Professor Joyce **C3.3 Differentiable Manifolds** **MT 2018**

Solutions to initial problem sheet

1. The n-sphere is $\mathcal{S}^n = \{(x_0,\ldots,x_n) \in \mathbb{R}^{n+1} : x_0^2+\cdots+x_n^2 = 1\}$. It has an atlas $\{(U_1,\phi_1),(U_2,\phi_2)\}$ with two charts, where $U_1 = U_2 = \mathbb{R}^n$, $\phi_1(U_1) = \mathcal{S}^n \setminus \{(-1,0,\ldots,0)\}$, $\phi_2(U_2) = \mathcal{S}^n \setminus \{(1,0,\ldots,0)\}$, and ϕ_1,ϕ_2 are the inverses of

$$\phi_1^{-1} : (x_0,\ldots,x_n) \longmapsto \tfrac{1}{1+x_0}(x_1,\ldots,x_n) = (y_1,\ldots,y_n)$$
$$\phi_2^{-1} : (x_0,\ldots,x_n) \longmapsto \tfrac{1}{1-x_0}(x_1,\ldots,x_n) = (z_1,\ldots,z_n).$$

Show that $\mathcal{S}^n$ is a Hausdorff, second countable topological space. Compute the transition function $\phi_2^{-1}\circ\phi_1$ between (U_1,ϕ_1) and (U_2,ϕ_2), and show that it is smooth with smooth inverse.

Thus $\{(U_1,\phi_1),(U_2,\phi_2)\}$ is an atlas on $\mathcal{S}^n$, which extends to a unique maximal atlas, making $\mathcal{S}^n$ into a smooth n-dimensional manifold.

Answer. To show $\mathcal{S}^n$ is Hausdorff, suppose $(x_0,\ldots,x_n),(y_0,\ldots,y_n) \in \mathcal{S}^n$ are distinct. Then $x_i \neq y_i$ for some $i = 0,\ldots,n$. If $x_i < y_i$, pick $c \in (x_i,y_i)$ and set

$$U = \{(z_0,\ldots,z_n) \in \mathcal{S}^n : z_i < c\}, \quad V = \{(z_0,\ldots,z_n) \in \mathcal{S}^n : z_i > c\}.$$

Then U,V are disjoint open sets in $\mathcal{S}^n$ with $x \in U$, $y \in V$. If $x_i > y_i$, swap U,V. Thus $\mathcal{S}^n$ is Hausdorff.

To show $\mathcal{S}^n$ is second countable, note that $\mathbb{R}^{n+1}$ is second countable, since

$$\mathcal{B} = \{(a_0,b_0)\times(a_1,b_1)\times\cdots\times(a_n,b_n) : a_i,b_i \in \mathbb{Q},\ \ a_i < b_i\}$$

is a countable basis for its topology. Hence $\{U \cap \mathcal{S}^n : U \in \mathcal{B}\}$ is a countable basis for the topology of $\mathcal{S}^n$.

The transition map is $\phi_2^{-1}\circ\phi_1 : \mathbb{R}^n\setminus\{0\} \to \mathbb{R}^n\setminus\{0\}$ mapping $(y_1,\ldots,y_n) \longmapsto (z_1,\ldots,z_n) = \frac{1}{y_1^2+\cdots+y_n^2}(y_1,\ldots,y_n)$. This is clearly smooth. The inverse $(\phi_2^{-1}\circ\phi_1)^{-1}$ is given by the same formula, and is also smooth.

2. The n-dimensional projective space $\mathbb{RP}^n$ is the set of 1-dimensional vector subspaces of $\mathbb{R}^{n+1}$. Points in $\mathbb{RP}^n$ are written $[x_0, x_1, \ldots, x_n]$ for $(x_0, \ldots, x_n)$ in $\mathbb{R}^{n+1} \setminus \{0\}$, where $[x_0, \ldots, x_n] = \mathbb{R} \cdot (x_0, \ldots, x_n) \subseteq \mathbb{R}^{n+1}$, and $[\lambda x_0, \ldots, \lambda x_n] = [x_0, \ldots, x_n]$ for $\lambda \in \mathbb{R} \setminus \{0\}$. It has the quotient topology induced from the surjective projection $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$, $\pi : (x_0, \ldots, x_n) \mapsto [x_0, \ldots, x_n]$.
Define a chart (V_i, ψ_i) on $\mathbb{RP}^n$ for $i = 0, \ldots, n+1$ by $V_i = \mathbb{R}^n$ and

$$\psi_i(y_1, \ldots, y_n) = [y_1, \ldots, y_{i-1}, 1, y_i, \ldots, y_n].$$

Compute the transition functions $\psi_j^{-1} \circ \psi_i$ between (V_i, ψ_i) and (V_j, ψ_j), for $0 \le i < j \le n+1$, and that they are smooth with smooth inverses.
Thus $\{(V_i, \psi_i) : i = 0, \ldots, n\}$ is an atlas on $\mathbb{RP}^n$, which extends to a unique maximal atlas, making $\mathbb{RP}^n$ into a smooth n-dimensional manifold.

