Homological algebra

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Week 1

$\mathbb{Z}$-modules are the same thing as abelian groups. The direct sum of R-modules M_i is defined by

$$\bigoplus_{i\in\mathcal{I}} M_i := \Big\{ f:\mathcal{I}\to\bigcup_{i\in\mathcal{I}} M_i \,\Big|\, f(i)\in M_i \text{ and } \#\{i\in\mathcal{I}: f(i)=0_{M_i}\}<\infty \Big\}.$$

The product of modules M_i is given by

$$\prod_{i\in\mathcal{I}} M_i := \Big\{ f:\mathcal{I}\to\bigcup_{i\in\mathcal{I}} M_i \,\Big|\, f(i)\in M_i \Big\}.$$

The R-module structure is given by $(f+g)(i)=f(i)+_{M_i} g(i)$ and $(r\cdot f)(i)=r\cdot_{M_i}(f(i))$.

An inclusion of of R-modules $N\subset M$ is called *split* if there exists another submodule $N'\subset M$ such every element of M can be uniquely written as a sum of an element of N and an element of N'. *Example:* the inclusion $\mathbb{Z}/2\hookrightarrow\mathbb{Z}/4$ is not split, but the inclusion $\mathbb{Z}/2\hookrightarrow\mathbb{Z}/6$ is split. A sequence

$$0\to A\overset{i}{\to}B\overset{\pi}{\to}C\to 0$$

is called a *short exact sequence* if i is injective, π is surjective, and $\ker(\pi)=\mathrm{im}(i)$. A short exact sequence is split if it is isomorphic to one of the form $0\to A\to A\oplus C\to C\to 0$.

Lemma 1. *A short exact sequence $0\to A\overset{i}{\to}B\overset{\pi}{\to}C\to 0$ is split iff there exists a retraction $r:B\to A$ (a map satisfying $ri=1_A$) iff there exists a section $s:C\to B$ (a map satisfying $\pi s=1_C$).*

Proof. Assuming the existence of a retraction $r:B\to A$, we construct a section $s:C\to B$. Consider the map $1-ir:B\to B$. This map is zero on $i(A)\subset B$, and therefore descends to a map $s:B/i(A)\cong C\to B$. We check that for $c\in C$, we have $\pi s(c)=(c)$:

$$\pi s(c) \overset{\substack{\text{pick } b\in B,\\ \pi(b)=c\\ \downarrow}}{=} \pi\big(b-ir(b)\big)=\pi(b)-\pi ir(b) \overset{\substack{\pi\circ i=0\\ \downarrow}}{=} \pi(b)=c$$

Assuming the existence of a section $s:C\to B$, we construct a retraction $r:B\to A$. Consider the map $1-s\pi:B\to B$. The composite of this map with the projection $\pi:B\to C$ is zero. Its image therefore lands in $\ker(\pi)=i(A)\subset B$. Let $r:=i^{-1}(1-s\pi)$. We check:

$$ri(a)=i^{-1}\big(i(a)-s\pi i(a)\big)\overset{\substack{\pi i=0\\ \downarrow}}{=} i^{-1}i(a)=a.$$

Finally, assuming the existence of a section s and a retraction r, we can identify B with the direct sum $A\oplus C$ via the maps $B\to A\oplus C: b\mapsto (r(b),\pi(b))$ and $A\oplus C\to B:(a,c)\mapsto i(a)+s(c)$. □

Given a ring R, the tensor product over R of a right module M with a left module N is denoted $M\otimes_R N$. It is the abelian group generated by symbols $m_1\otimes n_1+\ldots+m_k\otimes n_k$, under the equivalence relation generated by

$$(m+m')\otimes n = m\otimes n+m'\otimes n,$$
$$m\otimes(n+n') = m\otimes n+m\otimes n',$$
$$\text{and}\qquad mr\otimes n = m\otimes rn.$$

If R is non-commutative, then $M \otimes_R N$ is just an abelian group. If R is commutative, then it is an R-module, via $r \cdot (\sum m_i \otimes n_i) := \sum rm_i \otimes n_i$.

Given two left R-modules M and N, we write $\mathrm{Hom}_R(M, N)$ for the set of R-module homomorphism from M to N. If R is non-commutative, then $\mathrm{Hom}_R(M, N)$ is just an abelian group. If R is commutative, then it is an R-module, with $(r \cdot f)(m) := r \cdot (f(m))$.

There are canonical isomorphisms

$$(\bigoplus A_i) \otimes B \cong \bigoplus (A_i \otimes B), \qquad \mathrm{Hom}\,(\bigoplus A_i, B) \cong \prod \mathrm{Hom}(A_i, B), \qquad \mathrm{Hom}\,(A, \prod B_i) \cong \prod \mathrm{Hom}(A, B_i).$$

There are also canonical isomorphisms $R \otimes_R N \cong N$, $M \otimes_R R \cong M$, and $\mathrm{Hom}_R(R, N) \cong N$. More generally, if $I < R$ is a left ideal, then there are canonical isomorphisms

$$M \otimes_R R/I \cong M/MI, \qquad \text{and} \qquad \mathrm{Hom}_R(R/I, N) \cong \{n \in N \,|\, rn = 0\ \forall r \in I\}.$$

We provide a proof for the first isomorphism:

Proof. The isomorphism $M \otimes_R R/I \to M/MI$ is given by

$$\sum m_i \otimes [r_i] \mapsto \big[\sum m_i \otimes r_i\big].$$

This map is well defined because (1) $(m + m') \otimes [r]$ and $m \otimes [r] + m' \otimes [r]$ map to the same element $[(m+m')r]$ of M/MI, (2) $m \otimes ([r]+[r'])$ and $m \otimes [r] + m \otimes [r']$ map to the same element $[m(r+r')]$of M/MI, (3) $mr_1 \otimes [r]$ and $m \otimes r_1[r]$ map to the same element $[mr_1r]$ of M/MI, and (4) for any $a \in I$, the elements $m \otimes [r]$ and $m \otimes [r + a]$ map to the same element $[mr] = [m(r + a)]$ of M/MI. The inverse map is given by

$$M/MI \to M \otimes_R R/I \,:\, [m] \mapsto m \otimes [1].$$

It is well defined because for $m = m'a$ with $a \in I$, the image of $[m]$ under that map is given by $m'a \otimes [1] = m' \otimes a[1] = m' \otimes 0$, which is zero in $M \otimes_R R/I$.

The composite $M/MI \to M \otimes_R R/I \to M/MI$ is obviously the identity. The other composite $M \otimes_R R/I \to M/MI \to M \otimes_R R/I$ sends $\sum m_i \otimes [r_i]$ to $\big(\sum m_i r_i\big) \otimes [1]$. It is the identity since

$$\big(\sum m_i r_i\big) \otimes [1] = \sum \big(m_i r_i \otimes [1]\big) = \sum m_i \otimes r_i[1] = \sum m_i \otimes [r_i].$$

□

Exercise 1. Show that if p and q are distinct prime numbers, then $\mathbb{Z}/p \otimes_{\mathbb{Z}} \mathbb{Z}/q = 0$.

Exercise 2. Let $R = \mathbb{R}[x]/x^n$. Prove that the obvious inclusion of R-modules $R/x \hookrightarrow R$ is not split. Compute the quotient, and write down the short exact sequence formed by those three modules.

Exercise 3. Let R be ring, let $\{A_i\}_{i \in \mathcal{I}}$ be a collection of right R-modules, and let B be a left R-module. Show that $\big(\bigoplus A_i\big) \otimes_R B \cong \bigoplus (A_i \otimes_R B)$.

Exercise 4. Let $R := \mathbb{Z}[x]$. Compute $\mathrm{Hom}_R\big(R/(2x), R/(4)\big)$ as an R-module. Show that it is isomorphic to R/I for some ideal $I \subset R$.

Exercise 5. Let $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \,|\, a, b, c \in k\}$ be the ring of upper triangular 2×2 matrices with coefficients in some field k. Show that R is a direct sum of two smaller R-modules: $R = P \oplus Q$. Compute $\mathrm{Hom}_R(P, Q)$ and $\mathrm{Hom}_R(Q, P)$.

