; and use (2) domain + algebraic/k $\Rightarrow$ field extension of k.

15.4. LOCALISATION: RESTRICTING TO OPEN SETS

Remark. We already encountered localisation in Section 10, so we will be brief.

Question: What are the functions on a basic open set?
Recall $D_f = \{\wp : f(\wp) \neq 0\} \subset \operatorname{Spec} A$, so we should allow the function $\frac{1}{f}$ on D_f. Thus we "define"

$$\boxed{\mathcal{O}(D_f) = A_f = \text{localisation of } A \text{ at } f}$$

which will ensure that $\operatorname{Spec} A_f \cong D_f$. When A is an integral domain,

$$A_f = \{\tfrac{a}{f^m} \in \operatorname{Frac} A : a \in A, m \in \mathbb{N}\}.$$

Example. $\mathbb{A}^1 \setminus 0 = D_x \subset \mathbb{A}^1$, and we view $\mathbb{A}^1 \setminus 0 \cong \mathbb{V}(xy-1) \subset \mathbb{A}^2$ as an affine variety via $t \leftrightarrow (t, t^{-1})$. By definition, $k[\mathbb{A}^1 \setminus 0] = k[x,y]/(xy-1) \cong k[x, x^{-1}] \cong k[x]_x$ is the localisation at x.

Question: What are the functions on a general open set $U \subset \operatorname{Spec} A$*?*
We know $U = \cup D_f$ is a union of basic open sets. Loosely,[1] the "functions" in $\mathcal{O}(U)$, called **sections** s_U, are defined as the family of functions $s_f \in \mathcal{O}(D_f) = A_f$ which agree on the overlaps

$$s_f|_{D_g} = s_g|_{D_f} \in \mathcal{O}(D_f \cap D_g) = \mathcal{O}(D_{fg}) = A_{fg}.$$

Remark. Not all open sets are basic open sets. For $X = \mathbb{V}(xw - yz) \subset \mathbb{A}^4$, the union $D_y \cup D_w \subset X$ is not basic and $\mathcal{O}(D_y \cup D_w)$ does not arise as a localisation of $k[X]$. Indeed $f = \frac{x}{y} = \frac{z}{w} \in \mathcal{O}(D_y \cup D_w)$ cannot be written as a fraction which is simultaneously defined on both D_y, D_w.

Question: What are the germs of functions?
Recall the **germ of a function** near a point $a \in X$ of a topological space, means a function $U \to k$ defined on a neighbourhood $U \subset X$ of a, and we identify two such functions $U \to k$, $U' \to k$ if they agree on a smaller neighbourhood of a (so the germ is an equivalence class of functions). Write $\mathcal{O}_\wp$ for the germs of functions at $\wp \in \operatorname{Spec} A$, this is called the **stalk** of $\mathcal{O}$ at $\wp$. It turns out that[2]

$$\boxed{\mathcal{O}_\wp = A_\wp = \text{localisation of } A \text{ at } A \setminus \wp}$$

i.e. we localise at all $f \notin \wp$, by allowing $\frac{1}{f}$ to be a function whenever f does not vanish at $\wp$. We explain this in greater detail in Sec.15.10. When A is an integral domain,[3]

$$A_\wp = \{\tfrac{a}{b} \in \operatorname{Frac} A : b \notin \wp \text{ (i.e. } b(\wp) \neq 0)\} = \prod_{f \notin \wp} A_f \subset \operatorname{Frac} A.$$

Example. Let $A = k[x,y]/(xy)$. The affine variety $X = \operatorname{Specm}(A) \cong \mathbb{V}(xy) \subset \mathbb{A}^2$ consists of the x-axis and y-axis. The x-axis is the vanishing locus of the prime ideal $\wp = (y)$. The function $f = x$ does not vanish at $\wp$, since $\overline{x} \neq 0 \in (k[x,y]/(xy))/\wp \cong k[x]$, so $\frac{1}{x} \in A_\wp$ is a germ of a function on $\operatorname{Spec}(A)$ defined near $\wp$. This should not be confused with germs of functions defined near the closure $\mathbb{V}(\wp)$, i.e. germs of functions defined near the x-axis. Indeed, the germs of functions 0 and y are different on any neighbourhood[4] of $\mathbb{V}(\wp)$. However, in the localisation $A_\wp$ the functions 0 and y are identified, because $xy = 0$ forces $0 = \frac{1}{x} \cdot xy = y$. Also, $\frac{1}{x}$ is not a well-defined function on all of $\mathbb{V}(\wp)$, as it is not defined at $x = 0$, it is only defined on the open subset $\mathbb{V}(\wp) \cap D_f$ of $\wp$. So functions in $A_\wp$ are defined near the generic point $\wp$ of $\mathbb{V}(\wp)$ but need not extend to a function on all of $\mathbb{V}(\wp)$.

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[1] Formally: $\mathcal{O}(U) = \varprojlim \mathcal{O}(D_f)$ is the inverse limit for $D_f \subset U$, taken over the restriction maps $\mathcal{O}(D_{f'}) \leftarrow \mathcal{O}(D_f)$ for $D_f' \subset D_f \subset U$ (these maps are the localisation maps $A_f' \leftarrow A_f$). This means precisely that for each basic open set inside U we have a function, and these functions are compatible with each other under restrictions to overlaps.

[2] Formally: $\mathcal{O}_\wp = \varinjlim \mathcal{O}(U)$ is the direct limit for open subsets U containing $\wp$, taken over the restriction maps $\mathcal{O}(U) \to \mathcal{O}(U')$ for $U \supset U' \ni \wp$. So we have sections $s_U \in \mathcal{O}(U)$ and we identify sections $s_U \sim s_V$ whenever $s_U|_W = s_V|_W$ for some open $\wp \in W \subset U \cap V$.

[3] This requires care: $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$ (the definition of Frac), so there may be many expressions for the same element. In $A_\wp$ we want *some* expression to have a denominator which does not vanish at $\wp$. Example: $\wp = (2) \subset A = \mathbb{Z}$, then $\frac{2}{3} \in A_{(2)} \subset \operatorname{Frac} \mathbb{Z} = \mathbb{Q}$ since $3 \notin (2)$, whereas $\frac{4}{6}$ fails the condition $6 \notin (2)$ even though it equals $\frac{2}{3}$.

[4] such neighbourhoods contain all but finitely many points of the y-axis, so $0 \neq y$ as functions.

The ring $\mathcal{O}_\wp = A_\wp$ is a **local ring**, meaning it has precisely one maximal ideal, namely

$$\mathfrak{m}_\wp = A_\wp \cdot \wp \subset A_\wp.$$

So $\operatorname{Specm} A_\wp$ = one point, namely $\mathfrak{m}_\wp$, which you should think of as "representing $\wp$" because $\operatorname{Specm} A_\wp \to \operatorname{Spec} A$ maps the point to $\wp$.
Exercise. Show that, indeed, at the algebra level $A_\wp \leftarrow A$ maps $\mathfrak{m}_\wp \leftarrow\!\shortmid \wp$.
The value of $f \in A$ at $\wp$ lives in the **residue field**[1] of that local ring

$$f(\wp) \in \mathcal{O}_\wp/\mathfrak{m}_\wp = A_\wp/\mathfrak{m}_\wp \cong \mathbb{K}(\wp).$$

Exercise. Prove that $A_\wp/\mathfrak{m}_\wp \cong \operatorname{Frac} A/\wp = \mathbb{K}(\wp)$.
Example. Consider $A = \mathbb{Z}$. Either $\wp = (p)$ for prime p, or $\wp = (0)$:
$\mathcal{O}_p = \mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : p \nmid b\}$, $\mathfrak{m}_p = p \cdot \mathbb{Z}_{(p)} = \{\frac{a}{b} \in \mathbb{Q} : p|a, p \nmid b\}$, and[2] $\mathbb{K}(p) = \mathcal{O}_p/\mathfrak{m}_p \cong \mathbb{F}_p = \mathbb{Z}/p$.
$\mathcal{O}_0 = \mathbb{Z}_{(0)} = \operatorname{Frac}\mathbb{Z} = \mathbb{Q}$, $\mathfrak{m}_0 = (0) \subset \mathbb{Q}$, and $\mathbb{K}(0) = \mathcal{O}_0/\mathfrak{m}_0 \cong \mathbb{Q}$.

15.5. SHEAVES

Given a topological space X, a **sheaf $\mathcal{S}$ of rings** on X means an association[3]

$$(\text{open subset } U \subset X) \mapsto (\text{ring } \mathcal{S}(U)).$$

Elements of $\mathcal{S}(U)$ are called **sections over** U. We require that for all open $U \supset V$ there is a restriction, namely a ring homomorphism

$$\mathcal{S}(U) \to \mathcal{S}(V),\ s \mapsto s|_V$$

satisfying two obvious requirements: $\mathcal{S}(U) \to \mathcal{S}(U)$ is the identity map, and "restricting twice is the same as restricting once".[4] We also require two **local-to-global conditions**:
(1). "Two sections equal if they equal locally".[5]
(2). "You can build global sections by defining local sections which agree on overlaps".[6]
Without the local-to-global conditions, it would be called a **presheaf**.
Given a sheaf (or presheaf) $\mathcal{S}$ on X, the **stalk** $\mathcal{S}_p$ at $p \in X$ is the ring of germs of sections at p.[7]

EXAMPLES.
1. $X = \operatorname{Spec} A$, and $\mathcal{S}(U) = \mathcal{O}(U)$ as in Section 15.4. For example, $\mathcal{O}(D_f) = A_f$, and $D_f \supset D_{fg}$ determines the restriction which "localises further",

$$A_f \to A_{fg},\ \ \frac{a}{f^m} \mapsto \frac{ag^m}{(fg)^m}.$$

2. Sheaf of continuous functions: $\mathcal{S}(U) = C(U,k)$ =(continuous functions $U \to k$).
3. Sheaf of sections of a map[8] $\pi : E \to B$: take $\mathcal{S}(U)$ = sections[9] $s : U \to \pi^{-1}(U) \subset E$.
4. Skyscraper sheaf at $p \in X$ for the ring A: $\mathcal{S}(U) = A$ if $p \in U$, and $\mathcal{S}(U) = 0$ if $p \notin U$. Exercise: show the stalks are $\mathcal{S}_p = A$ and $\mathcal{S}_q = 0$ for $q \neq p$.