Answer. The transition function from (V_i, ψ_i) and (V_j, ψ_j) for $i < j$ is

$$\psi_j^{-1} \circ \psi_i : \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_{j-1} \neq 0\} \longmapsto \{(z_1, \ldots, z_n) \in \mathbb{R}^n : z_i \neq 0\},$$
$$\psi_j^{-1} \circ \psi_i : (y_1, \ldots, y_n) \longmapsto \left(\tfrac{y_1}{y_{j-1}}, \ldots, \tfrac{y_{i-1}}{y_{j-1}}, \tfrac{1}{y_{j-1}}, \tfrac{y_i}{y_{j-1}}, \ldots, \tfrac{y_{j-2}}{y_{j-1}}, \tfrac{y_j}{y_{j-1}}, \ldots, \tfrac{y_n}{y_{j-1}}\right).$$

This is clearly smooth, with smooth inverse given by

$$(\psi_j^{-1} \circ \psi_i)^{-1} : (z_1, \ldots, z_n) \longmapsto \left(\tfrac{z_1}{z_i}, \ldots, \tfrac{z_{i-1}}{z_i}, \tfrac{z_{i+1}}{z_i}, \ldots, \tfrac{z_{j-1}}{z_i}, \tfrac{1}{z_i}, \tfrac{z_j}{z_i}, \ldots, \tfrac{z_n}{z_i}\right).$$

3. Define $f : \mathcal{S}^n \to \mathbb{RP}^n$ by $f(x_0, \ldots, x_n) = [x_0, \ldots, x_n]$. Show that f is a smooth surjective map of differentiable manifolds, and that for each $y \in \mathbb{RP}^n$, the inverse image $f^{-1}(y)$ consists of two points.

Answer. Note that f is the composition $\mathcal{S}^n \xhookrightarrow{\iota} \mathbb{R}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{RP}^n$, where ι is the inclusion and π the projection. But ι, π are continuous by definition of the topologies on $\mathcal{S}^n, \mathbb{RP}^n$, so f is continuous.
To show that f is smooth, consider for example the charts (U_1, ϕ_1) on $\mathcal{S}^n$ and (V_i, ψ_i) on $\mathbb{RP}^n$ for $i > 0$. Then we have a diagram

$$\begin{array}{ccc}
(f \circ \phi_1)^{-1}(\psi_i(V_i)) = \{(y_1, \ldots, y_n) \in \mathbb{R}^n : y_i \neq 0\} & \xrightarrow[\phi_1|\cdots]{} & \mathcal{S}^n \\
\downarrow \psi_i^{-1} \circ f \circ \phi_1 & & f \downarrow \\
V_i = \mathbb{R}^n & \xrightarrow{\psi_i} & \mathbb{RP}^n.
\end{array}$$

Here for $(y_1,\dots,y_n)\in(f\circ\phi_1)^{-1}(\psi_i(V_i))$, if $\phi_1(y_1,\dots,y_n)=(x_0,\dots,x_n)$, and $f\circ\phi_1(y_1,\dots,y_n)=[x_0,\dots,x_n]$, and $\psi_i^{-1}\circ f\circ\phi_1(y_1,\dots,y_n)=(z_1,\dots,z_n)$, then

$$(y_1,\dots,y_n)=\frac{1}{1+x_0}(x_1,\dots,x_n),$$
$$(z_1,\dots,z_n)=\big(\tfrac{x_0}{x_i},\dots,\tfrac{x_{i-1}}{x_i},\tfrac{x_{i+1}}{x_i},\dots,\tfrac{x_n}{x_i}\big).$$

Solving the first equation using $x_0^2+\dots+x_n^2=1$ gives

$$x_0=\frac{1-y_1^2-\dots-y_n^2}{1+y_1^2+\dots+y_n^2},\quad 1+x_0=\frac{2}{1+y_1^2+\dots+y_n^2},$$
$$x_i=\frac{2y_i}{1+y_1^2+\dots+y_n^2},\qquad i=1,\dots,n.$$

Putting all this together shows that $\psi_i^{-1}\circ f\circ\phi_1$ maps

$$\psi_i^{-1}\circ f\circ\phi_1:(y_1,\dots,y_n)\mapsto\big(\tfrac{1-y_1^2-\dots-y_n^2}{2y_i},\tfrac{y_1}{y_i},\dots,\tfrac{y_{i-1}}{y_i},\tfrac{y_{i+1}}{y_i},\dots,\tfrac{y_n}{y_i}\big).$$

This is smooth on $y_i\neq 0$. Similarly we show that the transition functions for each chart from $\big\{(U_1,\phi_1),(U_2,\phi_2)\big\}$ and each chart from $\big\{(V_i,\psi_i):i=0,\dots,n\big\}$ is smooth. So f is smooth on the charts of atlases for $\mathcal{S}^n,\mathbb{RP}^n$, and thus it is smooth.

To see that $f^{-1}([y_0,\dots,y_n])$ is 2 points, note that for a fixed representative $(y_0,\dots,y_n)$ for $[y_0,\dots,y_n]$, we have

$$f^{-1}([y_0,\dots,y_n])=\big\{(\lambda y_0,\dots,\lambda y_n):\lambda\in\mathbb{R}\setminus\{0\},\ \ \lambda^2(y_0^2+\dots+y_n^2)=1\big\},$$

and there are two solutions $\lambda=\pm(y_0^2+\dots+y_n^2)^{-1/2}$.