Exercise 6. Let k be a field. Find an exact sequence of $k[x]$-modules $0 \to A \to B \to C \to 0$ such that the induced sequence $A \otimes_{k[x]} k[x]/(x) \to B \otimes_{k[x]} k[x]/(x) \to C \otimes_{k[x]} k[x]/(x)$ is not exact.

Exercise 7. Find an example of a ring R and two modules M and N such that the abelian group $M \otimes_R N$ does not carry the structure of an R-module. *Hint:* try a 2×2 matrix algebra.

Week 2

A chain complex of R-modules $C_\bullet = (C_n, d_n)_{n\in\mathbb{Z}}$ is a collection of R-modules C_n and R-module maps $d_n : C_n \to C_{n-1}$, called 'differentials', subject to the axiom $d_n \circ d_{n+1} = 0$. This axiom is sometimes abusively abbreviated $d^2 = 0$. A chain complex is called *exact* if $\ker(d_n) = \operatorname{im}(d_{n+1})$.

The *homology* of a chain complex of R-modules $C_\bullet = (C_n, d_n : C_n \to C_{n-1})_{n\in\mathbb{Z}}$ is defined by

$$H_n(C_\bullet) := \frac{Z_n}{B_n} := \frac{\ker(d_n : C_n \to C_{n-1})}{\operatorname{im}(d_{n+1} : C_{n+1} \to C_n)}$$

Here Z_n are called the *cycles*, and B_n are called the *boundaries*. If $C_\bullet$ is a chain complex in an arbitrary abelian category (to be defined later), the object $H_n(C_\bullet)$ can be defined in purely categorical terms, as the cokernel of the canonical map $C_{n+1} \to \ker(d_n : C_n \to C_{n-1})$.

A morphism of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ is called a *quasi-isomorphism* if it induces an isomorphism at the level of homology: $H_n(f_\bullet) : H_n(C_\bullet) \xrightarrow{\cong} H_n(D_\bullet), \forall n \in \mathbb{Z}$.

An additive functor between abelian categories (to be defined later) is called *exact* if it sends exact sequences to exact sequences, equivalently, if it sends short exact sequences to short exact sequences. Note that the functor $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ is not exact: it sends the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \to \mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \xrightarrow{\simeq} \mathbb{Z}/2 \to 0$ which is not exact. Similarly, the functor $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ sends the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \to 0 \to 0 \to \mathbb{Z}/2 \to 0$ which is not exact. Finally, the contravariant functor $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$ sends the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 \to 0$$

to the sequence $0 \leftarrow \mathbb{Z}/2 \xleftarrow{0} \mathbb{Z}/2 \xleftarrow{\simeq} \mathbb{Z}/2 \leftarrow 0$ which is not exact. The functors $- \otimes_{\mathbb{Z}} \mathbb{Z}/2$ and $\operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/2, -)$ and $\operatorname{Hom}_{\mathbb{Z}}(-, \mathbb{Z}/2)$ are therefore not exact.

A functor F is *right exact* if for every short exact sequence $0 \to A \to B \to C \to 0$, the sequence $F(A) \to F(B) \to F(C) \to 0$ is exact. Similarly, a functor F is *left exact* if whenever $0 \to A \to B \to C \to 0$ is exact, then $0 \to F(A) \to F(B) \to F(C)$ is exact.

Lemma 2. *Let $\mathcal{A}$ be an abelian category, and let $M \in \mathcal{A}$ be an object. Then the functor* $\operatorname{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \text{AbGrp}$ *is left exact.*

Proof. Let $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$ be a short exact sequence in $\mathcal{A}$. We need to show that $0 \to \operatorname{Hom}(M, A) \xrightarrow{\iota_*} \operatorname{Hom}(M, B) \xrightarrow{\pi_*} \operatorname{Hom}(M, C)$ is exact.

- ι_* is injective. Let $\alpha \in \operatorname{Hom}(M, A)$ be an element that maps to zero in $\operatorname{Hom}(M, B)$. Since $\iota \circ \alpha = 0$, and ι is a monomorphism (see Lemma 4 below), $\alpha = 0$. So ι_* is injective.
- $\operatorname{im}(\iota_*) \subset \ker(\pi_*)$. Follows trivially from the fact that $\pi \circ \iota = 0$.
- $\ker(\pi_*) \subset \operatorname{im}(\iota_*)$. Let $\beta \in \operatorname{Hom}(M, B)$ be an element that maps to zero in $\operatorname{Hom}(M, C)$. Since $\pi \circ \beta = 0$, the map $\beta : M \to B$ factors through $\ker(\pi) = A$. So we can write β as $\iota \circ \alpha$ for some $\alpha \in \operatorname{Hom}(M, A)$. We have $\beta = \iota_*(\alpha)$, and hence $\beta \in \operatorname{im}(\iota_*)$. □

Corollary. Let R be a ring and let M be an R-module. Then the functors

$$\operatorname{Hom}_R(M, -) : R\text{-Mod} \to \text{AbGrp} \qquad \text{and} \qquad \operatorname{Hom}_R(-, M) : R\text{-Mod}^{op} \to \text{AbGrp}$$

are left exact.

Lemma 3. *The functor* $-\otimes_R N$ *is right exact.*

Proof. Given a short exact sequence of right R-modules $0 \to A \xrightarrow{\iota} B \xrightarrow{\pi} C \to 0$, we need to show that $A \otimes_R N \to B \otimes_R N \to C \otimes_R N \to 0$ is exact. The surjectivity of $B \otimes_R N \to C \otimes_R N$ is easy, so let us focus on the harder argument: given an element $\sum b_i \otimes n_i \in B \otimes_R N$ that goes to zero in $C \otimes_R N$, we need to show that it comes from $A \otimes_R N$.

Since $\sum \pi(b_i) \otimes n_i = 0$ in $A \otimes_R N$, there exist elements c'_α, c''_α, n_α, c_β, n'_β, n''_β, c_γ, r_γ, n_γ such that

$$\begin{aligned}\sum_i \pi(b_i) \otimes n_i + \sum_\alpha (c'_\alpha + c''_\alpha) \otimes n_\alpha - c'_\alpha \otimes n_\alpha - c''_\alpha \otimes n_\alpha \\ + \sum_\beta c_\beta \otimes (n'_\beta + n''_\beta) - c_\beta \otimes n'_\beta - c_\beta \otimes n''_\beta \\ + \sum_\gamma c_\gamma r_\gamma \otimes n_\gamma - c_\gamma \otimes r_\gamma n_\gamma\end{aligned}$$

is zero in the free abelian group on the set of symbols "$c \otimes n$". If we mod out that free abelian group by the first set of relations $(c' + c'') \otimes n = c' \otimes n + c'' \otimes n$, then we get the abelian group $\bigoplus_{n \in N} C$. So, another way of saying that $\sum \pi(b_i) \otimes n_i$ is zero in $A \otimes_R N$ is to say that there exist elements c_β, n'_β, n''_β, c_γ, r_γ, n_γ such that

$$\sum_i \pi(b_i) \otimes n_i + \sum_\beta c_\beta \otimes (n'_\beta + n''_\beta) - c_\beta \otimes n'_\beta - c_\beta \otimes n''_\beta + \sum_\gamma c_\gamma r_\gamma \otimes n_\gamma - c_\gamma \otimes r_\gamma n_\gamma = 0 \text{ in } \bigoplus_{n \in N} C,$$

where "$c \otimes n$" now stands for the element c put in the n-th copy of C.

Pick preimages b_β, $b_\gamma \in B$ of $c_\beta, c_\gamma \in C$, and consider the element

$$y := \sum_i b_i \otimes n_i + \sum_\beta b_\beta \otimes (n'_\beta + n''_\beta) - b_\beta \otimes n'_\beta - b_\beta \otimes n''_\beta + \sum_\gamma b_\gamma r_\gamma \otimes n_\gamma - b_\gamma \otimes r_\gamma n_\gamma \quad \in \bigoplus_{n \in N} B.$$

This element goes to 0 in $\bigoplus_{n \in N} C$ and therefore comes from some $x \in \bigoplus_{n \in N} A$.