Non-example. The presheaf of constant functions (or constant presheaf): $\mathcal{S}(U) = A$ for open $U \neq \emptyset$, and $\mathcal{S}(\emptyset) = 0$, is not a sheaf for $A = \mathbb{Z}/2$ and $X = \{p, q\}$ with the discrete topology. Indeed,

[1] the field obtained by quotienting a local ring by its unique maximal ideal.
[2] via $\frac{a}{b} \leftrightarrow ab^{-1} \bmod p$.
[3] Categorically: a **presheaf** is a functor $\mathrm{Open}_X^{\mathrm{op}} \to \mathrm{Rings}$ where the objects of Open_X are the open sets and the only morphisms allowed are inclusion maps; and a **morphism of presheaves** is a natural transformation of such functors. For **sheaves** we impose the above local-to-global conditions for sections, but no extra condition on morphs.
[4] For $U \supset V \supset W$, $\mathcal{S}(U) \to \mathcal{S}(V) \to \mathcal{S}(W)$ agrees with $\mathcal{S}(U) \to \mathcal{S}(W)$.
[5] For $f, g \in \mathcal{S}(U)$, $U = \cup U_i$, $f|_{U_i} = g|_{U_i}$ for all $i \Rightarrow f = g$.
[6] $U = \cup U_i$, $s_i \in \mathcal{S}(U_i)$, $s_i|_{U_j} = s_j|_{U_i} \in \mathcal{S}(U_i \cap U_j) \Rightarrow$ there is some $s \in \mathcal{S}(U)$ with $s|_{U_i} = s_i$ (and s is unique by (1)).
[7] A germ at p is an equivalence class of sections. It is determined by some section $s_U \in \mathcal{S}(U)$, for an open $p \in U$. We identify two sections $s_U \sim s_V$ if $s_U|_W = s_V|_W$, for an open $p \in W \subset U \cap V$.
[8] for example, a vector bundle E over a manifold B.
[9] here "section" means it is compatible with the projection π, so $\pi(s(u)) = u$. So at each u in the base, the section s picks an element in the fibre $s^{-1}(u)$ over u.

take $s|_{\{p\}}(p) = 0$, $s|_{\{q\}}(q) = 1$: these local sections do not globalise to a global constant function $s : X \to A$ contradicting (2).

15.6. SHEAFIFICATION

One can always **sheafify** a presheaf $\mathcal{P}$ to obtain a sheaf $\mathcal{S}$ by artificially imposing local-to-global:

$$\mathcal{S}(U) = \{s = (s_p) \in \prod_{p \in U} \mathcal{P}_p : \forall p \in U \text{ there is an open } p \in V \subset U \text{ and } s_V \in \mathcal{P}(V) \text{ with } s_V|_p = s_p\}.$$

Notice how we impose that locally all germs arise from restricting a local section. We now explain this in more detail.

For any sheaf $\mathcal{S}$ on a topological space X, there is an obvious restriction $\mathcal{S}(U) \to \mathcal{S}_x$, $f \mapsto f_x$ to stalks, for each $x \in U$. Being a sheaf ensures the local-to-global property:

$$\text{If } f_x = g_x \text{ at all } x \in U, \text{ then } f = g \in \mathcal{S}(U)$$

because $f_x = g_x$ means that f, g equal on a small neighbourhood of x. So f is completely determined by the data $(f_x)_{x \in U}$. Not all data $(f_x)_{x \in U}$ arises in this way, the data has to be **compatible**: locally, on some open V around any given point, the f_x arise from restricting some $F \in \mathcal{S}(V)$. So $\mathcal{S}(U)$ consists of compatible families $(f_x)_{x \in U}$ and the restriction map for open $V \subset U$ extracts subfamilies:

$$\mathcal{S}(U) \to \mathcal{S}(V),\ (f_x)_{x \in U} \mapsto (f_x)_{x \in V}.$$

So the sheafification of a pre-sheaf $\mathcal{P}$ is

$$\mathcal{S}(U) = \{\text{compatible families of germs } \{s_x\}_{x \in U} \text{ where } s_x \in \mathcal{P}_x\}.$$

This is a very useful trick, we will use it in Sections 15.8 and 15.12.
Exercise. Show that the sheafification of the pre-sheaf of constant k-valued functions on a topological space X is the sheaf of *locally constant* functions (i.e. constant on each connected component).
Example. For X an affine variety, let $\mathcal{P}(U) = \{$functions $f : U \to k : f = \frac{g}{h}$ some $g, h \in k[X]$, with $h(u) \neq 0$ for all $p \in U\}$. This is a presheaf, whose sheafification defines $\mathcal{O}(U)$, see Sec.10.2.

15.7. MORPHISMS OF SHEAVES

A morphism $\psi : \mathcal{S}_1 \to \mathcal{S}_2$ of sheaves over X means an association

$$(\text{open subset } U \subset X) \mapsto (\text{ring hom } \psi_U : \mathcal{S}_1(U) \to \mathcal{S}_2(U))$$

which is compatible with restriction maps.[1]
Exercise. Show that this induces a ring hom on stalks: $\psi_p : \mathcal{S}_{1,p} \to \mathcal{S}_{2,p}$.
Exercise. $\psi : \mathcal{S}_1 \to \mathcal{S}_2$ is an isomorphism $\Leftrightarrow$ it is an isomorphism on stalks (all ψ_p are isos).
Exercise. If $\psi : X \to Y$ is a continuous map of topological spaces, and $\mathcal{S}$ is a sheaf on X, then ψ induces a sheaf on Y called **direct image sheaf** $\psi_*\mathcal{S}$, defined by

$$(\psi_*\mathcal{S})(U) = \mathcal{S}(\psi^{-1}(U)).$$

15.8. RINGED SPACES

A **ringed space** $(X, \mathcal{S})$ is a topological space X together with a sheaf of rings, $\mathcal{S}$.
Example. The affine scheme $(\operatorname{Spec} A, \mathcal{O})$ is a ringed space.
A **morphism of ringed spaces** $(X_1, \mathcal{S}_1) \to (X_2, \mathcal{S}_2)$ means a continuous map $f : X_1 \to X_2$ together with a morphism of sheaves over X_2,

$$f^* : f_*\mathcal{S}_1 \leftarrow \mathcal{S}_2$$

so explicitly $f^*(U)$ maps $\mathcal{S}_1(f^{-1}(U)) \leftarrow \mathcal{S}_2(U)$ for $U \subset X_2$, and on stalks $f_p^* : (\mathcal{S}_1)_p \leftarrow (\mathcal{S}_2)_{f(p)}$.
Example. $\varphi : A \to B$ a ring hom $\Rightarrow f = \varphi^* : \operatorname{Spec} A \leftarrow \operatorname{Spec} B$ and

$$\psi = f^* : \mathcal{O}_A \to f_*\mathcal{O}_B$$

[1] For $V \subset U \subset X$, a commutative diagram relates ψ_U, ψ_V with the restriction maps res_V^U, so: $\mathrm{res}_V^U \circ \psi_U = \psi_V \circ \mathrm{res}_V^U$.

so $\psi_U : \mathcal{O}_A(U) \to \mathcal{O}_B((\varphi^*)^{-1}(U))$. Notice $\psi_{\operatorname{Spec} A} : A \to B$ is just φ, on basic open sets ψ is the relevant localisation of φ, and on stalks we get the localised map[1] $\psi_{\varphi^*\wp} : A_{\varphi^*\wp} \to B_\wp$ for $\wp \in \operatorname{Spec} B$.

A **locally ringed space** means we additionally require the stalks $\mathcal{S}_p$ to be local rings, so they have a unique maximal ideal $\mathfrak{m}_p \subset \mathcal{S}_p$. A morphism of locally ringed spaces is additionally required to preserve maximal ideals, i.e. $f^* : \mathfrak{m}_p \leftarrow \mathfrak{m}_{f(p)}$ (but this need not be bijective).

Example.[2] Show that $\operatorname{Spec} A \leftarrow \operatorname{Spec} B$ is a morph of locally ringed spaces.

15.9. SCHEMES

An **affine scheme** is a locally ringed space isomorphic to $(\operatorname{Spec} A, \mathcal{O})$ for some ring A.

A **scheme** $(X, \mathcal{S})$ is a locally ringed space which is locally an affine scheme.[3]

We now describe the affine scheme $X = \operatorname{Spec}(A)$ as a locally ringed space $(X, \mathcal{O}_X)$ (Lemma 10.4 will prove that the stalks $\mathcal{O}_{X,\wp}$ of the structure sheaf are local rings). By definition,

$$\{\text{ring homs } \varphi : A \to B\} \stackrel{1:1}{\longleftrightarrow} \{\text{morphisms } \varphi^* : \operatorname{Spec}(A) \leftarrow \operatorname{Spec}(B)\}$$

where $\varphi^*\wp = \varphi^{-1}(\wp)$. One can check that a ring hom $A \to B$ induces a local ring hom on stalks $\mathcal{O}_{A,\varphi^*\wp} \to \mathcal{O}_{B,\wp}$ (Equation (10.2)).

We sketched one definition of the structure sheaf $\mathcal{O} = \mathcal{O}_X$ on $X = \operatorname{Spec}(A)$ in Section 15.4. We now explain an equivalent definition using sheafification (Sec.15.6). For $U \subset X$ an open subset, $\mathcal{O}(U)$ consists of **compatible** families of elements $\{f_\wp \in \mathcal{O}_\wp\}_{\wp \in U}$. Recall $\mathcal{O}_\wp \cong A_\wp$ is the localisation of A at the prime ideal $\wp$, so we formally invert all elements in $A \setminus \wp$. So equivalently, these are functions

$$f : U \to \bigsqcup_{\wp \in U} A_\wp, \quad \wp \mapsto f_\wp.$$

Compatible means: for any $\mathfrak{q} \in U$, there is a basic open set $\mathfrak{q} \in D_g \subset U$ (so $g \notin \mathfrak{q}$) and some $F \in A_g = \mathcal{O}(D_g)$ such that the $f_\wp$ are the restrictions of F (meaning, $A_g \to A_\wp$, $F \mapsto f_\wp$ for all $\wp \in D_g$). The restriction homs for open $V \subset U$, are simply defined by taking subfamilies:

$$\mathcal{O}(U) \to \mathcal{O}(V), \ (f_\wp)_{\wp \in U} \mapsto (f_\wp)_{\wp \in V}.$$

The "value" $f(\wp) \in \mathbb{K}(\wp)$ of f (Sec.15.1) is the image of $f_\wp$ via the natural map $\mathcal{O}_\wp \to \mathcal{O}_\wp/\mathfrak{m}_\wp \cong \mathbb{K}(\wp)$.

Exercise. After reading Section 11, check that the above is consistent with the explicit definition of $\mathcal{O}_X, \mathcal{O}_{X,p}$ for a quasi-projective variety X, carried out in Sections 11.3 and 11.5.

15.10. LOCALISATION REVISITED: affine varieties

For X an affine variety and $\wp \subset k[X]$ a prime ideal, the stalk $\mathcal{O}_{X,\wp}$ means "germs of functions on $\operatorname{Spec} k[X]$ defined near $\wp$", which we now explain. It suffices to consider basic neighbourhoods D_f, for $f \in k[X]$ with $f \neq 0 \in k[X]/\wp$. Then $\mathcal{O}_{X,\wp}$ consists of pairs (D_f, F) with $f \neq 0 \in k[X]/\wp$, U open, $F : U \to k$ regular, and identifying $(D_f, F) \sim (D_g, G) \Leftrightarrow F|_{D_h} = G|_{D_h}$ on an open D_h with $D_h \subset D_f \cap D_g$ and $h \neq 0 \in k[X]/\wp$. Algebraically this is the **direct limit**

$$\mathcal{O}_{X,\wp} = \varinjlim_{\wp \in D_f} \mathcal{O}_X(D_f) = \varinjlim_{f \notin \wp} k[X]_f$$

over all basic open neighbourhoods D_f of $\wp$. It is easy to verify algebraically that

$$\varinjlim_{f \notin \wp} k[X]_f \cong k[X]_\wp,$$

indeed we are formally inverting all elements that do not belong to $\wp$. This is the analogue of Lemma 10.5, which showed $\mathcal{O}_{X,\mathfrak{m}_p} \cong k[X]_{\mathfrak{m}_p}$, namely the case when $\wp$ is a maximal ideal (corresponding to a geometric point in X). Recall that analogously to (10.3), we get a field extension of k:

$$\mathbb{K}(\wp) = \operatorname{Frac}(A/\wp).$$

[1] Explicitly: $\frac{a}{a'} \mapsto \frac{\varphi(a)}{\varphi(a')}$ where $a' \in A \setminus \varphi^{-1}(\wp)$ (so $\varphi(a') \in B \setminus \wp$).

