Solutions to Problem Sheet 4

1. Let $f : X \to Z$ be a smooth map of a compact oriented manifold X of dimension k to a manifold Z, and $\alpha \in \Omega^k(Z)$ be a closed k-form on Z. Show by integrating $f^*(\alpha)$ on X that f defines a linear map $L_f : H^k(Z) \to \mathbb{R}$.

Let $g : Y \to Z$ be a smooth map from a compact oriented $(k+1)$-manifold with boundary Y, such that $\partial Y = X$ and $g|_{\partial Y} = f$. Show using Stokes' Theorem that $L_f = 0$.

Answer.

$$f^*(\lambda_1\alpha_1 + \lambda_2\alpha_2) = \lambda_1 f^*\alpha_1 + \lambda_2 f^*\alpha_2$$

and integration of forms is linear, so

$$\alpha \mapsto \int_X f^*(\alpha)$$

is a linear map on all forms. Now if $\alpha = \mathrm{d}\beta$ then

$$\int_X f^*(\mathrm{d}\beta) = \int_X \mathrm{d}f^*(\beta) = 0$$

by Stokes' theorem, so the linear map vanishes on exact forms and gives a well-defined map L_f from closed forms modulo exact forms which is the k-th cohomology group.

If g restricts to f on $\partial Y = X$, then by Stokes' Theorem

$$\int_X f^*(\alpha) = \int_{\partial Y} g^*(\alpha) = \int_Y \mathrm{d}g^*(\alpha) = \int_Y g^*(\mathrm{d}\alpha) = 0,$$

since $\mathrm{d}\alpha = 0$.

2(a) On the circle $\mathcal{S}^1 \subset \mathbb{R}^2$ denote by $\mathrm{d}\theta$ the 1-form

$$\mathrm{d}\theta = \frac{\mathrm{d}x_2}{x_1} = -\frac{\mathrm{d}x_1}{x_2}.$$

Now consider the product manifold $T^n = \mathcal{S}^1 \times \cdots \times \mathcal{S}^1$, and let $\pi_i : T^n \to \mathcal{S}^1$ be the projection onto the i^{th} factor. By considering the exterior product of all the forms $\pi_i^*(\mathrm{d}\theta)$, deduce that the de Rham cohomology classes $\pi_i^*([\mathrm{d}\theta])$ for $i = 1, \ldots, n$ are linearly independent in $H^1(T^n)$.

(b) Let $n > 1$ and let $f : \mathcal{S}^n \to T^n$ be a smooth map. Noting that $H^1(\mathcal{S}^n) = 0$, prove that the degree of f is zero.

Answer. (a) On the product manifold T^n, the map $(\theta_1, \ldots, \theta_n) \in (0, 2\pi)^n \mapsto (e^{i\theta_1}, \ldots, e^{i\theta_n})$ is a coordinate chart with open dense image. In these coordinates $\pi_1^*(\mathrm{d}\theta) \wedge \cdots \wedge \pi_n^*(\mathrm{d}\theta) = \mathrm{d}\theta_1 \wedge \mathrm{d}\theta_2 \wedge \cdots \wedge \mathrm{d}\theta_n$. This is non-vanishing and defines an orientation. Integrating gives $(2\pi)^n$ which is non-zero. By Stokes' theorem for a compact orientable manifold the cohomology class $\pi_1^*[\mathrm{d}\theta] \cup \cdots \cup \pi_n^*[\mathrm{d}\theta]$ is therefore non-zero.

If $\pi_n^*[\mathrm{d}\theta]$, say, is linearly dependent on the others then

$$\pi_1^*[\mathrm{d}\theta] \ldots \pi_n^*[\mathrm{d}\theta] = \sum_{i<n} \pi_1^*[\mathrm{d}\theta] \ldots \pi_{n-1}^*[\mathrm{d}\theta] c_i \pi_i^*[\mathrm{d}\theta]$$

but then there are repeated $\pi_i^*[\mathrm{d}\theta]$'s in each term and since these are of degree one and anticommute we get zero: contradiction.