Let $[x]$ denote the image of x in $A \otimes_R N$ and let $[y]$ denote the image of y in $B \otimes_R N$. Since $x \mapsto y$, it follows that $[x] \mapsto [y]$. We are done since $[y] = \sum_i b_i \otimes n_i$ in $B \otimes_R N$. □

Week 3

A *terminal object* is an object that admits exactly one morphism to it from any other object. An *initial object* is an object that admits exactly one morphism from it to any other object. A *zero object* is an object that admits exactly one morphism to it from any other object and exactly one morphism from it to any other object, i.e., is both initial and terminal.

A *monomorphism* is a morphism f that satisfies $(f \circ g_1 = f \circ g_2) \Rightarrow (g_1 = g_2)$. Equivalently, it is a morphism $f : X \to Y$ with the property that whenever two morphisms $g_1, g_2 : Z \to X$ are distinct, they remain distinct after composing them with f. Dually, an *epimorphism* is a map f that satisfies $(g_1 \circ f = g_2 \circ f) \Rightarrow (g_1 = g_2)$.

The *direct sum* of two objects X_1 and X_2 is an object Z equipped with maps $i_1 : X_1 \to Z$, $i_2 : X_2 \to Z$, $p_1 : Z \to X_1$, $p_2 : Z \to X_2$ satisfying $p_1 \circ i_1 = \text{id}$, $p_2 \circ i_2 = \text{id}$, $p_1 \circ i_2 = 0$, $p_2 \circ i_1 = 0$, and $i_1 \circ p_1 + i_2 \circ p_2 = \text{id}$.

An pre-additive category is a category such that all the hom-sets are equipped with the structure of abelian groups and such that composition $\text{Hom}(x, y) \times \text{Hom}(y, z) \to \text{Hom}(x, z)$ is bilinear. An *additive category* is a category which is preadditive, admits a zero object, and admits all direct sums.

The *kernel* of a map $f : X \to Y$ is a morphism $i : K \to X$ which is universal w.r.t the property that $f \circ i = 0$. This means the following: it's an object K along with a morphism $i : K \to X$

satisfying $f \circ i = 0$, such that for every object $\tilde{K}$ and every morphism $\tilde{i} : \tilde{K} \to X$ satisfying $f \circ \tilde{i} = 0$, there exists a unique morphism $g : \tilde{K} \to K$ such that $\tilde{i} = i \circ g$.

Dually, the *cokernel* of a map $f : X \to Y$ is a morhpism $q : Y \to C$ which is universal w.r.t the property that $q \circ f = 0$.

An additive category is called *abelian* if for every monomorphism $f : A \rightarrowtail B$, the pair (A, f) is a kernel of the morphism $B \to \mathrm{coker}(f)$, and for every epimorphism $f : A \twoheadrightarrow B$ the pair (B, f) is a cokernel of the morphism $\ker(f) \to A$.

A sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is exact iff (A, f) is a kernel of g and (C, g) is a cokernel of f.

The homology of a chain complex $\ldots C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \ldots$ is the cokernel of the map $C_{n+1} \to \ker(d_n)$.

Lemma 4. *Kernels are monomorphisms; cokernels are epimorphisms.*

Proof. Let $f : A \to B$ be a morphism. Consider two morphisms $a, b : X \to \ker(f)$ with the property that $\iota a = \iota b$:

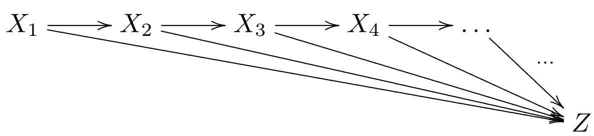
$$X \underset{b}{\overset{a}{\rightrightarrows}} \ker(f) \xrightarrow{\iota} A \xrightarrow{f} B$$

Since $f \iota a = 0$, by the universal property of $\ker(f)$, there exists a unique morphism $X \to \ker(f)$ whose composition with ι yields ιa. Both a and b satisfy that property. So they're equal. $\square$

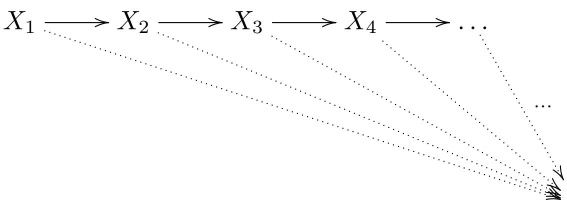
Lemma 5 (exercise)**.** *A morphism f is an epimorphism if and only if* $\mathrm{coker}(f) = 0$.

A *colimit* (also called *direct limit*) of a sequence of morphisms $X_1 \to X_2 \to X_3 \to \ldots$ is an object Z along with morphisms $X_i \to Z$ such that

all little triangles commute

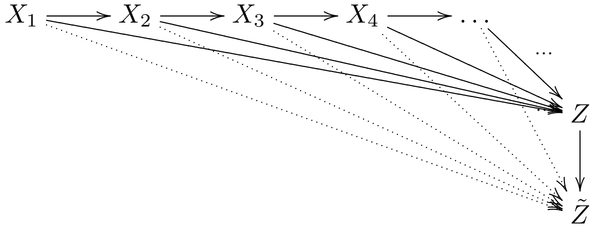


and such that for every other diagram



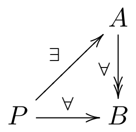
all of whose little triangles commute

there exists a unique morphism $Z \to \tilde{Z}$ such that all the triangles in this big diagram commute:



The colimit can be denoted $\mathrm{colim}\, X_i$ or $\varinjlim X_i$. Quite often 'colimit' means the same thing as 'union'. The dual notion is called a *limit*. It is denoted $\lim X_i$ or $\varprojlim X_i$.

An R-module is called *projective* if it is a direct summand of a free module. An object P of an abelian category is called *projective* if for every epimorphism $A \to B$ and every morphism $P \to B$, there exists a morphism $P \to A$ such that the triangle commutes:



Exercise 8. Let k be a field, and let C be the abelian category of k-vector spaces. Let D be an arbitrary abelian category. Prove that every additive functor $C \to D$ is exact.

Exercise 9. Let R and S be rings, let $C := R$-Mod and $D := S$-Mod be the associated abelian categories of modules, and let $F : C \to D$ be an additive functor.

Assume that F sends short exact sequences to short exact sequences. Prove that it sends exact sequences (or any length) to exact sequences.

Exercise 10. Let R be a ring. Prove that an R-module P is projective iff every surjective map $A \to P$ admits a section.

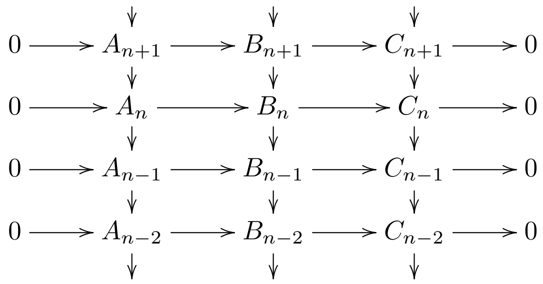
Exercise 11. Let $R := \mathbb{Z}[\sqrt{-5}]$. Prove that the ideal generated by 2 and $1+\sqrt{-5}$ is a projective R-module which is not free. *Hint:* show that the map $\left(\begin{smallmatrix} 2 & \frac{1-\sqrt{-5}}{2} \\ 1+\sqrt{-5} & 2 \end{smallmatrix}\right) : R^{\oplus 2} \to M^{\oplus 2}$ is an isomorphism.

Exercise 12. Let R be a ring. Prove that for every sequence of R-modules $(M_i)_{n\in\mathbb{Z}}$, there exists a chain complex of free modules $C_\bullet$ such that $H_i(C_\bullet) \cong M_i$ for all $i \in \mathbb{Z}$.