[2] You need to check that $\varphi^*\wp \cdot A_{\varphi^*\wp}$ maps into $\wp \cdot B_\wp$ via $f_{\varphi^*\wp}$.

[3] $X = \cup U_i$, $U_i \cong \operatorname{Spec} A_i$ some rings A_i, $\mathcal{S}|_{U_i} \cong \mathcal{O}_{A_i}$ (the structure sheaf for A_i).

We think of the unique prime ideal (0) of this field as corresponding to the point $\wp \in \operatorname{Spec}(A) = X$: the ring hom $\varphi : A \to A/\wp \hookrightarrow \mathbb{K}(\wp)$ corresponds to the point-inclusion $\varphi^* : \operatorname{Spec}(\mathbb{K}(\wp)) \hookrightarrow \operatorname{Spec}(A)$, $(0) \mapsto \wp$. In Section 15.1 we used $\mathbb{K}(\wp)$ to define the "value" of "functions" $f \in A$, by saying that

$$f(\wp) = \overline{f} \in A/\wp \hookrightarrow \mathbb{K}(\wp).$$

Exercise. $f(\wp) \neq 0 \in \mathbb{K}(\wp) \Leftrightarrow f \notin \wp \Leftrightarrow \wp \in D_f$.

Example. $A = \mathbb{Z}$, $p \in \mathbb{Z}$ prime, $\mathbb{K}(p) = \mathbb{Z}/p = \mathbb{F}_p$. For $f \in A$, $f(p) = (f \bmod p) \in \mathbb{F}_p$.

Example. Consider $X = (x\text{-axis}) \cup (y\text{-axis})$, $k[X] = k[x,y]/(xy)$ and $\wp = (y)$, so $\mathbb{V}(\wp) = (x\text{-axis})$. Then $k[X]_\wp \cong k(x)$, indeed we invert everything outside of (y), we already saw that inverting x gives $k[X]_x \cong k[x, x^{-1}]$, but now we also invert any polynomial in x so we get $k(x) = \operatorname{Frac}(k[x])$. One should not interpret "germs near $\wp$" as meaning "germs near $\mathbb{V}(\wp)$", since the functions y and 0 are not equal on any neighbourhood of $\mathbb{V}(\wp) = (x\text{-axis})$. In particular, $\frac{1}{x}$ is not well-defined on all of $\mathbb{V}(\wp)$. The correct interpretation of $k[X]_\wp$ is: rational functions defined on a non-empty (dense) open subset of $\mathbb{V}(\wp)$.

Exercise. For X an irreducible affine variety, i.e. $A = k[X]$ an integral domain, show that

$$\mathcal{O}_X(U) = \bigcap_{D_f \subset U} \mathcal{O}(D_f) = \bigcap_{D_f \subset U} k[X]_f \subset \operatorname{Frac}(k[X]) = k(X),$$

using that the D_f are a basis for the topology, and that a function is regular iff it is locally regular.

When X is not irreducible, then we cannot define the fraction field of $A = k[X]$ in which to take the above intersection ($k[X]_f$ and $k[X]_g$ don't live in a larger common ring where we can intersect). So instead, algebraically, one has to take the **inverse limit**:

$$\mathcal{O}_X(U) = \varprojlim_{D_f \subset U} \mathcal{O}_X(D_f) = \varprojlim_{D_f \subset U} k[X]_f$$

taken over all restriction maps $k[X]_f \to k[X]_g$ where $D_g \subset D_f \subset U$. Explicitly, these are families of functions $F_f \in k[X]_f$ which are compatible in the sense that $F_f|_{D_g} = F_g$ (where $F_f|_{D_g}$ is the image of F_f via the natural map $k[X]_f \to k[X]_g$). This definition makes sense also for any q.p.v. X.

Finally, the FACT from Section 10.1, implies a 1:1 correspondence

$$\{\text{irreducible subvarieties } Y \subset X \text{ containing } \mathbb{V}(\wp)\} \stackrel{1:1}{\longleftrightarrow} \{\text{prime ideals of } k[X]_\wp\}.$$

15.11. WORKED EXAMPLE: THE SCHEME $\operatorname{Spec} \mathbb{Z}[x]$

Some basic algebra implies that

$$\begin{aligned} \operatorname{Spec} \mathbb{Z}[x] \quad = \quad & \{(0)\} \cup \{(p) : p \in \mathbb{Z} \text{ prime }\} \cup \\ & \cup \{(f) : f \in \mathbb{Z}[x] \text{ non-constant irreducible}\} \cup \\ & \cup \{(p, f) : p \in \mathbb{Z} \text{ prime}, f \in \mathbb{Z}[x] \text{ irreducible mod } p\} \end{aligned}$$

Consider the projection $\pi : \operatorname{Spec} \mathbb{Z}[x] \to \operatorname{Spec} \mathbb{Z}$ induced by the inclusion $\mathbb{Z} \to \mathbb{Z}[x]$.

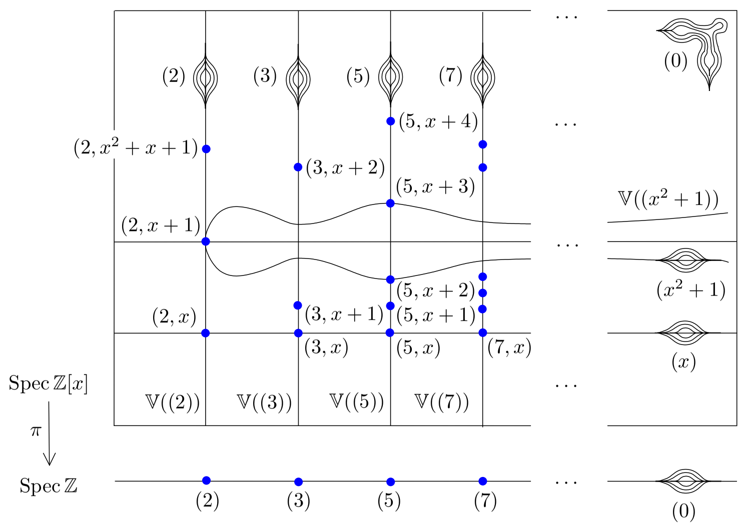
Exercise. Explicitly $\pi(\wp) = $ (all constant polynomials in $\wp$).

Below is an imaginative geometric picture[1] of π.

The base $\operatorname{Spec} \mathbb{Z}$ has prime ideals (p) and (0). Since (0) is a generic point it is drawn by a squiggly symbol to remind ourselves that (0) is dense in $\operatorname{Spec} \mathbb{Z}$. The fibre over (p) is $\pi^{-1}((p)) = \mathbb{V}((p))$, i.e. prime ideals in $\mathbb{Z}[x]$ which contain p, and $\pi^{-1}((0))$ consists of all other prime ideals, i.e. those which do not contain a non-zero constant polynomial. The fibre $\pi^{-1}((p))$ contains the generic point (p), and we draw it by a squiggly symbol because it is dense in $\mathbb{V}((p))$. The point $(0) \in \operatorname{Spec} \mathbb{Z}[x]$ is generic, because every ideal in $\mathbb{Z}[x]$ contains 0, so we use a large squiggly symbol.

When looking for generators of an ideal in $\pi^{-1}(p)$ (apart from p), we may reduce the polynomial coefficients mod p. Example: for $(5, x+j) \in \pi^{-1}(5)$ we only need to consider the cases $j = 0, 1, \ldots, 4$.

[1] an adaptation of a famous picture by David Mumford, *The Red Book of Varieties and Schemes*.



Exercise. $\pi^{-1}(p) = \mathbb{V}((p)) \cong \operatorname{Spec}\mathbb{F}_p[x]$ are homeomorphic, where $\mathbb{F}_p = \mathbb{Z}/p$.

By definition, (x^2+1) is dense (hence a generic point) in $\mathbb{V}((x^2+1))$, so we draw it by a squiggly symbol lying on the "curve" $\mathbb{V}((x^2+1))$. This "curve" contains the points $(2, x+1), (5, x+2), (5, x+3)$, etc., that is: we claim (x^2+1) is contained in those ideals.

Example. $\mathbb{Z}[x]/(5, x+2) \cong \mathbb{F}_5[x]/(x+2)$ by first quotienting by (5). This iso is given by "reduce mod 5". Now x^2+1 is divisible by $(x+2)$ mod 5, because -2 is a root of x^2+1 mod 5. So $x^2+1 = 0 \in \mathbb{F}_5[x]/(x+2) \cong \mathbb{Z}[x]/(5, x+2)$, so $(x^2+1) \subset (5, x+2)$. The roots of x^2+1 mod 5 are precisely $2, 3$, which explains the points $(5, x+2)$, $(5, x+3)$ on the "curve" $\mathbb{V}((x^2+1))$.

Remark. Notice the points on $\mathbb{V}((x^2+1))$ encode the square roots of -1 over $\mathbb{F}_p$. A classical result in number theory says that solutions exist $\Leftrightarrow p \equiv 1 \bmod 4$ or $p = 2$.

We want to prove the above description of $\operatorname{Spec}\mathbb{Z}[x]$, using the fibre product machinery.[1]

In Section 6.4, working with affine varieties over an algebraically closed field k, we explained that the fibre of $X \to Y$ over $a \in Y$ is Specm of

$$k[X] \otimes_{k[Y]} k$$

where $k \cong k[Y]/\mathfrak{m}_a = \operatorname{Frac} k[Y]/\mathfrak{m}_a = \mathbb{K}(a)$, where $\mathfrak{m}_a$ is the maximal ideal corresponding to a. When working with rings, and the map $\operatorname{Spec} A \to \operatorname{Spec} B$ induced by some ring hom $A \leftarrow B$, the scheme-theoretic fibre over $\wp \in \operatorname{Spec} B$ is the Spec of the following ring:

$$A \otimes_B \mathbb{K}(\wp)$$

where the **residue field** $\mathbb{K}(\wp)$ at $\wp$ is

$$\boxed{\mathbb{K}(\wp) = \operatorname{Frac}(B/p) \cong B_\wp/\mathfrak{m}_\wp}$$

Remark. (Later in the course.) Prime ideals in the localisation $B_\wp$ are in 1:1 correspondence with prime ideals of B contained in $\wp$, and $\wp$ corresponds to the unique max ideal $\mathfrak{m}_\wp \subset B_\wp$.

Exercise. After reading about localisation in Section 10, prove $\operatorname{Frac}(B/\wp) \cong B_\wp/\mathfrak{m}_\wp$.

