C2.7 Category theory: problem sheet 0

You are welcome to attempt these questions before the course starts as an introduction to some of the ideas involved, but your solutions will not be marked. For more details see the introduction to the book 'Basic Category Theory' by T. Leinster which is available online at arXiv: 1612.09375.

1. Let R be a r1ng (i.e. a ring with multiplicative identity). Show that there is a unique r1ng homomorphism from $\mathbb{Z}$ to R.
2. Show that if A is a r1ng such that for every r1ng R there exists a unique r1ng homomorphism $A \to R$, then $A \cong \mathbb{Z}$.
3. Let V and W be vector spaces over a field k, and let $\mathcal{B}$ be a basis for V. Show that there is a bijection from the set of k-linear maps $V \to W$ to the set of functions $\mathcal{B} \to W$ given by $\theta \mapsto \theta|_{\mathcal{B}}$. In what sense is this bijection 'natural'?
4. Let $\{U_i : i \in I\}$ be an open cover indexed by I of a topological space X. Show that, given a topological space Y with continuous maps $f_i : U_i \to Y$ for each $i \in I$ which satisfy $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ whenever $i, j \in I$, there exists a unique continuous map $h : X \to Y$ such that $h|_{U_i} = f_i$ for each $i \in I$.
5. Let $\theta : G \to H$ be a homomorphism of groups and let $\epsilon : G \to H$ send every $g \in G$ to the identity in H. Show that
   (a) the inclusion $\iota : \ker\theta \to G$ satisfies $\theta \circ \iota = \epsilon \circ \iota$, and that
   (b) if $\phi : K \to G$ is any other group homomorphism satisfying $\theta \circ \phi = \epsilon \circ \phi$, then there is a unique group homomorphism $\psi : K \to \ker\theta$ such that $\phi = \iota \circ \psi$.
6. Let U, V, W be vector spaces over a field k. Recall that a map $f : U \times V \to W$ is called bilinear if it is linear in each variable when the other is fixed.
   (a) Show that there is a vector space $U \otimes V$ over k and a bilinear map $b : U \times V \to U \otimes V$ satisfying the 'universal property' that every bilinear map $f : U \times V \to X$ to a vector space X over k factors uniquely as $f = \bar{f} \circ b$ where $\bar{f} : U \otimes V \to X$ is linear. [One way to construct $U \otimes V$ is as the quotient by a suitable subspace of the vector space with basis given by the set of symbols $u \otimes v$ for $(u, v) \in U \times V$].
   (b) Show (using the universal property of $b : U \times V \to U \otimes V$) that if Y is any other vector space over k with a bilinear map $B : U \times V \to Y$ such that every bilinear map $f : U \times V \to X$ to a vector space X over k factors uniquely as $f = \bar{f} \circ b$ where $\bar{f} : Y \to X$ is linear, then there is a unique isomorphism $h : Y \to U \otimes V$ such that $h \circ B = b$.
   (c) Show (again using the universal property) that $(U \otimes V) \otimes W$ is isomorphic to $U \otimes (V \otimes W)$.