Exercise 13. Let $\mathcal{A}$ be an arbitrary abelian category, and let $Ch(\mathcal{A})$ be the category of chain complexes of objects of $\mathcal{A}$. Given a morphism $f_\bullet : C_\bullet \to D_\bullet$ in $Ch(\mathcal{A})$, prove that the kernel of $f_\bullet$ is the chain complex $\big(\ldots \to \ker(f_n) \to \ker(f_{n-1}) \to \ldots\big)$.

The next exercise is a long and painful one which I don't expect you (or want you) to finish. But I do want you to start it. Write down what you think is approximately 50% of the proof, and then write "I give up" (or, if you don't want to give up, you may hand in a complete answer):

Exercise 14. Prove that a short exact sequence of chain complexes (of R-modules)



induces a long exact sequence in homology

$$\ldots \to H_{n+1}(C_\bullet) \to H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \to H_{n-1}(A_\bullet) \to \ldots$$

[For the definition of the so-called 'connecting homomorphism' $H_{n+1}(C_\bullet) \to H_n(A_\bullet)$, you may have a look at e.g. https://ncatlab.org/nlab/show/connecting+homomorphism*]*

Week 4

Let M be a right R-module and N a left R-module. Then:

$$\mathrm{Tor}_i^R(M,N) := H_i(P_\bullet \otimes_R N) = H_i(M \otimes_R Q_\bullet)$$

where $P_\bullet$ is a projective resolution of M, or $Q_\bullet$ is a projective resolution of N. Implicit in the above definition is the fact that $\mathrm{Tor}_i^R(M,N)$ doesn't depend on the choice of projective resolution, and doesn't depend on whether one resolves M or N.

Let M and N be R-modules (either both right modules or both left modules). Then:

$$\mathrm{Ext}^i_R(M,N) := H^i(\mathrm{Hom}_R(P_\bullet, N)) = H^i(\mathrm{Hom}_R(M, I^\bullet)).$$

Here, $P_\bullet$ is a projective resolution of M and $I^\bullet$ is an injective resolution of N (injective objects are defined below). Once again, the choice of resolution doesn't matter, neither does the choice of which of the two modules one decides to resolve.

If $R = \mathbb{Z}$, then every module admits a resolution of length 1. This implies that $\mathrm{Tor}_i^{\mathbb{Z}}$ and $\mathrm{Ext}^i_{\mathbb{Z}}$ vanishes as soon as $i > 1$. This property is called '$\mathbb{Z}$ has cohomological dimension one'.

Let A and B be abelian categories. Assume that A has enough projectives. Let $F : A \to B$ be an additive functor (often assumed to be right exact). The nth *left derived functor* of F, denoted $L_nF : A \to B$ is defined by $X \mapsto H_n(F(P_\bullet))$, where $P_\bullet \to X$ is a projective resolution.

Assume now that A has enough injectives and that $F : A \to B$ is an additive functor (often assumed to be left exact). The nth *right derived functor* of F, denoted $R^nF : A \to B$ is defined by $X \mapsto H^n(F(I^\bullet))$, where $X \to I^\bullet$ is an injective resolution.

Lemma 6. *If F is right exact, then $L_0F = F$. (If F is left exact, then $R^0F = F$.)*

Proof. Let $P_\bullet \to M$ be a projective resolution, so that $P_1 \overset{d}{\to} P_0 \overset{\varepsilon}{\to} M \to 0$ is exact. By definition, $L_0F(M) = \mathrm{coker}(F(d))$. Consider the short exact sequence $0 \to K \to P_0 \to M \to 0$, where $K := \ker(\varepsilon)$. The comparison map $P_1 \to K$ is an epimorphism by the exactness of $P_\bullet \to M$. Since right exact functors send epimorphisms to epimorphisms, the map $F(P_1) \to F(K)$ is then also an epimorphism.

By the right exactness of F, the sequence $F(K) \to F(P_0) \to F(M) \to 0$ is exact. So $F(M) = \ker(F(K) \to F(P_0)) = \ker(F(P_1) \to F(P_0)) = L_0F(M)$. The middle equality holds true because composing with an epimorphism (namely with the map $F(P_1) \to F(K)$) doesn't change cokernels; see the next lemma. □

Lemma 7 (exercise)**.** *Given composable morphisms $A \overset{f}{\twoheadrightarrow} B \overset{g}{\to} C \overset{h}{\to} D$, the morphism h is a cokernel of g if and only if it is a cokernel of $g \circ f$.*

A morphism of chain complexes $f_\bullet : C_\bullet \to D_\bullet$ induces a corresponding morphism at the level of cohomology groups $H_n(f_\bullet) : H_n(C_\bullet) \to H_n(D_\bullet)$. Two chain maps $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ are called *chain homotopic* if there exists a degree -1 map $h : C_\bullet \to D_\bullet$ satisfying $hd + dh = f - g$.

There are two ways of making the operation "take a projective resolution" into a functor:

(1) Take P_0 to be the free R-module on the underlying set of M. Take P_1 to be the free R-module on the underlying set of $\ker(P_0 \to M)$. Take P_2 to be the free R-module on the underlying set of $\ker(P_1 \to P_0)$. Etc.

(2) View the operation "take a projective resolution" as a functor from our abelian category $\mathcal{A}$ to its derived category $D(\mathcal{A})$.

Definition: Let $\mathcal{A}$ be an abelian category. Its *derived category* $D(\mathcal{A})$ has:
- Object = positively graded chain complexes of projectives of $\mathcal{A}$
- Morphisms = chain maps modulo chain homotopy.

The notion of chain homotopy is made so that whenever $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : C_\bullet \to D_\bullet$ are chain homotopic maps, then $H_*(f_\bullet) = H_*(g_\bullet) : H_*(C_\bullet) \to H_*(D_\bullet)$.

Here's a way of defining the nth derived functor of an additive functor $F : \mathcal{A} \to \mathcal{B}$:

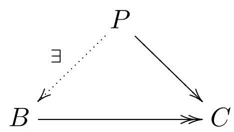
$$L_nF \;:\; \mathcal{A} \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{A}) \xrightarrow{\text{apply } F} \left(\mathrm{Ch}(\mathcal{B});\substack{\text{chain maps modulo}\\ \text{chain homotopy}}\right) \xrightarrow{H_n} \mathcal{B}$$

The *total derived functor* of F, or simply "the derived functor of F" is the functor

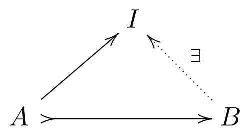
$$LF \;:\; \mathcal{A} \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{A}) \xrightarrow{\text{apply } F} \left(\mathrm{Ch}(\mathcal{B});\substack{\text{chain maps modulo}\\ \text{chain homotopy}}\right) \xrightarrow[\text{resolution}]{\text{take projective}} D(\mathcal{B}).$$

Here, a projective resolution of a chain complex $C_\bullet$ is the data of a chain complex of projectives $P_\bullet$ together with a map of chain complexes $P_\bullet \to C_\bullet$ which is a quasi-isomorphism.

Week 5 Recall that a module (or object of some arbitrary abelian category) P is projective if for every solid arrow diagram there exists a dotted arrow making the diagram commute:



In the same vein, a module (or object of some arbitrary abelian category) I is called injective if for every solid arrow diagram there exists a dotted arrow making the diagram commute:



A module P is *projective* iff $\mathrm{Hom}_R(P, -)$ is exact. A module I is *injective* iff $\mathrm{Hom}_R(-, I)$ is exact. A module F is *flat* if $- \otimes_R F$ is exact. Every projective module is flat. Indeed, if $M = M' \oplus M''$, then we have (M is flat) $\Leftrightarrow$ (M' is flat and M'' is flat). Starting from the obvious fact that free modules are flat, we conclude that every projective module is flat.