[1] Of course, $\operatorname{Spec}\mathbb{Z}[x]$ is the union of the fibres of π, explicitly: $\wp \in \operatorname{Spec}\mathbb{Z}[x]$ lies in $\pi^{-1}(\pi(\wp))$.

The diagram for the fibre product is

$$\begin{array}{ccc} \operatorname{Spec}(A \otimes_B \mathbb{K}(\wp)) & \longrightarrow & \operatorname{Spec} A \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{K}(\wp) & \longrightarrow & \operatorname{Spec} B \\ (\text{one point } (0)) & \longmapsto & (\wp) \end{array}$$

In our case, $\pi : \operatorname{Spec}\mathbb{Z}[x] \to \operatorname{Spec}\mathbb{Z}$, so

$$A = \mathbb{Z}[x] \qquad B = \mathbb{Z}.$$

For $\wp = (p)$: $\quad \mathbb{K}((p)) = \operatorname{Frac}(B/p) \cong \mathbb{F}_p \qquad A \otimes_B \mathbb{K}((p)) = \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{F}_p[x]$.
For $\wp = (0)$: $\quad \mathbb{K}((0)) = \operatorname{Frac}(B/0) \cong \mathbb{Q} \qquad A \otimes_B \mathbb{K}((0)) = \mathbb{Z}[x] \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[x]$.
So the two diagrams for the fibre product are:

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{F}_p[x] & \longrightarrow & \operatorname{Spec}\mathbb{Z}[x] \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec}\mathbb{F}_p & \longrightarrow & \operatorname{Spec}\mathbb{Z} \\ (0) & \longmapsto & (p) \end{array} \qquad \begin{array}{ccc} \operatorname{Spec}\mathbb{Q}[x] & \longrightarrow & \operatorname{Spec}\mathbb{Z}[x] \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec}\mathbb{Q} & \longrightarrow & \operatorname{Spec}\mathbb{Z} \\ (0) & \longmapsto & (0) \end{array}$$

Recall $\mathbb{F}_p[x]$ is a principal ideal domain, so $\operatorname{Spec}\mathbb{F}_p[x] = \{(0)\} \cup \{(f) : f \in \mathbb{F}_p[x] \text{ irred}\}$.
Recall $\mathbb{Q}[x]$ is a principal ideal domain, so $\operatorname{Spec}\mathbb{Q}[x] = \{(0)\} \cup \{(f) : f \in \mathbb{Q}[x] \text{ irred}\}$.
Finally, recall **Gauss's lemma**: a non-constant polynomial $f \in \mathbb{Z}[x]$ is irreducible if and only if it is irreducible in $\mathbb{Q}[x]$ and it is primitive[1] in $\mathbb{Z}[X]$.

Combining these two calculations, we deduce the above description of $\operatorname{Spec}\mathbb{Z}[x]$.

15.12. PROJ: the analogue of Spec for projective varieties

Recall we associated an affine scheme to a ring, which for $k[x_1, \ldots, x_n]$ recovers $\mathbb{A}^n$. Can we associate a scheme to an $\mathbb{N}$-graded ring, which for $k[x_0, \ldots, x_n]$ with grading by degree recovers $\mathbb{P}^n$?

Let $A = \oplus_{m \geq 0} A_m$ be a graded ring. The **irrelevant ideal** is $A_+ = \oplus_{m>0} A_m$ (in analogy with $(x_0, \ldots, x_n) \subset k[x_0, \ldots, x_n]$ in Section 3.6). Then define

$$\boxed{\operatorname{Proj}(A) = \{\text{homogeneous prime ideals in } A \text{ not containing the irrelevant ideal } A_+\}}$$

with the Zariski topology: closed sets are $\mathbb{V}(I) = \{\wp \in \operatorname{Proj}(A) : \wp \supset I\}$ for all homogeneous ideals $I \subset A$. The basic open sets are $D_f = \{\wp \in \operatorname{Proj}(A) : f \notin \wp\}$ for homogeneous $f \in A$.

Example. For $A = k[x_0, \ldots, x_n]$, the maximal ideals in $\operatorname{Proj} A$ correspond to the (closed) points $\{[a]\} = \mathbb{V}_{\text{classical}}(\mathfrak{m}_a)$ of $\mathbb{P}^n$, where $\mathfrak{m}_a = \langle a_i x_j - a_j x_i : \text{all } i, j \rangle$ (Sec.3.6). The full $\operatorname{Proj} A$ corresponds geometrically to the collection of all the irreducible projective subvarieties $\mathbb{V}_{\text{classical}}(\wp) \subset \mathbb{P}^n$ of $\mathbb{P}^n$.

Example. We will describe blow-ups in terms of Proj in Section 15.13.

We define the **structure sheaf** $\mathcal{O} = \mathcal{O}_X$ on $X = \operatorname{Proj}(A)$: for $U \subset X$ an open subset, $\mathcal{O}(U)$ consists of **compatible** families $\{f_\wp \in \mathcal{O}_\wp\}_{\wp \in U}$, equivalently functions

$$f : U \to \bigsqcup_{\wp \in U} A_{(\wp)}, \quad \wp \mapsto f_\wp,$$

where $\mathcal{O}_\wp \cong A_{(\wp)}$ is the homogeneous localisation which we defined in Section 10.3. Recall $A_{(\wp)}$ consists of all fractions $\frac{F}{G}$ of homogeneous elements of A of the same degree, whose denominator G is not in $\wp$, equivalently $G(\wp) \neq 0 \in \mathbb{K}(\wp) = \operatorname{Frac}(A/\wp)$. Compatibility is defined as before: locally, on a basic neighbourhood D_G, there is a common function $\frac{F}{G} \in A_{(G)} = \mathcal{O}(D_G)$ whose restriction gives the elements $f_\wp$, for $\wp \in D_G$ (here $A_{(G)}$ is the homogeneous localisation at the multiplicative set generated by a homogeneous element G of A, so we formally invert G).

[1] A polynomial is **primitive** if the g.c.d. of the coefficients is a unit.

15.13. THE BLOW-UP AS A PROJ

The modern definition of blow-ups is via the Proj construction. Let $R = k[x_1, \ldots, x_n]$. For an aff.var. $Y \subset \mathbb{A}^n = \operatorname{Specm} R$, with defining ideal $I = \mathbb{I}^h(Y)$, the blow-up of $\mathbb{A}^n$ along Y (i.e. along the ideal I) is

$$\mathrm{B}_Y\mathbb{A}^n = \operatorname{Proj} \bigoplus_{d=0}^{\infty} I^d = \operatorname{Proj}(R \oplus I \oplus I^2 \oplus \cdots)$$

where $I^0 = R$, so the homogeneous coordinate ring is $S = \oplus_{d\geq 0} I^d$. The exceptional divisor is

$$E = \operatorname{Proj} \bigoplus_{d=0}^{\infty} I^d/I^{d+1} = \operatorname{Proj}(R/I \,\oplus\, I/I^2 \,\oplus\, I^2/I^3 \,\oplus \cdots)$$

which can be interpreted as follows: I/I^2 can be thought[1] of as the vector space which is "normal" to Y, and we want to take the projectivisation of this vector space. Compare $\mathbb{P}^n = \mathbb{P}(\mathbb{A}^{n+1})$: we take the irrelevant ideal $J = \langle x_0, x_1, \ldots, x_n\rangle \subset k[x_0, \ldots, x_n]$, then the k-vector space J/J^2 can be identified with $\mathbb{A}^{n+1}$, and to projectivise we take Proj $\oplus_{d\geq 0} J^d/J^{d+1}$. Equivalently, this is the Proj of the symmetric algebra $\operatorname{Sym}_R J/J^2 \equiv k[x_0, \ldots, x_n]$.

Example. For $Y = \{0\}$, $I = \langle x_1, \ldots, x_n\rangle$, we have a surjective hom

$$\varphi : R[y_1, \ldots, y_n] \to S = \oplus I^d/I^{d+1}, \;\; y_i \mapsto x_i.$$

Then $J = \ker\varphi = \langle x_i y_j - x_j y_i\rangle$ defines an aff.var. $\mathbb{V}(J) \subset \mathbb{A}^n \times \mathbb{A}^n$ which is how we originally defined the blow-up $B_0\mathbb{A}^n$ (after projectivising the second $\mathbb{A}^n$ factor, i.e. $\mathbb{V}(J) \subset \mathbb{A}^n \times \mathbb{A}^n$ is the cone of $B_0\mathbb{A}^n \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$).

16. APPENDIX 1: Irreducible decompositions and primary ideals

This Appendix is non-examinable.
Recall, if X is an affine variety, then it has a decomposition into irreducible affine varieties

$$X = X_1 \cup X_2 \cup \cdots \cup X_N \tag{16.1}$$

which is unique up to reordering, provided[2] we impose $X_i \not\subset X_j$ for all $i \neq j$. This implies

$$\mathbb{I}(X) = \mathbb{I}(X_1) \cap \mathbb{I}(X_2) \cap \cdots \cap \mathbb{I}(X_N) \tag{16.2}$$

where $P_j = \mathbb{I}(X_j) \subset R = k[x_1, \ldots, x_n]$ are distinct prime ideals (in particular, radical).
Question. *Can we recover* (16.2) *by algebra methods? (then recover* (16.1) *by taking* $\mathbb{V}(\cdot)$).
The answer is yes, and the aim of this discussion is to explain the following:

[1] A tangent vector $v \in T_p\mathbb{A}^n$ normal to T_pY acts on functions by taking the directional derivative of f at p in the direction v. In the normal space (the quotient of vector spaces T_pX/T_pY), we view v as zero if $v \in T_pY$. By only allowing functions $f \in I$ (i.e. vanishing along Y) we ensure that v acts as zero if $v \in T_pY$, since f does not vary in the T_pY directions. Since differentiation only cares about first order terms, we only care about the quotient class $f \in I/I^2$ (because $d(I^2) \ni d(\sum a_i b_i) = \sum a_i\, db_i + \sum b_i\, da_i = 0$ along Y as the $a_i, b_i \in I$ vanish on Y). So the normal space is the dual vector space $(I/I^2)^* =$ (linear functionals $v : I/I^2 \to k$).

[2] e.g. silly ways to make it non-unique are: take $X_{N+1} = \emptyset$ or $X_{N+1} = \{p\}$ for some $p \in X_N$.

FACT. (Lasker[1]-Noether Theorem) *For any Noetherian ring A, and any ideal $I \subset A$,*

$$I = I_1 \cap \cdots \cap I_N \tag{16.3}$$

where I_j are primary ideals *(Definition 16.1).*
The decomposition is called ***reduced*** *if the $P_j = \sqrt{I_j}$ are all distinct and the I_j are* ***irredundant***[2].
A reduced decomposition always exists, and the P_j are unique up to reordering. The prime ideals P_j are called the ***associated primes*** *of I, denoted*[3]

$$\operatorname{Ass}(I) = \{P_1, \ldots, P_N\}.$$

Moreover, viewing $M = A/I$ as an A-module,

$$\operatorname{Ass}(I) = \{\textit{all annihilators } \operatorname{Ann}_M(m) \subset A \textit{ which are prime ideals of } A\}.$$

Recall $\operatorname{Ann}_M(m) = \{a \in A : am = 0 \in M\}$, so for some non-unique $a_j \in A$,

$$P_j = \operatorname{Ann}_M(\overline{a_j}) = \{r \in A : r \cdot \overline{a_j} = 0 \in M\} = \{r \in A : r \cdot a_j \in I\}.$$

Definition 16.1 (Primary ideals). *$I \subsetneq A$ is a* ***primary ideal*** *if all zero divisors of A/I are nilpotent. Such an I is* ***P-primary*** *if $\sqrt{I} = P$. The decomposition* (16.3) *is a* ***primary decomposition*** *of I.*

Remarks. Being primary is weaker than being prime (in which case zero divisors of A/I are zero).
Exercise.[1] *I primary $\Rightarrow P = \sqrt{I}$ is prime, in fact the smallest prime ideal containing I.*

Examples of primary ideals.

