C2.7 Category theory: problem sheet 2

Starred questions are optional.

1. Let $\mathbb{Z}$ and $\mathbb{R}$ be the posets of integers and real numbers (with their usual orderings) considered to be categories as in Problem 1(a) from Sheet 1. Describe right and left adjoints for the inclusion functor $\mathbb{Z} \to \mathbb{R}$.
2. Let $F\colon \mathcal{C}^{op} \to \mathrm{Set}$ be a presheaf on a category $\mathcal{C}$. Define the category $(* \Rightarrow F)^{op}$ to have objects consisting of pairs (x, y) where $x \in \mathcal{C}$ and $y \in F(x)$, and morphisms from (x_1, y_1) to (x_2, y_2) given by morphisms $x_1 \to x_2$ in $\mathcal{C}$ inducing maps of pointed sets $(F(x_1), y_1) \leftarrow (F(x_2), y_2)$ (that is, morphisms $g : x_1 \to x_2$ such that $F(g)(y_2) = y_1$).
   Let $\Delta\colon \mathcal{C} \to \mathrm{Fun}((* \Rightarrow F)^{op}, \mathcal{C})$ be the diagonal functor sending each object $c \in \mathcal{C}$ to the constant functor $(x, y) \mapsto c$, and let $P\colon (* \Rightarrow F)^{op} \to \mathcal{C}$ be the projection functor which assigns $P(x, y) = x$. [Why are these functors?]
   Prove the *co-Yoneda lemma*:
   $$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op},\mathrm{Set})}(F, Y(c)) \cong \mathrm{Hom}_{\mathrm{Fun}((*\Rightarrow F)^{op},\mathcal{C})}(P, \Delta(c))$$
   naturally in $c \in \mathcal{C}$, where $Y(c)$ is the representable presheaf associated to c.
3. Consider an adjunction $F \dashv G$. Show that the unit of the adjunction is an isomorphism iff F is fully faithful. Deduce that, dually, the counit of the adjunction is an isomorphism iff G is fully faithful.
4. Show that abelianization $G \mapsto G/[G, G]$ defines a left adjoint for the forgetful functor $\mathrm{Ab} \to \mathrm{Grp}$.
5. (*) Show that the forgetful functor $\mathrm{Ab} \to \mathrm{Grp}$ does not have a right adjoint.