Example: $\mathbb{Q}$ is a flat $\mathbb{Z}$-module. That's because $\mathbb{Q} = \mathrm{colim}(\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \xrightarrow{\cdot 5} \dots)$ and for every abelian group A we have

$$\mathbb{Q} \otimes_{\mathbb{Z}} A = \mathrm{colim}(A \xrightarrow{\cdot 2} A \xrightarrow{\cdot 3} A \xrightarrow{\cdot 4} A \xrightarrow{\cdot 5} \dots).$$

In order to check that $\mathbb{Q}$ is flat, one needs to check that an injective map $f : A \to B$ remains injective after applying the functor $\mathbb{Q} \otimes_{\mathbb{Z}} -$. This is a diagram chase in the diagram:

$$\begin{array}{ccccccc} A & \xrightarrow{\cdot 2} & A & \xrightarrow{\cdot 3} & A & \xrightarrow{\cdot 4} & \dots \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} & & \\ B & \xrightarrow{\cdot 2} & B & \xrightarrow{\cdot 3} & B & \xrightarrow{\cdot 4} & \dots \end{array}$$

Lemma 8. *A short exact sequence of chain complexes* $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ *(which, by definition, means that for each* n *the sequence* $0 \to A_n \to B_n \to C_n \to 0$ *is exact) induces a long exact sequence in homology.* See p. 117 of Hatcher's book for a proof.

A bigraded chain complex $C_{\bullet\bullet}$ is a sequence of abelian groups $C_{p,q}$ (or objects of some abelian category) together with maps $d_h : C_{p,q} \to C_{p-1,q}$ and $d_v : C_{p,q} \to C_{p,q-1}$ satisfying $d_h d_h = 0$, $d_v d_v = 0$, and $d_h d_v = d_v d_h$. The total chain complex $\mathrm{Tot}(C_{\bullet\bullet})$ is defined by

$$\big[\mathrm{Tot}(C_{\bullet\bullet})\big]_n = \bigoplus_{p+q=n} C_{p,q}$$

The differential $d^{\mathrm{Tot}} : \big[\mathrm{Tot}\big(C_{\bullet\bullet}\big)\big]_n \to \big[\mathrm{Tot}\big(C_{\bullet\bullet}\big)\big]_{n-1}$ is the sum of the maps $d_h : C_{p,q} \to C_{p-1,q}$ and $(-1)^p \cdot d_v : C_{p,q} \to C_{p,q-1}$ over all p, q such that $p + q = n$. There's also a variant of Tot where one uses direct products instead of direct sums

$$\big[\mathrm{Tot}^{\prod}\big(C_{\bullet\bullet}\big)\big]_n = \prod_{p+q=n} C_{p,q}$$

Lemma 9. *Let $C_{\bullet\bullet}$ be a double complex such that for every n there exists only finitely many pairs (p,q), $p+q=n$, such that $C_{p,q} \neq 0$. Then we have*

$$\Big(C_{\bullet\bullet} \text{ has exact rows}\Big) \Rightarrow \Big(\mathrm{Tot}(C_{\bullet\bullet}) \text{ is exact}\Big)$$

More generally, if $C_{\bullet\bullet}$ is a double complex such that for every n the set $\{p \in \mathbb{Z} \,|\, C_{p,n-p} \neq 0\}$ is bounded below, then

$$\Big(C_{\bullet\bullet} \text{ has exact rows}\Big) \Rightarrow \Big(\mathrm{Tot}^{\prod}(C_{\bullet\bullet}) \text{ is exact}\Big)$$

$\mathrm{Tor}_i^R(M,N)$ and $\mathrm{Ext}^i_R(M,N)$ are independent of the choice of resolution. They can be computed by resolving either M or N.

Let M be a right R-module and N a left R-module, let $P_\bullet$ be a projective resolution of M and $Q_\bullet$ a projective resolution of N. Then we have quasi-isomorphisms

$$P_\bullet \otimes_R N \;\leftarrow\; \mathrm{Tot}(P_\bullet \otimes_R Q_\bullet) \;\to\; M \otimes_R Q_\bullet$$

inducing isomorphisms

$$H_i(P_\bullet \otimes_R N) \;\cong\; H_i(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet)) \;\cong\; H_i(M \otimes_R Q_\bullet).$$

The isomorphism $H_i(P_\bullet \otimes_R N) \xleftarrow{\cong} H_i(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet))$ is the connecting homomorphism in the LES associated to the short exact sequence

$$0 \to P_\bullet \otimes_R N \to \mathrm{Tot}\big(P_\bullet \otimes_R Q_\bullet \to P_\bullet \otimes_R N\big) \to \mathrm{Tot}\big(P_\bullet \otimes_R Q_\bullet\big) \to 0.$$

The fact that the middle term is acyclic (the words 'acyclic' and 'exact' are synonyms) follows from Lemma 9 below.

Let now M and N be R-modules (either both right modules or both left modules). Let $P_\bullet$ be a projective resolution of M and $I^\bullet$ an injective resolution of N. Then we have quasi-isomorphisms

$$\mathrm{Hom}_R(P_\bullet, N) \;\to\; (\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet)) \;\leftarrow\; \mathrm{Hom}_R(M, I^\bullet)$$

and $\mathrm{Ext}^i_R(M,N)$ can be computed in any one of the following ways:

$$H^i(\mathrm{Hom}_R(P_\bullet, N)) \;\cong\; H^i(\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet))) \;\cong\; H^i(\mathrm{Hom}_R(M, I^\bullet)).$$

If instead one takes a projective resolution $Q_\bullet$ of N, then one has yet another chain complex that computes $\mathrm{Ext}^*_R(M,N)$, namely $\mathrm{Tot}^{\prod}(\mathrm{Hom}_R(P_\bullet, Q_\bullet))$.

Exercise 15. Compute $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, and $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ first using projective resolutions, and then using injective resolutions.

Exercise 16. Let k be a field. Compute $\mathrm{Tor}^R_*(k,k)$ and $\mathrm{Ext}^*_R(k,k)$ for $R = k[x]$, $R = k[x,y]$, $R = k[x]/x^n$, $R = k[x,y]/(x^n, y^m)$, $R = k[x,y]/(x^2, y^2, xy)$, $R = k[x,y]/(x^3 - y^2)$.

Exercise 17. Compute $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ using the formula $H^*(\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet)))$.

Exercise 18. Compute $\mathrm{Tor}^{\mathbb{C}[x]/x^2}_*(\mathbb{C}, \mathbb{C})$ using the formula $H_*(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet))$.

Exercise 19. Write down an example of a bigraded chain complex $C_{\bullet\bullet}$ which fails the condition *"for every n there exists only finitely many pairs (p,q) such that $p+q=n$ and $C_{p,q}\neq 0$"*, and for which the implication

$$\Big(C_{\bullet\bullet} \text{ has exact rows}\Big) \Rightarrow \Big(\mathrm{Tot}(C_{\bullet\bullet}) \text{ is exact}\Big)$$

fails. In other words, you must find a bigraded chain complex $C_{\bullet\bullet}$ which has has exact rows, but such that $\mathrm{Tot}(C_{\bullet\bullet})$ is not exact.

Exercise 20. Prove that an abelian group A is *flat* as a $\mathbb{Z}$-module if and only if it is *torsion-free*. *Hint:* Write A as a colimit of free abelian groups.

Exercise 21. Prove that an abelian group A is *injective* as a $\mathbb{Z}$-module if and only if it is *divisible* (here, divisible means $\forall a\in A, \forall n\in\mathbb{N}, \exists x\in A$ s.t. $nx=a$).

Week 6

The *pullback* of a diagram of modules $A\xrightarrow{f} C \xleftarrow{g} B$ is the set $\{(a,b)\in A\oplus B : f(a)=g(b)\}$. It is also the limit of the diagram $A\to C\leftarrow B$. The *pushout* of a diagram of modules $A\xleftarrow{f} C \xrightarrow{g} B$ is the quotient $A\oplus B/\{(f(c),-g(c)) : c\in C\}$. It is also the colimit of the diagram $A\leftarrow C\to B$.