1). The primary ideals of $\mathbb{Z}$ are (0) and (p^m) for p prime, any $m \geq 1$. The (p^m) are (p)-primary.
2). In $k[x,y]$, $I = (x, y^2)$ is (x,y)-primary. Indeed the zero divisors of $k[x,y]/I \cong k[y]/(y^2)$ lie in (y) and are nilpotent since $y^2 = 0$. Notice $(x,y)^2 \subsetneq (x,y^2) \subsetneq (x,y)$, so primary ideals need not be a power of a prime ideal. (Conversely, a power of a prime ideal need not be primary, although it is true for powers of maximal ideals).

Exercise. Show the following are equivalent definitions for I to be primary:

- zero divisors of A/I are nilpotent
- $\forall f, g \in A$, if $fg \in A$ then $f \in I$ or $g \in I$ or both $f, g \in \sqrt{I}$.
- $\forall f, g \in A$, if $fg \in A$ then $f \in I$ or $g^m \in I$ for some $m \in \mathbb{N}$.
- $\forall f, g \in A$, if $fg \in A$ then $f^m \in I$ or $g \in I$ for some $m \in \mathbb{N}$.

Exercise. *I,J both P-primary $\Rightarrow I \cap J$ is P-primary.*
If $I = \cap I_j$ is a primary decomposition with $P_i = \sqrt{I_i} = \sqrt{I_j} = P_j$, then we can replace I_i, I_j with $I_i \cap I_j$ since that is again P_i-primary (by the last exercise). This way, one can always adjust a primary decomposition so that it becomes reduced (see the statement of Lasker-Noether).

Examples of primary decompositions.

[1] This is in fact also the famous chess player, Emanuel Lasker, world chess champion for 27 years.
[2] meaning no smaller subcollection of the I_j gives $I = \cap I_j$.
[3] This unfortunate notation seems to be standard. Allegedly, the Bourbaki group was thinking of "assassins".
[1] **LEMMA.** For any Noetherian ring A,

$$\begin{aligned} \textbf{nilradical of } A \;=\; \operatorname{nil}(A) &\overset{def}{=} \{\text{all nilpotent elements of } A\} \\ &= \text{intersection of the prime ideals of } A \\ \textbf{radical of } I \;=\; \sqrt{I} &\overset{def}{=} \{f \in A : f^m \in I \text{ for some } m\} \\ &= \text{intersection of the prime ideals containing } I \\ &= \text{preimage of } \operatorname{nil}(A/I) \text{ via the quotient hom } A \to A/I \end{aligned}$$

Proof. For the first claim, suppose $f \in A$ is not nilpotent. Let P be an ideal that is maximal (for inclusion) amongst ideals satisfying $f^n \notin P$ for all $n \geq 1$ (using A Noetherian). Then P is prime because: if $xy \in P$ with $x, y \notin P$, then $(x) + P$ and $(y) + P$ are larger than P, hence some $f^n \in (x) + P$, $f^m \in (y) + P$, hence $f^{nm} \in (xy) + P \subset P$, contradiction. So $\operatorname{nil}(A) \subset \cap$(prime ideals), and the converse is easy. The second claim follows by the correspondence theorem: prime ideals in A/I correspond precisely to the prime ideals in A containing I. □

1). $A = \mathbb{Z}$, $I = (n)$, say $n = p_1^{a_1} \cdots p_N^{a_N}$ is the factorization into distinct primes p_j. Then $I = (p_1^{a_1}) \cap \cdots \cap (p_N^{a_N})$ is the primary decomposition. So $I_j = (p_j^{a_j})$ and $P_j = (p_j) = \mathrm{Ann}_{\mathbb{Z}/(n)}(\frac{n}{p_j})$.

2). $I = (y^2, xy) \subset k[x,y]$, here are several possible primary decompositions

$$I = (y) \cap (x,y)^2 = (y) \cap (x, y^2) = (y) \cap (x+y, y^2).$$

In each case, $P_1 = \sqrt{(y)} = (y) = \mathrm{Ann}(x)$ and $P_2 = \sqrt{I_2} = (x,y) = \mathrm{Ann}(y)$.

3). $A = \mathbb{Z}[\sqrt{-5}]$ is an integral domain but not a UFD: unique factorization into irreducibles fails:

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$

where you can check that $2, 3, 1 \pm \sqrt{-5}$ are all irreducibles (but not primes.[1]) Notice that $(1+\sqrt{-5})$ is not primary: $2 \cdot 3 = 0 \in A/(1+\sqrt{-5})$ but the zero divisor 2 is not nilpotent.[2] Whereas (2), (3) are primary.[3] In this case, $I = (6) = I_1 \cap I_2$ for $I_1 = (2)$, $I_2 = (3)$, and[4]

$$P_1 = \sqrt{(2)} = (2, 1-\sqrt{-5}) = \mathrm{Ann}_{A/(6)}(3+3\sqrt{-5})$$
$$P_2 = \sqrt{(3)} = (3, 1-\sqrt{-5}) = \mathrm{Ann}_{A/(6)}(2+2\sqrt{-5}).$$

The original goal of the Lasker-Noether theorem was to recover a "unique factorization" theorem in such situations. Note: it is a unique factorization theorem for ideals, rather than elements.

Exercise.[5] *A Noetherian ⇒ primary decompositions always exist.*

The minimal[6] elements of $\mathrm{Ass}(I)$ are called **minimal prime ideals** or **isolated prime ideals** in I, the others are called **embedded prime ideals** in I. The $\mathbb{V}(P_i) \subset \mathbb{V}(I)$ are called **associated reduced components** of $\mathbb{V}(I)$, and it is called an **embedded** component if $\mathbb{V}(P_i) \neq \mathbb{V}(I)$.

Geometrically, for $X = \mathbb{V}(I)$ and $I \subset R = k[x_1, \ldots, x_n]$, the minimal P_i are the irreducible components $X_i = \mathbb{V}(P_i) = \mathbb{V}(I_i)$, and the embedded P_i are irreducible subvarieties contained inside the irreducible components (if $P_1 \subset P_2$ then $\mathbb{V}(P_1) \supset \mathbb{V}(P_2)$).

Example. $I = (y^2, xy) \subset k[x,y]$ then $I = (y) \cap (x,y)^2$ so $\mathrm{Ass}(I) = \{(y), (x,y)\}$. So $P_1 = (y)$ is minimal, and $P_2 = (x,y)$ is embedded. Geometrically, $\mathbb{V}(I) = X_1 = \{(a,0) : a \in k\} \cong \mathbb{A}^1$ is already irreducible, $\mathbb{V}(y) = \mathbb{V}(I)$ is an associated component, the origin $\mathbb{V}(x,y) = \{(0,0)\} \subsetneq \mathbb{V}(I)$ is an embedded component. Notice $X_2 = \{(0,0)\}$ does not arise in the irreducible decomposition (16.1) since $X_2 \subset X_1$, and in (16.2) we get $\mathbb{I}(X) = (y) = P_1$ because we decomposed $\mathbb{I}(X) = \sqrt{I}$ not I.

GEOMETRIC MOTIVATION.

As you can see from the last example, primary decomposition is not very interesting in classical algebraic geometry (i.e. reduced k-algebras). It becomes important in modern algebraic geometry, when you consider the ring of "functions" $\mathcal{O}(\mathrm{Spec}(A)) = A$ (Section 15.1).

Examples.

1). $I = k[x^2, y]$ and $A = k[x,y]/I$. Then I is P-primary, where $P = (x,y)$ corresponds to the origin $(0,0) \in \mathbb{A}^2$. What do the functions A on $\mathrm{Spec}(A)$ mean geometrically?
Write $f = a_0 + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \text{higher} \in k[x,y]$. Reducing modulo I gives

$$\overline{f} = a_0 + a_{10}x \in A.$$

[1] e.g. $1 \pm \sqrt{-5}$ are zero divisors in $A/(2)$.
[2] brute force: $2^m = (a + b\sqrt{-5})(1+\sqrt{-5}) = (a-5b) + (a+b)\sqrt{-5}$ forces $b = -a$ and $2^m = 6a$, impossible.
[3] e.g. $A/(2)$ has a zero divisor $1+\sqrt{-5}$, but it is nilpotent $(1+\sqrt{-5})^2 = -4 + 2\sqrt{-5} = 0 \in A/(2)$.
[4] by Lasker-Noether, we just need to verify that those annihilators are prime. This holds as both quotients are integral domains: $\mathbb{Z}/3 \cong A/(2, 1-\sqrt{-5})$ via $2 \mapsto \sqrt{-5}$, and $\mathbb{Z}/3 \cong A/(3, 1-\sqrt{-5})$ via $2 \mapsto 2$.
[5] Hints: first show that every ideal is an intersection of **indecomposable ideals** ($I \subset A$ is **indecomposable** if $I = J \cap K$ implies $I = J$ or $I = K$). Do this by considering a maximal element amongst indecomposable ideals (that a maximal element exists uses that A is Noetherian). Then show that for Noetherian A, indecomposable implies primary. For this notice that $I \subset A$ is indecomposable/primary iff $0 \subset A/I$ is indecomposable/primary, so you reduce to studying the case: $fg = 0$ and $\mathrm{Ann}(g) \subset \mathrm{Ann}(g^2) \subset \cdots \subset \mathrm{Ann}(g^m) \subset \cdots$ (again now use that A is Noetherian).
[6] minimal with respect to inclusion. One can show that these are in fact minimal amongst all prime ideals containing I, and all such minimal prime ideals arise in the $\mathrm{Ass}(I)$.

The "values" of f at prime ideals $\wp \in \mathrm{Spec}(A)$ only[1] "see" a_0. But the abstract function $\overline{f} \in A$ also remembers the partial derivative $a_{10} = \partial_x f|_{(0,0)}$. So $\mathrm{Spec}(A)$ should be thought of as a point $(0,0) \in \mathbb{A}^2$ together with the tangent vector ∂_x in the horizontal x-direction.

2). For $I = (x,y)^2 = (x^2, xy, y^2)$, $\overline{f} \in A = k[x,y]/I$ remembers $\partial_x f$ and $\partial_y f$ at zero (namely a_{10}, a_{01}) and thus by linearity it remembers all first order directional derivatives. Thus $\mathrm{Spec}(A)$ should be thought of as the origin $(0,0) \in \mathbb{A}^2$ together with a first order infinitesimal neighbourhood of 0. (Similarly, $\mathrm{Spec}(A)$ for $I = (x,y)^n$ is an $(n-1)$-th order infinitesimal neighbourhood of zero: the ring of functions remembers the Taylor expansion of f up to order $n-1$).

