Professor Joyce **C3.3 Differentiable Manifolds** **MT 2018**

Problem Sheet 2

1(a) Let X be the sphere $\mathcal{S}^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \cdots + x_n^2 = 1\}$. Explain why we may identify

$$T_{(x_0,\ldots,x_n)}\mathcal{S}^n \cong \{(y_0, \ldots, y_n) \in \mathbb{R}^{n+1} : x_0 y_0 + \cdots + x_n y_n = 0\}.$$

(b) By identifying $\mathbb{R}^{2k+2} \cong \mathbb{C}^{k+1}$, show that any odd-dimensional sphere $\mathcal{S}^{2k+1}$ has a nonvanishing vector field $v \in C^\infty(T\mathcal{S}^{2k+1})$ (i.e. $v \neq 0$ at every point).

For discussion: can the same thing hold for even-dimensional spheres $\mathcal{S}^{2k}$?

2*. Let X^n be a *compact* n-dimensional manifold covered by coordinate neighbourhoods U_α, with coordinate maps $\psi_\alpha : U_\alpha \to \mathbb{R}^n$. Show by using compactness that there exists a finite set of bump functions $\varphi_1, \ldots, \varphi_k$ such that for each $x \in X$ at least one φ_i is identically 1 in a neighbourhood of x, and the support of each φ_i is contained in $U_{\alpha(i)}$ for some $\alpha(i)$.
Now define $F : X \to \mathbb{R}^{k(n+1)}$ by

$$F(x) = \big(\varphi_1, \ldots, \varphi_k, \varphi_1 \psi_{\alpha(1)}, \varphi_2 \psi_{\alpha(2)}, \ldots \varphi_k \psi_{\alpha(k)}\big)(x).$$

Show that:

(i) F is injective.

(ii) the derivative of F is injective at each point $x \in X$.

(iii) the manifold topology is the induced topology.

Deduce that $F : X \to \mathbb{R}^{k(n+1)}$ is an embedding.
(This is a special case of the Whitney Embedding Theorem.)

3. Let X be a manifold and $v, w \in C^\infty(TX)$. Suppose $(x_1, \dots, x_n)$ are local coordinates on an open set $U \subseteq X$, so that we may write $v = v_1 \frac{\partial}{\partial x_1} + \cdots + v_n \frac{\partial}{\partial x_n}$ and $w = w_1 \frac{\partial}{\partial x_1} + \cdots + w_n \frac{\partial}{\partial x_n}$ on U, for $v_i, w_j : U \to \mathbb{R}$ smooth.
Define the Lie bracket $[v, w] \in C^\infty(TX)$ by

$$[v, w] = \sum_{i,j=1}^{n} \left(v_i \frac{\partial w_j}{\partial x_i} \cdot \frac{\partial}{\partial x_j} - w_j \frac{\partial v_i}{\partial x_j} \cdot \frac{\partial}{\partial x_i} \right) \qquad \text{on } U. \tag{1}$$

Prove that this is independent of choice of local coordinates. That is, if $(y_1, \dots, y_n)$ is another local coordinate system on $V \subseteq X$, then (1) and its analogue for $(y_1, \dots, y_n), V$ define the same vector field on $U \cap V$.

4. Find the vector fields u, v, w on $\mathbb{R}^2$ given by the following three one-parameter groups of diffeomorphisms:

- $\varphi_t(x_1, x_2) = (x_1 + t, x_2)$.
- $\varphi_t(x_1, x_2) = (x_1, x_2 + t)$.
- $\varphi_t(x_1, x_2) = ((\cos t)x_1 + (\sin t)x_2, (-\sin t)x_1 + (\cos t)x_2)$.

Show that the Lie bracket of any pair of u, v, w is a linear combination of u, v, w (with constant coefficients).

5. Let A be an $n \times n$ matrix and consider the vector field v in $\mathbb{R}^n$ defined by

$$v = \sum_{i,j} A_{ij} x_j \frac{\partial}{\partial x_i}$$

Use the exponential of a matrix:

$$\exp C = I + C + \frac{C^2}{2} + \cdots + \frac{C^n}{n!} + \cdots$$

to integrate this vector field to a one-parameter group of diffeomorphisms.