A *diagram* of R-modules indexed by a poset P is just a functor $P\to R$-Mod. Concretely, this is the data of R-modules M_α indexed by P, and maps $f_{\alpha\beta}: M_\alpha\to M_\beta$ for all $\alpha<\beta\in P$, satisfying $f_{\beta\gamma}f_{\alpha\beta}=f_{\alpha\gamma}$.

The *limit* of a a diagram $P\to R$-Mod (where P is a poset) can be described concretely as $\big\{(m_\alpha)\in\prod_{\alpha\in P} M_\alpha : f_{\alpha\beta}(m_\alpha)=m_\beta, \forall\alpha<\beta\in P\big\}$. The *colimit* of a a diagram $P\to R$-Mod is given by $\bigoplus_{\alpha\in P} M_\alpha/\mathrm{Span}\{m-f_{\alpha\beta}(m) : m\in M_\alpha\}$. Limits and colimits can alternatively be defined by means of a universal property.

A poset is called *directed* if for every $x,y\in P$, there exists $z\in P$ such that $z\geq x$ and $z\geq y$. If P is a directed poset, then every element of $\mathrm{colim}_{\alpha\in P}M_\alpha$ is represented by some element m of some M_α. Moreover, if P is a direct poset, then an element $m\in M_\alpha$ represents the zero element in $\mathrm{colim}_{\alpha\in P}M_\alpha$ iff there exists some $\beta\geq\alpha$ in P such that m becomes zero in M_β.

The latter fails miserably for e.g. pushout$(\mathbb{Z}/2\leftarrow\mathbb{Z}\to\mathbb{Z}/3)$.

Theorem (Baer's criterion)

An R-module E is injective if and only if every left ideal $I<R$ and any map $I\to E$, the extension problem

$$\begin{array}{ccc} E & & \\ \uparrow & \nwarrow & \\ I & \longrightarrow & R \end{array}$$

admits a solution.

See e.g. https://ncatlab.org/nlab/show/Baer's+criterion for a proof.

Corollary of Baer's criterion: if R is a PID, then a module M is injective iff it is *divisible*, i.e. iff for every $x\in M$ and every non-zero $r\in R$ there exists $y\in M$ such that $ry=x$.

An abelian category is said to *have enough projectives* if for every object X, there exists a projective object P and an epimorphism $P\to X$. Dually, an abelian category is said to *have enough injectives* if for every object X, there exists an injective object I and a monomorphism $X\to I$.

It is easy to see that for any ring R, the category of R-modules has enough projectives: take P to be free R module on the underlying set of X (any generating set would also do).

Showing the R-mod has enough injectives is much harder. Given an R-module M, let S denote the set of all pairs (I,f), where I is an ideal of R, and $f: I\to M$ is an R-module homomorphism.

We write M' for the following pushout:

$$\begin{array}{ccc} M & \longrightarrow & M' \\ {\scriptstyle \bigoplus f}\uparrow & & \uparrow \\ \bigoplus\limits_{(I,f)\in S} I & \longrightarrow & \bigoplus\limits_{(I,f)\in S} R \end{array}$$

Write $M_0 := M$ and $M_{n+1} := (M_n)'$. If every ideal is finitely generated, then $M_\infty := \operatorname{colim}(M_0 \to M_1 \to M_2 \to \ldots)$ is an injective module. It obviousely contains M as a submodule. To show that M_∞ is injective, we use Baer's criterion. Using the fact that every ideal is finitely generated, every map $f : I \to M_\infty$ factors through some finite stage of the colimit, let's say $f : I \to M_n$. The extension problem will then admit a solution at the next stage: $\begin{array}{ccc} M_{n+1} & & \\ {\scriptstyle f}\uparrow & \nwarrow{\scriptstyle \exists} & \\ I & \longrightarrow & R \end{array}$. Here, the map $R \to M_{n+1}$ comes from $\begin{array}{ccc} M_n & \longrightarrow & M_{n+1} \\ {\scriptstyle \bigoplus f}\uparrow & & \uparrow \\ \bigoplus\limits_{(I,f)} I & \longrightarrow & \bigoplus\limits_{(I,f)} R \\ \uparrow & & \uparrow \\ I & \longrightarrow & R \end{array}$, where the bottom vertical maps $I \to \bigoplus_{(I,f)} I$ and $R \to \bigoplus_{(I,f)} R$ are the inclusions of the summands indexed by (I, f).

For general rings, i.e. without the condition that every ideal is finitely generated, then a similar construction can be made to work, provided one replaces $\operatorname{colim}_{n\in\mathbb{N}} M_n$ by a colimit indexed over all ordinals which are small than a suitably chosen cardinal. Let λ be the smallest cardinal which is bigger than the cardinality of R. For every ordinal α with $|\alpha| < \lambda$, define inductively $M_0 := M$, $M_\alpha := (M_\beta)'$ if $\alpha = \beta + 1$, and $M_\alpha := \operatorname{colim}_{\beta<\alpha} M_\beta$ if α is a limit ordinal. Then $\operatorname{colim}_{|\alpha|<\lambda} M_\alpha$ is an injective that contains M as a submodule.

Week 7

Recall that a ring is called Noetherian if every ideal is finitely generated. Using Bear's criterion, one can prove:

Lemma 10 (exercise)**.** *Let R be a Noetherian ring, and let $\{I_i\}_{i\in\mathcal{I}}$ be a collection of injective modules. Then $\bigoplus_{i\in\mathcal{I}} I_i$ is injective.*

In the absence of the Noetherian condition, one can still show that $\prod_{i\in\mathcal{I}} I_i$ is injective.

Proposition. A $\mathbb{Z}$-module is injective if and only if it is a direct sum of the following groups: $\mathbb{Q}$, and $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$, for p a prime.

Proof. Let I be an injective $\mathbb{Z}$-module. Consider the collection of submodules M equipped with a direct sum decomposition into pieces isomorphic to $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. This is a poset under inclusion respecting the direct sum decompositions. By an application of Zorn's lemma, this poset admits a maximal element. If the maximal element is I, we're done.

Assume by contradiction that the maximal element M is not I. Since M is injective, the short exact sequence $0 \to M \to I \to I/M \to 0$ splits. So it's enough to find a submodule of $N := I/M$

which is isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$. Note that N is injective as it's a direct summand of an injective module.

Pick $x \in N$, non-zero, and let C_0 be the cyclic subgroup generated by x. Let $C \subset C_0$ be a subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or $\mathbb{Z}$. Let $D := \mathbb{Z}[\frac{1}{p}]/p\mathbb{Z} \cong \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ if $C \cong \mathbb{Z}/p\mathbb{Z}$, and $D := \mathbb{Q}$ if $C \cong \mathbb{Z}$. Since N is injective, the map $C \to N$ extends to a map $D \to N$.

It remains to show that the map $D \to N$ is injective. Indeed, for every non-zero element $d \in D$, there exists $n \in \mathbb{N}$ such that $nd \in C$. The map $D \to N$ is injective when restricted to C. So it's injective on all of D. □

Similarly, if k is an algebraically closed field, a $k[x]$-module is injective if and only if it is a direct sum of copies of the fraction field $k(x)$, and of the modules $k[\tilde{x}, \tilde{x}^{-1}]/k[\tilde{x}]$ for $\tilde{x} := x - a$ and $a \in k$.