3). $I = (x^2) \subset k[x,y]$ corresponds to the y-axis in $\mathbb{A}^2$ together with a first order infinitesimal neighbourhood of the y-axis. It remembers all coefficients a_{0m}, a_{1m} of f, all $m \geq 0$, so it remembers all values of f and $\partial_x f$ at any point on the y-axis.

4). The primary decomposition $I = (x^2, xy) = (x) \cap (x,y)^2$ corresponds to the y-axis in $\mathbb{A}^2$ together with a first-order neighbourhood of the origin. The fact that $I = (x) \cap (x^2, y)$ is another primary decomposition reflects the geometric fact that if a "function" $f \in A = k[x,y]/I$ remembers all the values on the y-axis, then it automatically remembers all the values of $\partial_y f$ along the y-axis, so the only additional information coming from the first-order neighbourhood of the origin is the horizontal derivative $\partial_x f|_{(0,0)}$ (compare the discussion of (x^2, y) in 1) above).

The remainder of this Section is less important (and non-examinable).

We explain below the last piece of the proof of the Lasker-Noether theorem: why $\mathrm{Ass}(A/I)$ are the prime annihilators of the A-module $M = A/I$.

Lemma 16.2. *If J is a P-primary ideal for I, then $P = \sqrt{J} = \sqrt{\mathrm{Ann}_M(\overline{a})}$ for any $a \in A \setminus J$.*

Proof. If $ra \in J$ then, since J is primary, either $r^m \in J$ (so $r \in \sqrt{J} = P$) or $a \in J$ (false, $a \in A \setminus J$). Thus $\mathrm{Ann}(\overline{a}) \subset P$. Conversely, if $r \in P$ then some $r^m \in J$, so $r^m \in \mathrm{Ann}(\overline{a})$, so $r \in \sqrt{\mathrm{Ann}(\overline{a})}$. □

Exercise. *If $a \in A \setminus P$ then $\mathrm{Ann}_M(\overline{a}) = J$. If $a \in J$ then $\mathrm{Ann}_M(\overline{a}) = A$.*
Exercise. *Show that $\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$. Hence it follows from* (16.3) *that:*

$$\sqrt{I} = P_1 \cap P_2 \cap \cdots \cap P_N.$$

Now, for $I = \cap I_j$, notice that: $\mathrm{Ann}_M(\overline{a}) = \bigcap \mathrm{Ann}_{A/\cap I_j}(\overline{a}) = \bigcap \mathrm{Ann}_{A/I_j}(\overline{a})$ so by the two exercises,

$$\sqrt{\mathrm{Ann}_M(\overline{a})} = \bigcap_j \sqrt{\mathrm{Ann}_{A/I_j}(\overline{a})} = \bigcap_{a \notin I_j} P_j.$$

Exercise. *Let A be a ring, $I_j \subset A$ ideals, $P \subset A$ a prime ideal. Then:*
If $P = \cap J_j$ then $P = J_j$ for some j. If $P \supset \cap J_j$ then $P \supset J_j$ for some j.

By the exercise, it follows that if $\sqrt{\mathrm{Ann}_M(\overline{a})}$ is prime, then it equals some P_j. This is the converse of Lemma 16.2. It also follows by the last two exercises that any prime ideal of A containing I must contain a minimal prime ideal: $P \supset I = \cap I_j$ then $P = \sqrt{P} \supset \cap \sqrt{I_j} = \cap P_j$ so $P \supset P_j$.

Lemma 16.3. *A maximal[2] element of the collection $\{\mathrm{Ann}_M(\overline{a}) : \overline{a} \neq 0 \in M\}$ is a prime ideal in A.*

Proof. Notice that $\overline{a} \neq 0$ ensures that $1 \notin \mathrm{Ann}_M(\overline{a}) \subset A$ are proper ideals. Suppose $P = \mathrm{Ann}(\overline{a})$ is maximal amongst annihilators. If $xy \in P$ and $y \notin P$, then $xy\overline{a} = 0 \in M$, $y\overline{a} \neq 0$. So $P \subset \mathrm{Ann}(\overline{ya})$ must be an equality, by maximality. But $x \in \mathrm{Ann}(\overline{ya})$, so $x \in P$. □

[1] explicitly: $f(\wp) = (f \bmod \wp) = a_0 \in \mathbb{K}(\wp) = \mathrm{Frac}(A/\wp)$ since $x^2 \in I \subset \wp$ implies $x \in \wp$, because $\wp$ is prime.
[2] under inclusion.

For A Noetherian, the Lemma implies[1] that

$$\bigcup_{P_j \in \mathrm{Ass}(I)} P_j = \{\text{all zero divisors of } A/I\}.$$

Lemma 16.4. *For the A-module $M = A/I$,*

$$(P = \mathrm{Ann}_M(m) \text{ is prime, for some } m \in M) \iff (M \text{ contains a submodule } N \text{ isomorphic to } A/P)$$

for example $N = Am \subset M$. Moreover, $P = \mathrm{Ann}_M(n)$ for any $n \in N$.

Proof. The A-module hom $A \to Am, 1 \mapsto m$ by definition has kernel P, so $A/P \cong Am$ as A-mods. As P is prime, A/P has no zero divisors so $an = 0 \in Am$ forces $a \in P$, so $\mathrm{Ann}_M(n) = P$. Conversely an iso $A/P \cong N \subset M$ is a surjective hom $\varphi : A \to N, 1 \mapsto m$ with $P = \ker \varphi = \mathrm{Ann}_M(m)$. □

Lemma 16.5.
1). I is P-primary $\Leftrightarrow \mathrm{Ass}(I) = \{P\}$.
2). If A is Noetherian, and I is P-primary, then $P = \mathrm{Ann}_{A/I}(\beta)$ for some $\beta \in A/I$.

Proof. (1) follows by definition: $I = I$ is a primary decomposition. Lemma 16.3 implies (2). □

Lemma. *For A Noetherian, let $M = A/I$,*

$$\mathrm{Ass}(I) = \{\text{all annhiliators } \mathrm{Ann}_M(\overline{a}) \text{ which are prime ideals in } A\}$$

Remark. Notice we don't need to take the radicals of the annihilators.

Proof. Consider a reduced primary decomposition $I = \cap I_j$, so $P_j = \sqrt{I_j}$ are the elements in $\mathrm{Ass}(I)$. Consider the injective hom[2]

$$\varphi : M = A/I \hookrightarrow \bigoplus A/I_j.$$

By Lemma 16.4 applied to I_i, $A/P \cong N \subset A/I_i$. Notice that $\varphi(M) \cap N \neq \emptyset$ because by irredundancy there is some $m \in \cap_{j \neq i} I_j \setminus I_i$, so $\varphi(m)$ is only non-vanishing in the A/I_i summand. Pick any such $m \in \varphi^{-1}(N \setminus \{0\})$, then φ defines an iso of A-mods $A/I \supset Am \cong A\varphi(m) = N \subset A/I_i$ (by Lemma 16.4, $N = A\varphi(m)$). So A/I also contains an A-submod iso to A/P, so by Lemma 16.4 $P = \mathrm{Ann}_M(m)$. □

17. APPENDIX 2: Differential methods in algebraic geometry

This Appendix is non-examinable.

THE TANGENT SPACE IN DIFFERENTIAL GEOMETRY

In physics, we think of a tangent vector to a smooth manifold M (e.g. a smooth surface) at a point $p \in M$ as the velocity vector $\gamma'(0)$ of a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to M$ passing through $\gamma(0) = p$. Mathematically, we define the tangent space T_pM as the collection of all equivalence classes $[\gamma]$ of smooth curves through $\gamma(0) = p$, identifying two curves if in local coordinates they have the same velocity $\gamma'(0)$. The Taylor expansion[3] of γ at $t = 0$ in local coordinates is

$$\gamma(t) = p + tv + (t^2\text{-terms and higher}) \tag{17.1}$$

so $\gamma(0) = p$, $\gamma'(0) = v$, and $v \in \mathbb{R}^n$ is the tangent vector in local coordinates.
Notice: reducing modulo t^2 we get $\gamma(t) = p + tv \in \mathbb{R}[t]/t^2$, and this determines the pair (p, v).

The curve γ also defines a differential operator: for a smooth function $f : M \to \mathbb{R}$, γ "operates" on f by telling us the rate of change of f along γ at p:

$$f \mapsto \tfrac{\partial}{\partial t}\big|_{t=0} f(\gamma(t)) = D_pf \cdot \gamma'(0) = D_pf \cdot v \in \mathbb{R}.$$

[1] If $ra = 0 \in A/I$, then the maximal annihilator containing $\mathrm{Ann}(\overline{a})$ will be an associated prime ideal containing r. Conversely, if $r \in \cup P_j$, then $r^m \in I_j$ for some j, m, so pick $a \in \cap_{i \neq j} I_i \setminus I_j$ (using irredundancy) then $r^m a = 0 \in A/I$ shows that r is a zero divisor of A/I.

[2] The quotient map $A \to \oplus A/I_j$ is surjective and has kernel $\cap I_j = I$.

[3] Not all smooth functions are equal to their Taylor series (e.g. e^{-1/x^2} has zero Taylor series at $x = 0$). This will not be an issue for us since we only care about the best linear approximation.

So we can also define T_pM as the vector space of **derivations** at p, meaning $\mathbb{R}$-linear maps $L : C^\infty(M) \to \mathbb{R}$ acting on smooth functions and satisfying the Leibniz rule:

$$L(fg) = L(f) \cdot g(p) + f(p) \cdot L(g). \tag{17.2}$$

The γ in (17.1) corresponds to the operator

$$L(f) = D_pf \cdot v = \langle \partial f(p), v \rangle = \sum \partial_{x_i} f(p) \cdot v_i$$

so the inner product between $v = \gamma'(0)$ and the vector $(\partial_{x_1} f, \ldots, \partial_{x_n} f)|_{x=p}$ of partial derivatives.

Example. For $M = \mathbb{R}^n$, $\gamma(t) = (t, 0, \ldots, 0)$ corresponds to the standard basis vector $v = e_1 = (1, 0, \ldots, 0)$ and it operates by $f \mapsto D_pf \cdot e_1 = \frac{\partial f}{\partial x_1}$, so we think of γ as the operator ∂_{x_1}.