(b) We have $H^n(\mathcal{S}^n) \cong \mathbb{R}$ and $H^n(T^n) \cong \mathbb{R}$ as both are connected and oriented, and by (a) we know that $\pi_1^*[\mathrm{d}\theta] \cup \cdots \cup \pi_n^*[\mathrm{d}\theta]$ is nonzero in $H^n(T^n)$. So

$$f^*\big(\pi_1^*[\mathrm{d}\theta] \cup \cdots \cup \pi_n^*[\mathrm{d}\theta]\big) = f^*\big(\pi_1^*[\mathrm{d}\theta]\big) \cup \cdots \cup f^*\big(\pi_n^*[\mathrm{d}\theta]\big) = 0,$$

since pullbacks $f^* : H^k(T^n) \to H^k(\mathcal{S}^n)$ commute with cup products, and $f^*\big(\pi_1^*[\mathrm{d}\theta]\big) = 0$ in $H^1(\mathcal{S}^n) = 0$. But $f^* : H^n(T^n) \to H^n(\mathcal{S}^n)$ is multiplication by $\deg f$, so $\deg f = 0$.

3. What is the degree of the map $\mathbf{x} \mapsto -\mathbf{x}$ on the sphere $\mathcal{S}^n$?

Answer. The standard non-vanishing n-form on $\mathcal{S}^n$ is

$$\omega = \frac{1}{x_{n+1}} \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n$$

so if $f : \mathcal{S}^n \to \mathcal{S}^n$ is given by $f(x) = -x$,

$$f^*\omega = (-1)^{n-1}\omega.$$

Since

$$\int_{\mathcal{S}^n} f^*\omega = \deg f \int_{\mathcal{S}^n} \omega$$

we have

$$\deg f = (-1)^{n-1}.$$

4. The *quaternions* consist of the four-dimensional associative algebra $\mathbb{H}$ of expressions $q = x_0 + ix_1 + jx_2 + kx_3$ where $x_i \in \mathbb{R}$ and i, j, k satisfy the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Show that if $\bar{q} = x_0 - ix_1 - jx_2 - kx_3$ then $q\bar{q} = \|q\|^2$ and $\|ab\|^2 = \|a\|^2\|b\|^2$.

Show that $f(q) = q^2$ defines a smooth map from $\mathbb{R}^4 \cup \{\infty\} \cong \mathcal{S}^4$ to itself. How many solutions are there to the equation $q^2 = 1$?

What is the degree of f?

How many solutions are there to the equation $q^2 = -1$?

Answer. First part is Algebra – just work it out.

$f(q) = q^2$ is polynomial so is smooth on $\mathbb{R}^4$. Now $f(\infty) = \infty$ and $q/\|q\|^2 = 1/\bar{q}$. But

$$f(1/\bar{q}) = (1/\bar{q})^2$$

and this is polynomial in $z = 1/\bar{q}$.

If $q^2 = 1$ then $q^2(\bar{q})^2 = 1$ and so

$$1 = qq\bar{q}\bar{q} = q\|q\|^2\bar{q} = \|q\|^2 q\bar{q} = \|q\|^4$$

so $\|q\|^2 = 1 = q\bar{q}$. Thus

$$0 = q^2 - q\bar{q} = q(q - \bar{q})$$

and since $q \neq 0$, $q = \bar{q}$, so that q is real. Since $q^2 = 1, q = \pm 1$, and there are two solutions.

The derivative of f at $q = \pm 1$ is

$$T_q f(\dot{q}) = \dot{q}q + q\dot{q} = \pm 2\dot{q}$$

which is invertible, so 1 is a regular value, with 2 inverse images. In 4-dimensions the map

$$\mathbf{x} \mapsto \pm 2\mathbf{x}$$

has positive determinant, so the orientation is preserved and the degree is therefore 2.

As before, if $q^2 = -1$, $q\bar{q} = 1$ and so

$$0 = q^2 + q\bar{q} = q(q + \bar{q}).$$

Since $q \neq 0$, $q + \bar{q} = 0$. Conversely, if $q + \bar{q} = 0$ and $q\bar{q} = 1$, $q^2 = -1$. So there is a 2-sphere of solutions

$$q = ix_1 + jx_2 + kx_3$$

where $x_1^2 + x_2^2 + x_3^2 = 1$.

5. Write down in coordinates $x_2, \ldots, x_n$ where $x_1 \neq 0$, the induced Riemannian metric on the sphere $\mathcal{S}^{n-1} \subset \mathbb{R}^n$. Show that its volume form is $\omega = x_1^{-1} \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n$.