Given two objects A and C in some abelian category, there is a canonical bijection between the abelian group $\mathrm{Ext}^1(C, A)$ and the set of isomorphism classes of extension $0 \to A \to B \to C \to 0$. Indeed, this is where the notation "Ext" comes from. Here, two extensions (= short exact sequences) are called *isomorphic* if they fit into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & A & \longrightarrow & B' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

The sum operation on the set of isomorphism classes of extensions of A by B is called the *Baer sum.* Given two extensions

$$0 \to B \to X_1 \xrightarrow{\pi_1} A \to 0 \qquad \text{and} \qquad 0 \to B \to X_2 \xrightarrow{\pi_2} A \to 0$$

let X_{12} be the pullback (or fibre product) $X_1 \times_A X_2 := \{(x_1, x_2) \in X_1 \times X_2 : \pi_1(x_1) = \pi_2(x_2)\}$. There's two copies of B inside X_{12}, namely $\{(b, 0)\}$ and $\{(0, b)\}$. The Baer sum is what one gets by identifying those two copies of B:

$$X_1 \boxplus X_2 := X_{12}/(b, 0) \sim (0, b) = X_{12}/\{(b, -b) : b \in B\}$$

and it again fits into a short exact sequence

$$0 \to B \to X_1 \boxplus X_2 \to A \to 0$$

This operation is obviously commutative. To see that $\boxplus$ is associative, one notes that both $(X_1 \boxplus X_2) \boxplus X_3$ and $X_1 \boxplus (X_2 \boxplus X_3)$ can be identified with $(X_1 \times_A X_2 \times_A X_3)/(b, 0, 0) \sim (0, b, 0) \sim (0, 0, b)$.

The main steps of the proof of the bijection

$$\left\{ \begin{array}{c} \text{Isomorphism classes of} \\ \text{extensions of } A \text{ by } B \end{array} \right\} \cong \mathrm{Ext}^1(A, B)$$

are as follows:

► Given an extension $0 \to A \to B \to C \to 0$ and a resolution $P_\bullet \to C$, construct a map of chain complexes from $(P_\bullet \to C \to 0)$ to $(0 \to A \to B \to C \to 0)$.
► Use this to define an element of $\mathrm{Ext}^1(C, A)$.
► Show that the resulting element of $\mathrm{Ext}^1(C, A)$ does not depend on the choice of map from $(P_\bullet \to C \to 0)$ to $(0 \to A \to B \to C \to 0)$.
► Given an element of $\mathrm{Ext}^1(C, A)$ represented by a map $P_1 \to A$ for some resolution $P_\bullet \to C$, define the group $B := (P_0 \oplus A)/P_1'$, where P_1' is the quotient of P_1 by the image of $d_2 : P_2 \to P_1$.
► Show that B fits into a short exact sequence $0 \to A \to B \to C \to 0$.
► Show that this short exact sequence does not depend on the choice of resolution $P_\bullet \to C$.
► Finally, show that the above constructions are each other's inverses.

Week 8
Recall that given projective resolutions $A \leftarrow P_\bullet$ and $B \leftarrow Q_\bullet$, the cochain complex

$$\underline{\mathrm{Hom}}(C_\bullet, D_\bullet) := \mathrm{Tot}^{\Pi}\Big(\mathrm{Hom}(P_\bullet, Q_\bullet)\Big)$$

computes $\mathrm{Ext}(A, B)$. (By this we mean that the nth cohomology group of this complex is canonically isomorphic to $\mathrm{Ext}(A, B)$.)

Using this fact, composition of homomorphisms $\circ : \mathrm{Hom}(A, B) \otimes \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$ induces a well-defined map $\mathrm{Ext}^i(A, B) \otimes \mathrm{Ext}^j(B, C) \to \mathrm{Ext}^{i+j}(A, C)$. In particular, this equips the graded abelian group

$$\mathrm{Ext}^*(A, A) := \bigoplus_{i=0}^{\infty} \mathrm{Ext}^i(A, A)$$

with the structure of a ring.

Upon identifying cycles in $\underline{\mathrm{Hom}}(C_\bullet, D_\bullet)$ with chain maps $C_\bullet \to D_\bullet$, one get the following convenient description of Ext:

$$H^n\Big(\underline{\mathrm{Hom}}(C_\bullet, D_\bullet)\Big) = \frac{\text{degree } (-n) \text{ chain maps } C_\bullet \to D_\bullet}{\text{chain maps which are chain-homotopic to zero}}$$

Here are some examples of Ext-ring computations:
- $\mathrm{Ext}_{k[x]}(k, k) = k[y]/y^2$, with y in degree 1.
- $\mathrm{Ext}_{k[x]/(x^2)}(k, k) = k[y]$, with y in degree 1.
- $\mathrm{Ext}_{k[x]/(x^3)}(k, k) = k[y, z]/(y^2)$, with y in degree 1 and z in degree 2.

Let's work out the last example in detail. Let $R := k[x]/(x^3)$ and let $P_\bullet := \Big(R \overset{x}{\leftarrow} R \overset{x^2}{\longleftarrow} R \overset{x}{\leftarrow} R \ldots\Big)$ be a resolution of k. Then the generator y of $\mathrm{Ext}^1(k, k)$ is given by

$$y := \begin{array}{ccccccccccccc}
0 & \longleftarrow & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R \;\cdots \\
 & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 1} \\
 & & 0 & \longleftarrow & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R \;\cdots
\end{array}$$

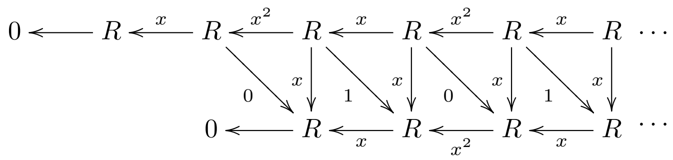
and the generator z of $\mathrm{Ext}^2(k, k)$ is given by

$$z := \begin{array}{ccccccccccccc}
0 & \longleftarrow & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R \;\cdots \\
 & & & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
 & & & & 0 & \longleftarrow & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R \;\cdots
\end{array}$$

To check that $y^2 = 0$ in the ring $\mathrm{Ext}^*(k, k)$, one composes the chain maps as follows:

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R \;\cdots \\
 & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 1} \\
 & & 0 & \longleftarrow & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R \;\cdots \\
 & & & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle x} \\
 & & & & 0 & \longleftarrow & R & \xleftarrow{x} & R & \xleftarrow{x^2} & R & \xleftarrow{x} & R \;\cdots
\end{array}$$

This gives $x \cdot z$, which is zero in the Ext ring (because $\mathrm{Ext}^2(k,k) = k$ as an R-module). Alternatively, one can construct an explicit null-homotopy of the above composite:



Exercise 22. *Let k be a field, and let $R := k[x,y]$. Write k for the R-module $R/(x,y)$.*
Let $n_1 > n_2 > \ldots > n_s = 0$, and $0 = m_1 < m_2 < \ldots < m_s$ be integers.
Compute $\mathrm{Tor}_*^R(R/(x^{n_1}y^{m_1}, x^{n_2}y^{m_2}, \ldots, x^{n_s}y^{m_s}), k)$
A projective resolution of $R/(x^{n_1}y^{m_1}, x^{n_2}y^{m_2}, \ldots, x^{n_s}y^{m_s})$ is given by

$$R^{s-1} \xrightarrow{\begin{pmatrix} y^{m_2-m_1} & 0 & \ldots & 0 \\ x^{n_1-n_2} & y^{m_3-m_2} & \ldots & 0 \\ 0 & x^{n_2-n_3} & y^{m_4-m_3} & \ldots \\ & & & \ddots \\ & \ldots & & \end{pmatrix}} R^s \xrightarrow{\begin{pmatrix} x^{n_1}y^{m_1} \\ \vdots \\ x^{n_s}y^{m_s} \end{pmatrix}} R$$

After tensoring by k, the differentials become zero and we get $\mathrm{Tor}_0 = k$, $\mathrm{Tor}_1 = k^s$, $\mathrm{Tor}_2 = k^{s-1}$.
Let $R = k[x,y]$ be as above, and let $a > n > 0$ be integers.
Compute $\mathrm{Tor}_*^R(R/(x^n, y^n), R/(x^a, xy, y^a))$.