Consider the ideal of smooth functions vanishing at p:

$$\mathfrak{m}_p = \mathbb{I}(p) = \{f \in C^\infty(M) : f(p) = 0\}.$$

Then consider the above linear map $L : \mathfrak{m}_p \to \mathbb{R}$ restricted to $\mathfrak{m}_p$. Notice that $L(\mathfrak{m}_p^2) = 0$ by Leibniz (17.2), since f, g vanish at p. Thus we get an $\mathbb{R}$-linear map:

$$\overline{L} : \mathfrak{m}_p/\mathfrak{m}_p^2 \to \mathbb{R}. \tag{17.3}$$

Conversely, given such a linear map $\overline{L}$ can we recover the derivation L? For $f \in C^\infty(M)$, write

$$f = f(p) + (f - f(p)) \in \mathbb{R} \oplus \mathfrak{m}_p. \tag{17.4}$$

A derivation L always vanishes on constant functions: $L(1) = L(1 \cdot 1) = L(1) \cdot 1 + 1 \cdot L(1) = 2L(1)$, so $L(1) = 0$, so by linearity $L(\mathbb{R}) = 0$. So given (17.3), we define L via $L(f) = \overline{L}(f - f(p))$. Is L a derivation? Abbreviating $f(p) = f_p$, $g(p) = g_p$, and using that $\overline{L}$ vanishes on $(f - f_p) \cdot (g - g_p) \in \mathfrak{m}_p^2$,

$$\begin{aligned} L(fg) &= \overline{L}(fg - f_pg_p) \\ &= \overline{L}((f - f_p) \cdot g_p + f_p \cdot (g - g_p) + (f - f_p) \cdot (g - g_p)) \\ &= \overline{L}(f - f_p) \cdot g_p + f_p \cdot \overline{L}(g - g_p) \\ &= L(f) \cdot g_p + f_p \cdot L(g). \end{aligned} \tag{17.5}$$

Example. For $M = \mathbb{R}^n$, $p = 0$, then $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \mathbb{R}\overline{x_1} + \cdots + \mathbb{R}\overline{x_n}$ (as a vector space), and knowing what $\overline{L}$ does on each $\overline{x_i}$ determines L. Indeed $L = \sum v_i \partial_{x_i}$ corresponds to $\overline{L}(\overline{x_i}) = v_i$.

So we can define T_pM as the vector space of linear functionals (17.3):

$$T_pM \cong (\mathfrak{m}_p/\mathfrak{m}_p^2)^*.$$

Suppose we Taylor expand $f \in C^\infty(M)$ at p in local coordinates,

$$f = f(p) + \sum a_i(x_i - p_i) + \sum a_{ij}(x_i - p_i)(x_j - p_j) + (\text{higher order } (x_i - p_i)).$$

where $a_i = \partial_{x_i} f|_{x=p} \in \mathbb{R}$. Composing with (17.1) and dropping t^2 terms:

$$f \circ \gamma(t) = f(p) + \sum a_i v_i t \in \mathbb{R}[t]/t^2. \tag{17.6}$$

We recover p, v by taking $f = x_i$: $f \circ \gamma = p_i + v_i t \in \mathbb{R}[t]/t^2$. So each γ defines an $\mathbb{R}$-algebra hom $C^\infty(M) \to \mathbb{R}[t]/t^2$, $f \mapsto f \circ \gamma$ and such a hom φ determines p, v via $\varphi(x_i) = p_i + v_i t$. Thus[1]

$$\varphi(f) = \varphi[f(p) + \textstyle\sum a_i(x_i - p_i) + \cdots] = f(p) + \textstyle\sum a_i \overline{L}(x_i - p_i)\, t. \tag{17.7}$$

So $L, \overline{L}, \varphi$ completely determined each other. So via $v_i t = \varphi(x_i - p_i)$ we get:

$$T_pM = \{\varphi \in \mathrm{Hom}_{\mathbb{R}\text{-alg}}(C^\infty(M), \mathbb{R}[t]/t^2) : (\varphi(x_i) \bmod t) = p_i \in \mathbb{R}[t]/t\}.$$

Suppose now that the manifold is already embedded in Euclidean space, so $M \subset \mathbb{R}^n$ (e.g. the unit sphere $S^2 \subset \mathbb{R}^3$), then we can think of T_pM as sitting inside $\mathbb{R}^n$ as follows.

Suppose $P : \mathbb{R}^m \hookrightarrow M \subset \mathbb{R}^n$ is a local parametrization of M, with $P(p_0) = p$.

Example. Spherical coordinates $(\theta, \varphi) \in \mathbb{R}^2$ give $P(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in S^2 \subset \mathbb{R}^3$.

[1] $\mathbb{R}$-algebra homs send 1 to 1, so $C^\infty \supset \mathbb{R} \cdot 1 \to \mathbb{R} \cdot 1 \subset \mathbb{R}[t]/t^2$ is the identity map.

A local curve $\gamma(t) = p_0 + v_0 t + \cdots \in \mathbb{R}^m$ then gives rise to a curve $P \circ \gamma(t) = p + vt + \cdots \in \mathbb{R}^n$. By the chain rule, $v = \partial_t|_{t=0} P \circ \gamma = D_{p_0}P \cdot v_0$. So local tangent vectors $v_0 \in \mathbb{R}^m = T_{p_0}\mathbb{R}^m$ correspond to vectors $D_{p_0}P \cdot v \in \mathbb{R}^n$ sitting inside $\mathbb{R}^n$. So

$$T_pM = \mathrm{Image}(D_{p_0}P) = D_{p_0}P \cdot \mathbb{R}^m \subset \mathbb{R}^n.$$

This is a vector subspace of $\mathbb{R}^n$. Finally, if M is locally defined by the vanishing of functions

$$M = \mathbb{V}(F_1, \dots, F_N) \text{ locally near } p$$

(e.g. $S^2 \subset \mathbb{R}^3$ is defined by $F = X^2 + Y^2 + Z^2 - 1 = 0$), then for any curve $\gamma \subset M \subset \mathbb{R}^n$, all $F_j(\gamma(t)) = 0$. Differentiating via the chain rule: all $D_pF_j \cdot \gamma'(0) = 0$. Equivalently:

$$\gamma'(0) = v \in \ker D_pF_1 \cap \cdots \cap \ker D_pF_N. \tag{17.8}$$

Conversely, a γ satisfying (17.8) is a curve $\gamma(t)$ on which each F_j vanishes to second order or higher. So T_pM can be identified with the vector subspace $\ker D_pF_1 \cap \cdots \cap \ker D_pF_N \subset \mathbb{R}^n$. The affine plane $p + T_pM \subset \mathbb{R}^n$ is the plane which best approximates $M \subset \mathbb{R}^n$ at p and it is the plane which we usually visualise in pictures as the tangent space.

Since γ and $\ell(t) = p + tv$ are equal modulo t^2, i.e. equivalent curves in $\mathbb{R}^n$,

$$p + T_pM = \bigcup \{\text{lines } \ell : \ell(t) = p + tv \in \mathbb{R}^n, \text{ each } F_j \circ \ell \text{ vanishes to order } \geq 2 \text{ at } t = 0\} \subset \mathbb{R}^n.$$

These ℓ are not curves in M usually, they are curves in $\mathbb{R}^n$. So we are describing T_pM as a vector subspace of $T_p\mathbb{R}^n$ by deciding which tangent vectors of $\mathbb{R}^n$ are also tangent to M. The above describes $p + T_pM$ as the union of straight lines which "touch" M at p (meaning, to order at least two, indeed tangent lines arise as limits of secant lines which intersect M at least twice near p).

One sometimes abbreviates by d_pf the linear part of the Taylor expansion of f at p, so

$$d_pf = \textstyle\sum \partial_{x_i} f(p) \cdot (x_i - p_i). \tag{17.9}$$

In this notation, the affine plane $p + T_pM \subset \mathbb{R}^n$ can be described succinctly as:

$$p + T_pM = \mathbb{V}(d_pF_1, \dots, d_pF_N) \subset \mathbb{R}^n.$$

THE TANGENT SPACE IN ALGEBRAIC GEOMETRY

For X an affine variety, recall the stalk $\mathcal{O}_{X,p} = k[X]_{\mathbb{I}(p)}$ consists of germs of regular functions at p, and this is a local ring whose unique maximal ideal is:

$$\mathfrak{m}_p = \mathbb{I}(p) \cdot \mathcal{O}_{X,p} = \{\tfrac{g}{h} : g, h \in k[X], g(p) = 0, h(p) \neq 0\}.$$

A k-**algebra** A is a k-vector space which is also a ring (commutative with 1), such that the operations are compatible in the obvious way. So in particular, A contains a copy of $k = k \cdot 1$.

A k-**algebra homomorphism** $\varphi : A \to B$ means: φ is k-linear and φ is a ring hom (in particular, this requires $\varphi(1) = 1$). So in particular φ is the identity map on $k \cdot 1 \to k \cdot 1$.

A k-**derivation** $L \in \mathrm{Der}_k(A, M)$ from a k-algebra A to an A-module M means a k-linear map $A \to M$ satisfying the Leibniz rule $L(ab) = L(a)b + aL(b)$.

Theorem 17.1. *Let $X = \mathbb{V}(F_1, \dots, F_N) \subset \mathbb{A}^n$. The following definitions are equivalent:*[1]

(1) *Writing $\ell_v(t) = p + tv$ for the straight line in $\mathbb{A}^n$ through p with velocity v,*

$$\boxed{p + T_pX = \bigcup \{\ell_v : \text{all } F_j(\ell_v(t)) \text{ vanish to order } \geq 2 \text{ at } t = 0\} \subset \mathbb{A}^n}$$

(2) *Recall the notation $d_pf = \sum \partial_{x_i} f(p) \cdot (x_i - p_i)$. Then $p + T_pX$ is an intersection of hyperplanes:*

$$\boxed{p + T_pX = \mathbb{V}(d_pF_1, \dots, d_pF_N) \subset \mathbb{A}^n}$$

(3) *Recall the notation $D_pf \cdot v = \sum \partial_{x_i} f(p) \cdot v_i$. Then T_pX is the vector space*

$$\boxed{T_pX = \ker D_pF_1 \cap \cdots \cap \ker D_pF_N \subset k^n}$$

[1] *Clarification. What we called T_pX in Section 13.1 corresponds to $p + T_pX$ in this Section (we now want T_pX to denote the vector space not the translated affine plane).*

(4) *Let $\mathrm{Jac}(F) = (\frac{\partial F_i}{\partial x_j})$ be the Jacobian matrix of $F = (F_1, \dots, F_N) : \mathbb{A}^n \to \mathbb{A}^N$, so $X = F^{-1}(0)$.*

$$\boxed{T_pX = \ker \mathrm{Jac}(F)}$$

(5) *Viewing k as an $\mathcal{O}_{X,p}$-module via $\mathbb{K}(p) = \mathcal{O}_{X,p}/\mathfrak{m}_p \cong k$, $\frac{g}{h} \mapsto \frac{g(p)}{h(p)}$,*

$$\boxed{T_pX = \mathrm{Der}_k(\mathcal{O}_{X,p}, k)}$$

(6) *The* ***cotangent space*** *at p is the k-vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$. Its dual is*

$$\boxed{T_pX = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*}$$

(7)
$$\boxed{T_pX = \mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,p}, k[t]/t^2)}$$

Remark. (6) *is the official definition. In scheme theory one replaces k by $\mathbb{K}(\wp) = \mathrm{Frac}(\mathcal{O}_{X,\wp}/\wp)$.*

