Professor Joyce **C3.3 Differentiable Manifolds** **MT 2018**

Initial problem sheet

Here is an initial problem sheet, for those who want to get started before the classes get going. The sheet is entirely voluntary, there will be no classes on it, and you should not hand answers in or ask people to mark them. Model answers are available on the Mathematics Institute web site.

1. The n-sphere is $\mathcal{S}^n = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\}$. It has an atlas $\{(U_1,\phi_1),(U_2,\phi_2)\}$ with two charts, where $U_1 = U_2 = \mathbb{R}^n$, $\phi_1(U_1) = \mathcal{S}^n \setminus \{(-1,0,\ldots,0)\}$, $\phi_2(U_2) = \mathcal{S}^n \setminus \{(1,0,\ldots,0)\}$, and ϕ_1,ϕ_2 are the inverses of

$$\begin{aligned} \phi_1^{-1} &: (x_0,\ldots,x_n) \longmapsto \tfrac{1}{1+x_0}(x_1,\ldots,x_n) = (y_1,\ldots,y_n) \\ \phi_2^{-1} &: (x_0,\ldots,x_n) \longmapsto \tfrac{1}{1-x_0}(x_1,\ldots,x_n) = (z_1,\ldots,z_n). \end{aligned}$$

Show that $\mathcal{S}^n$ is a Hausdorff, second countable topological space. Compute the transition function $\phi_2^{-1}\circ\phi_1$ between (U_1,ϕ_1) and (U_2,ϕ_2), and show that it is smooth with smooth inverse.

Thus $\{(U_1,\phi_1),(U_2,\phi_2)\}$ is an atlas on $\mathcal{S}^n$, which extends to a unique maximal atlas, making $\mathcal{S}^n$ into a smooth n-dimensional manifold.

2. The n-dimensional projective space $\mathbb{RP}^n$ is the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$. Points in $\mathbb{RP}^n$ are written $[x_0,x_1,\ldots,x_n]$ for $(x_0,\ldots,x_n)$ in $\mathbb{R}^{n+1}\setminus\{0\}$, where $[x_0,\ldots,x_n] = \mathbb{R}\cdot(x_0,\ldots,x_n) \subseteq \mathbb{R}^{n+1}$, and $[\lambda x_0,\ldots,\lambda x_n] = [x_0,\ldots,x_n]$ for $\lambda \in \mathbb{R}\setminus\{0\}$. It has the quotient topology induced from the surjective projection $\pi : \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{RP}^n$, $\pi : (x_0,\ldots,x_n) \mapsto [x_0,\ldots,x_n]$.

Define a chart (V_i,ψ_i) on $\mathbb{RP}^n$ for $i = 0,\ldots,n+1$ by $V_i = \mathbb{R}^n$ and

$$\psi_i(y_1,\ldots,y_n) = [y_1,\ldots,y_{i-1},1,y_i,\ldots,y_n].$$

Compute the transition functions $\psi_j^{-1}\circ\psi_i$ between (V_i,ψ_i) and (V_j,ψ_j), for $0 \le i < j \le n+1$, and that they are smooth with smooth inverses.

Thus $\{(V_i,\psi_i) : i = 0,\ldots,n\}$ is an atlas on $\mathbb{RP}^n$, which extends to a unique maximal atlas, making $\mathbb{RP}^n$ into a smooth n-dimensional manifold.

3. Define $f : \mathcal{S}^n \to \mathbb{RP}^n$ by $f(x_0,\ldots,x_n) = [x_0,\ldots,x_n]$. Show that f is a smooth surjective map of differentiable manifolds, and that for each $y \in \mathbb{RP}^n$, the inverse image $f^{-1}(y)$ consists of two points.