Homological algebra

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<u>**Sheet 3**</u>

Exercise 1. Compute $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$, and $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ first using projective resolutions, and then using injective resolutions.

Exercise 2. Let k be a field. Compute $\mathrm{Tor}^R_*(k,k)$ and $\mathrm{Ext}^*_R(k,k)$ for $R = k[x]$, $R = k[x,y]$, $R = k[x]/x^n$, $R = k[x,y]/(x^n, y^m)$, $R = k[x,y]/(x^2, y^2, xy)$, $R = k[x,y]/(x^3 - y^2)$.

Exercise 3. Compute $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ using the formula $H^*(\mathrm{Tot}(\mathrm{Hom}_R(P_\bullet, I^\bullet)))$.

Exercise 4. Compute $\mathrm{Tor}^{\mathbb{C}[x]/x^2}_*(\mathbb{C}, \mathbb{C})$ using the formula $H_*(\mathrm{Tot}(P_\bullet \otimes_R Q_\bullet))$.

Exercise 5. Write down an example of a bigraded chain complex $C_{\bullet\bullet}$ which fails the condition *"for every n there exists only finitely many pairs (p,q) such that $p+q=n$ and $C_{p,q} \neq 0$"*, and for which the implication

$$\Big(C_{\bullet\bullet} \text{ has exact rows}\Big) \Rightarrow \Big(\mathrm{Tot}(C_{\bullet\bullet}) \text{ is exact}\Big)$$

fails. In other words, you must find a bigraded chain complex $C_{\bullet\bullet}$ which has has exact rows, but such that $\mathrm{Tot}(C_{\bullet\bullet})$ is not exact.

Exercise 6. Prove that an abelian group A is *flat* as a $\mathbb{Z}$-module if and only if it is *torsion-free*. *Hint:* Write A as a colimit of free abelian groups.

Exercise 7. Prove that an abelian group A is *injective* as a $\mathbb{Z}$-module if and only if it is *divisible* (here, divisible means $\forall a \in A, \forall n \in \mathbb{N}, \exists x \in A$ s.t. $nx = a$).

(Hand in the Monday before the class, at 5pm)