Proof. We show (1)⇔(2). Note $F_j(\ell(0)) = F_j(p) = 0$ as $p \in X$. So ($F_j(\ell(t)) = $ order t^2) ⇔ (the derivative at 0 vanishes) ⇔ (the linear part d_pF_j in the Taylor series vanishes at $x = \ell(t) = p + tv$).
We show (1)⇔(3): $\partial_t|_{t=0} F_j(\ell(t)) = 0 \Leftrightarrow D_pF_j \cdot \ell'(0) = 0 \Leftrightarrow \sum \partial_{x_i} F_j(p) \cdot v_i = 0 \Leftrightarrow v \in \bigcap \ker D_pF_j$.
(alternatively (2)⇔(3) since $d_pF_j(\ell(t)) = d_pF_j(p + tv) = \sum \partial_{x_i} F_j(p) \cdot tv_i$).
That (3)⇔(4) is clear: the rows of the matrix $\mathrm{Jac}(F)$ are the linear functionals D_pF_i.
Now (5) ⇔ (6): derivations $L : \mathcal{O}_{X,p} \to k$ vanish on $k \cdot 1$ and $\mathfrak{m}_p^2$ by Leibniz (17.2). Just as (17.4),

$$\mathcal{O}_{X,p} \cong k \oplus \mathfrak{m}_p$$

as k-vector spaces, and $\mathfrak{m}_p \cong (\mathfrak{m}_p/\mathfrak{m}_p^2) \oplus \mathfrak{m}_p^2$. So, arguing as in (17.5), L is determined by a k-linear

$$\overline{L} : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k.$$

Now (6) ⇔ (7). Let $\varphi : \mathcal{O}_{X,p} \to k[t]/t^2$ be a k-alg hom $\varphi : \mathcal{O}_{X,p} \to k[t]/t^2$.
Claim. $\varphi(\mathfrak{m}_p) \subset (t)$.
Sub-proof. Compose φ with the quotient map $k[t]/t^2 \to k[t]/t \cong k$ to get $\overline{\varphi} : \mathcal{O}_{X,p} \to k$. Since $\varphi(1) = 1$, $\overline{\varphi}$ is surjective, so $\mathcal{O}_{X,p}/\ker \overline{\varphi} \cong k$. So $\ker \overline{\varphi} \subset \mathcal{O}_{X,p}$ is a maximal ideal so it must equal the unique maximal ideal $\mathfrak{m}_p$. Finally $\overline{\varphi}(\mathfrak{m}_p) = 0$ implies $\varphi(\mathfrak{m}_p) \subset (t)$ □
So $\varphi(f - f(p)) \in (t)$. We recover $\overline{L}$ via $\varphi(f - f(p)) = \overline{L}(f - f(p))\, t$. So:

$$\varphi(f) = \varphi\,[f(p) + (f - f(p))] = f(p) + \overline{L}(f - f(p))\, t \in k[t]/t^2.$$

Now (3) ⇔ (7): the analogue of (17.6), for $f \in k[X]$, is that

$$f(\ell(t)) = f(p + tv) = f(p) + \textstyle\sum \partial_{x_i} f(p) \cdot v_i\, t = \varphi(f) \in k[t]/t^2$$

defines a k-alg hom $\varphi : k[X] \to k[t]/t^2$. Indeed,

$$\varphi(fg) = f(p)g(p) + \textstyle\sum (\partial_{x_i} f(p) \cdot g(p) + f(p) \cdot \partial_{x_i} g(p)) \cdot v_i\, t = \varphi(f) \cdot \varphi(g) \text{ modulo } t^2.$$

Conversely, given φ, define v_i via $\varphi(\overline{x_i} - p_i) = v_i\, t$. Then since $\overline{F_j} = 0 \in k[X]$ (by definition $k[X] = k[x_1, \dots, x_n]/\sqrt{\langle F_1, \dots, F_N \rangle}$), we have $\varphi(\overline{F_j}) = 0$. So, using $F_j(p) = 0$ and $t^2 = 0$, we get

$$0 = \varphi(\overline{F_j}) = \varphi\,[F_j(p) + \textstyle\sum \partial_{x_i} F_j(p) \cdot (x_i - p_i) + (\text{terms in } \mathbb{I}(p)^2)] = \sum \partial_{x_i} F_j(p) \cdot v_i\, t. \quad \square$$

Lemma 17.2. *For $X = \mathbb{V}(J) \subset \mathbb{A}^n$, let $\mathcal{I}_p = \mathbb{I}(p) \cdot k[X] \subset k[X]$ then*

$$\boxed{\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \mathcal{I}_p/\mathcal{I}_p^2 \cong \mathbb{I}(p) \,/\, (\mathbb{I}(p)^2 + J)}$$

Proof. Apply the third isomorphism theorem[1] using that $J \subset \mathbb{I}(p)$ since $p \in X$. □

Theorem 17.3. *The disjoint union TX of all tangent spaces T_pX, as we vary $p \in X$, is:*

$$\boxed{TX = \mathrm{Hom}_{k\text{-alg}}(k[X], k[t]/t^2)} \quad (\textit{i.e. morphisms } \mathrm{Spec}(k[t]/t^2) \to X)$$

[1] For R-modules $S \subset M \subset B$ ("small,medium,big"), $B/M \cong (B/S)/(M/S)$. Apply this to $J \subset \mathbb{I}(p)^2 + J \subset \mathbb{I}(p)$.

Proof. Given a k-algebra hom $\varphi : k[X] \to k[t]/t^2$, compose with the quotient $k[t]/t^2 \to k[t]/t \cong k$ to get a k-alg hom $\overline{\varphi} : k[X] \to k$. This is surjective (since $1 \mapsto 1$) so the kernel is a maximal ideal of $k[X]$ (as $k[X]/\ker \cong k$). But the maximal ideals of $k[X]$ are precisely the $\mathbb{I}(p)$ for $p \in X$. Thus $\overline{\varphi}(\mathbb{I}(p)) = 0$, so $\varphi(\mathbb{I}(p)) \subset (t)$. Localising φ at $\mathbb{I}(p)$, gives $\varphi : \mathcal{O}_{X,p} \to k[t]/t^2$. □

Exercise. For a k-alg A, the **module of Kähler differentials** is the A-mod $\Omega_{A/k}$ generated over A by the symbols df for all $f \in A$, modulo the relations making

$$d : A \to \Omega_{A/k}, f \mapsto df$$

a k-derivation.[1] For any k-mod M, show there's a natural iso

$$\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M),\ L \mapsto (\Omega_{A/k} \to M, df \mapsto L(f)).$$

If A is also a local ring, with max ideal $\mathfrak{m}$ and residue field $A/\mathfrak{m} \cong k$, show[2] that there is an isomorphism

$$\mathfrak{m}/\mathfrak{m}^2 \cong \Omega_{A/k} \otimes_A k,\ f \mapsto df.$$

Denote $\Omega_{X,p} = \Omega_{\mathcal{O}_{X,p}/k}$ for affine X. Show that[3]

$$\boxed{\mathfrak{m}_p/\mathfrak{m}_p^2 \cong \Omega_{X,p} \otimes_{\mathcal{O}_{X,p}} k,\ f \mapsto df}$$

$$\boxed{\mathrm{Der}_k(\mathcal{O}_{X,p}, k) \cong \mathrm{Hom}_{\mathcal{O}_{X,p}}(\Omega_{X,p}, k),\ \tfrac{\partial}{\partial x_j}|_{x=p} \mapsto (dx_j)^*} \tag{17.10}$$

where $k \cong \mathcal{O}_{X,p}/\mathfrak{m}_p = \mathbb{K}(p)$ as $\mathcal{O}_{X,p}$-mod, and $(dx_j)^*$ is defined by $(dx_j)^*(dx_i) = dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$.

Remark. Globally, TX and Ω_X are sheaves (the tangent sheaf and the cotangent sheaf), and (17.10) says they are dual in the sense that:

$$TX = \mathrm{Der}(\mathcal{O}_X) \cong \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X).$$

The non-singular points of X are in fact those where $\Omega_{X,p}$ is a free $\mathcal{O}_{X,p}$-module, i.e. where Ω_X is a vector bundle.

Example. We describe $T_p\mathbb{A}^n = \mathbb{A}^n$.

Using (1): $\mathbb{I}(\mathbb{A}^n) = \{0\}$ and $(0 \circ \ell)(t)$ vanishes to infinite order for $\ell(p) = p + tv$, any $v \in \mathbb{A}^n$.
Using (2), (3) or (4): $\mathbb{I}(\mathbb{A}^n) = \{0\}$ so $\ker D_p 0 = \ker 0 = \mathbb{A}^n$.
Using (5): $\mathcal{O}_{\mathbb{A}^n,p} = \{f = \frac{g}{h} : h(p) \neq 0\} \subset k(x_1, \dots, x_n)$, so $\mathrm{Der}_k(\mathcal{O}_{\mathbb{A}^n,p}, k) \cong k\, L_1 \oplus \cdots \oplus k\, L_n$ where

$$L_j = \tfrac{\partial}{\partial x_j}|_{x=p}.$$

Using (6): $\mathfrak{m}_p = (x_1 - p_1, \dots, x_n - p_n) \cdot \mathcal{O}_{X,p} = \{\frac{g}{h} : g(p) = 0, h(p) \neq 0\} \subset k(x_1, \dots, x_n)$. Thus $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong k\, e_1 \oplus \cdots \oplus k\, e_n \cong k^n$ as vector spaces where the basis is $e_i = \overline{x_i - p_i}$. Thus

$$(\mathfrak{m}_p/\mathfrak{m}_p^2)^* \cong k\, \overline{L}_1 \oplus \cdots \oplus k\, \overline{L}_n \cong k^n$$

using the dual basis $\overline{L}_j = \frac{\partial}{\partial x_j}|_{x=p} : \mathfrak{m}_p/\mathfrak{m}_p^2 \to k$.

Using (7): $\mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,p}, k[t]/t^2) \cong k\, \varphi_1 \oplus \cdots \oplus k\, \varphi_n$ where $\varphi_j(f) = p + \overline{L}_j(f)\, t$.
Using (17.10): $\Omega_{X,p} \otimes_{\mathcal{O}_{X,p}} k \cong k\, dx_1 \oplus \cdots \oplus k\, dx_n$.

Exercise. Describe T_pX for the cuspidal cubic $X = \mathbb{V}(y^2 - x^3)$ at $p = 0$. Show that by the Lemma, $\mathfrak{m}_p/\mathfrak{m}_p^2 \cong (x, y)/(x^2, xy, y^2, y^2 - x^3) \cong k\overline{x} \oplus k\overline{y}$, and $\Omega_{X,p} \otimes_{\mathcal{O}_{X,p}} k = k\, d\overline{x} \oplus k\, d\overline{y}$.

[1] so d is k-linear and $d(fg) = f(dg) + (df)g$.
[2] To show injectivity it may be easier to show surjectivity of the dual map $\mathrm{Hom}_k(\Omega_{A/k}, k) \to \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$. If $a \in A$ equals $c + m \in k \oplus \mathfrak{m}$, consider $L(a) = \overline{L}(m)$ for $\overline{L} \in (\mathfrak{m}/\mathfrak{m}^2)^*$.
[3] For $f : X \to k$ think of df as the linear functional $D_pf : T_pX \to T_{f(p)}k \cong k$. Such D_pf satisfy relations, e.g. in $\mathbb{V}(y^2 - x^3)$, $D_p(y^2 - x^3) = 0$ implies $2p_2\, dy - 3p_1^2\, dx = 0$. The $\cdot \otimes_{\mathcal{O}_{X,p}} k$ just means evaluate coefficient functions at p.