Answer. We have

$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1$$

and so

$$x_1 \mathrm{d}x_1 = -\sum_2^n x_i \mathrm{d}x_i.$$

Hence

$$\sum_{i=1}^n \mathrm{d}x_i^2 = \sum_{i=2}^n \mathrm{d}x_i^2 + x_1^{-2} \left(\sum_{i=2}^n x_i \mathrm{d}x_i \right)^2$$

The matrix g_{ij} of this is

$$g_{ij} = \delta_{ij} + x_1^{-2} x_i x_j.$$

Take an orthonormal basis $e_1, \ldots, e_{n-1}$ in $\mathbb{R}^{n-1} \ni (x_2, \ldots, x_n)$ with its usual Euclidean metric such that

$$e_1 = (x_2, \ldots, x_n) / \sqrt{(\sum_{i=2}^n x_i^2)}$$

then the symmetric matrix g becomes diagonal with eigenvalues

$$1 + x_1^{-2}(\sum_{i=2}^{n} x_i^2), 1, 1, \dots, 1$$

i.e.

$$1 + (1 - x_1^2)/x_1^2 = 1/x_1^2, 1, 1, \dots, 1$$

The volume form is therefore

$$\sqrt{\det g}\, \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n = x_1^{-1} \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n.$$

6. Let

$$v = a(x, y)\frac{\partial}{\partial x} + b(x, y)\frac{\partial}{\partial y}$$

be a vector field in $\mathbb{R}^2$ such that

$$\mathcal{L}_v(\mathrm{d}x^2 + \mathrm{d}y^2) = 0$$

Solve this equation for a and b. Show that each vector field integrates to a one parameter group of diffeomorphisms, each of which is of the form

$$\varphi(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$$

where A is a rotation and $\mathbf{c}$ a constant vector.

Answer. The equation

$$\mathcal{L}_v(\mathrm{d}x^2 + \mathrm{d}y^2) = 0$$

spells out, after the Leibniz rule, to be

$$\mathrm{d}x\mathrm{d}a + \mathrm{d}a\mathrm{d}x + \mathrm{d}y\mathrm{d}b + \mathrm{d}b\mathrm{d}y = 0.$$

Expanding,

$$2\mathrm{d}x^2 a_x + \mathrm{d}x\mathrm{d}y(a_y + b_x) + \mathrm{d}y\mathrm{d}x(a_y + b_x) + 2\mathrm{d}y^2 b_y = 0$$

Thus $a_x = 0$ and hence $a = f(y)$. Similarly $b = g(x)$, and substituting in $a_y + b_x = 0$ we get

$$f'(y) + g'(x) = 0$$

This means that f' and g' are constants $\alpha, -\alpha$ so that

$$f(y) = \alpha y + \beta, g(x) = -\alpha x + \gamma$$

and

$$v = \alpha(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}) + \beta\frac{\partial}{\partial x} + \gamma\frac{\partial}{\partial y}.$$

Integrating the vector field means solving

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \alpha y + \beta, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -\alpha x + \gamma. \end{aligned}$$

Taking one more derivative of the first equation and substituting in the second gives

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \alpha(-\alpha x + \gamma),$$

which solves as

$$x = C\cos\alpha t + D\sin\alpha t + \gamma/\alpha$$

for constants C, D. This gives in the first equation

$$\alpha y + \beta = -C\alpha\sin\alpha t + D\alpha\cos\alpha t,$$

so altogether the general solution of the flow-lines of v is

$$\begin{aligned} x &= C\cos\alpha t + D\sin\alpha t + \gamma/\alpha, \\ y &= -C\sin\alpha t + D\cos\alpha t - \beta/\alpha. \end{aligned}$$

Now let $(x(t), y(t))$ be the flow-line with $(x(0), y(0)) = (x_0, y_0)$. Then $x_0 = C + \gamma/\alpha$, $y_0 = D - \beta/\alpha$, so

$$\begin{aligned} x(t) &= x_0\cos\alpha t + y_0\sin\alpha t + \big[\gamma/\alpha - (\gamma/\alpha)\cos\alpha t + (\beta/\alpha)\sin\alpha t\big], \\ y(t) &= -x_0\sin\alpha t + y_0\cos\alpha t + \big[-\beta/\alpha + (\gamma/\alpha)\sin\alpha t + (\beta/\alpha)\cos\alpha t\big]. \end{aligned}$$

Hence the 1-parameter group of diffeomorphisms $(\varphi_t)_{t\in\mathbb{R}}$ has

$$\varphi_t\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = A\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \mathbf{c},$$

where in matrix form

$$A = \begin{pmatrix} \cos\alpha t & \sin\alpha t \\ -\sin\alpha t & \cos\alpha t \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} \gamma/\alpha - (\gamma/\alpha)\cos\alpha t + (\beta/\alpha)\sin\alpha t \\ -\beta/\alpha + (\gamma/\alpha)\sin\alpha t + (\beta/\alpha)\cos\alpha t \end{pmatrix},$$

as required.