$R \xrightarrow{(\,y^n\ -x^n\,)} R^2 \xrightarrow{\begin{pmatrix} x^n \\ y^n \end{pmatrix}} R$ is a projective resolution of $R/(x^n,y^n)$. After tensoring with $R/(x^a,xy,y^a)$, this becomes

$$R/(x^a,xy,y^a) \xrightarrow{d_2 = (\,y^n\ -x^n\,)} \left[R/(x^a,xy,y^a)\right]^2 \xrightarrow{d_1 = \begin{pmatrix} x^n \\ y^n \end{pmatrix}} R/(x^a,xy,y^a).$$

The homology in degree zero is $\mathrm{Tor}_0 = \mathrm{coker}(d_1) = R/(x^n,xy,y^n)$. The kernel of d_1 has a k-basis given by $\{x^{a-n}, x^{a-n+1}, \ldots, x^{a-1}, y, y^2, \ldots, y^{a-1}\}$ in the first copy of $R/(x^a,xy,y^a)$ and by $\{x, x^2, \ldots, x^{a-1}, y^{a-n}, y^{a-n+1}, \ldots, y^{a-1}\}$ in second first copy of $R/(x^a,xy,y^a)$. Let us write

$$x_1^{a-n}, x_1^{a-n+1}, \ldots, x_1^{a-1}, y_1, y_1^2, \ldots, y_1^{a-1} \quad \textit{and} \quad x_2, x_2^2, \ldots, x_2^{a-1}, y_2^{a-n}, y_2^{a-n+1}, \ldots, y_2^{a-1}$$

to distinguish them. The elements in the image of d_2 are $y_1^n - x_2^n$, y_1^{n+1}, y_1^{n+2}, …, y_1^{a-1}, x_2^{n+1}, x_2^{n+2}, …, x_2^{a-1}. So a k-basis of $\mathrm{Tor}_1 = \ker(d_1)/\mathrm{im}(d_2)$ is given by

$$x_1^{a-n}, x_1^{a-n+1}, \ldots, x_1^{a-1}, y_1, y_1^2, \ldots, y_1^n = x_2^n, x_2^{n-1}, \ldots, x_2^3, x_2^2, x_2, y_2^{a-n}, y_2^{a-n+1}, \ldots, y_2^{a-1}$$

which decomposes as an R-module as

$$\underbrace{x_1^{a-n}, x_1^{a-n+1}, \ldots, x_1^{a-1}}_{\cong R/(y,x^n)}, \underbrace{y_1, y_1^2, \ldots, y_1^n = x_2^n, x_2^{n-1}, \ldots, x_2^3, x_2^2, x_2}_{\cong \frac{(x^{n-1},y^{n-1})}{(x^n,y^n)}}, \underbrace{y_2^{a-n}, y_2^{a-n+1}, \ldots, y_2^{a-1}}_{\cong R/(y^n,x)}$$

Finally, $\mathrm{Tor}_2 = \ker(d_2)$ has a k-basis given by $x^{a-n}, x^{a-n+1}, \ldots, x^{a-1}$ and $y^{a-n}, y^{a-n+1}, \ldots, y^{a-1}$, and is isomorphic to $R/(y,x^n) \oplus R/(y^n,x)$.

Exercise 23. *Consider the abelian category whose objects are diagrams $(M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \ldots)$ of abelian groups indexed by $\mathbb{N}$, and whose morphisms are natural transformations between such diagrams. Show, that the functor which sends an object $(M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \ldots)$ to its inverse limit $\varprojlim M_i$ is not right exact.*

Hint: *Construct a suitable morphism between the object $(\mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \overset{id}{\leftarrow} \mathbb{Z} \ldots)$ and the object $(\mathbb{Z}/2\mathbb{Z} \twoheadleftarrow \mathbb{Z}/4\mathbb{Z} \twoheadleftarrow \mathbb{Z}/8\mathbb{Z} \ldots)$, and analyse its properties.*

In order to show that a functor F is not right exact, it suffices to exhibit an epimorphism f such that $F(f)$ is not an epimorphism.
We consider the morphism

$$\begin{array}{ccccccccc} \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z}/2\mathbb{Z} & \leftarrow & \mathbb{Z}/4\mathbb{Z} & \leftarrow & \mathbb{Z}/8\mathbb{Z} & \leftarrow & \mathbb{Z}/16\mathbb{Z} & \leftarrow & \dots \end{array}$$

Its image under the functor $\varprojlim$ is the morphism of abelian groups $\mathbb{Z} \to \mathbb{Z}_2$ (the inclusion of the integers into the 2-adic integers). The latter is not be an epimorphism.

Consider the derived functors $\lim^i := R^i(\varprojlim)$ *of the inverse limit functor*

$$\varprojlim : (M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \dots) \mapsto (\varprojlim M_i).$$

[You may assume the knowledge that the inverse limit functor is left exact]
Assuming the knowledge that the functors $\lim^i$ *for* $i \geq 1$ *yield zero when evaluated on the object* $(\mathbb{Z} \xleftarrow{id} \mathbb{Z} \xleftarrow{id} \mathbb{Z} \dots)$*, compute the value of*

$$\lim{}^1 \big(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots\big).$$

The short exact sequence

$$0 \to (\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots) \to (\mathbb{Z} \xleftarrow{id} \mathbb{Z} \xleftarrow{id} \mathbb{Z} \dots) \to (\mathbb{Z}/2\mathbb{Z} \leftarrow \mathbb{Z}/4\mathbb{Z} \leftarrow \mathbb{Z}/8\mathbb{Z} \leftarrow \dots) \to 0$$

yields a long exact sequence of derived functors

$$\begin{aligned} 0 \to \varprojlim(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots) &\to \varprojlim(\mathbb{Z} \xleftarrow{id} \mathbb{Z} \xleftarrow{id} \mathbb{Z} \dots) \\ &\to \varprojlim(\mathbb{Z}/2\mathbb{Z} \leftarrow \mathbb{Z}/4\mathbb{Z} \leftarrow \mathbb{Z}/8\mathbb{Z} \leftarrow \dots) \\ &\to \lim{}^1(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots) \to 0 \end{aligned}$$

which reads

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z}_2 \to ? \to 0$$

It follows that $\lim^1(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots) = \mathbb{Z}_2/\mathbb{Z}$.

Exercise 24. *Given a possibly non-abelian group G, the nth homology group of G with coefficients in an abelian group A is defined to be the nth Tor-group* $\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A)$*. (Here,* $\mathbb{Z}[G]$ *denotes the group algebra of G i.e., the free abelian group on the elements of G, equipped with the ring structure inherited from the multiplication in G).*

Here, both $\mathbb{Z}$ *and A are equipped with the action of* $\mathbb{Z}[G]$ *in which all the generators of G act trivially.*

Let G be the cyclic group of order four, so that $\mathbb{Z}[G] = \mathbb{Z}[x]/(x^4 - 1)$*. Compute the group homology* $H_i(G, \mathbb{Z})$ *for all i.*

The group algebra $\mathbb{Z}[G]$ *is the same as the ring* $\mathbb{Z}[x]/(x^4 - 1)$*. So, by definition,* $H_i(G, \mathbb{Z}) = \mathrm{Tor}_i^R(\mathbb{Z}, \mathbb{Z})$.
A free resolution of $\mathbb{Z}$ *is given by*

$$\dots R \xrightarrow{1 \mapsto 1+x+x^2+x^3} R \xrightarrow{1 \mapsto 1-x} R \xrightarrow{1 \mapsto 1+x+x^2+x^3} R \xrightarrow{1 \mapsto 1-x} R \to \mathbb{Z}$$

Removing the last term and tensoring by $\mathbb{Z}$*, we get*

$$\dots \mathbb{Z} \xrightarrow{1+x+x^2+x^3} \mathbb{Z} \xrightarrow{1-x} \mathbb{Z} \xrightarrow{1+x+x^2+x^3} \mathbb{Z} \xrightarrow{1-x} \mathbb{Z} \to 0$$

which is

$$\dots \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

So the homology is $\mathbb{Z}$ in degree zero, $\mathbb{Z}/4$ is odd degrees, and zero otherwise.

Exercise 25. *Compute the structure of the graded ring* $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$.
Compute the structure of the graded ring $\mathrm{Ext}^*_{\mathbb{Z}/8}(\mathbb{Z}/4, \mathbb{Z}/4)$.