Homological algebra

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Sheet 4

Exercise 1. *Consider the abelian category whose objects are diagrams* $(M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \dots)$ *of abelian groups indexed by* $\mathbb{N}$*, and whose morphisms are natural transformations between such diagrams. Show, that the functor which sends an object* $(M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \dots)$ *to its inverse limit* $\varprojlim M_i$ *is not right exact.*

Hint: *Construct a suitable morphism between the object* $(\mathbb{Z} \xleftarrow{id} \mathbb{Z} \xleftarrow{id} \mathbb{Z} \dots)$ *and the object* $(\mathbb{Z}/2\mathbb{Z} \twoheadleftarrow \mathbb{Z}/4\mathbb{Z} \twoheadleftarrow \mathbb{Z}/8\mathbb{Z} \dots)$*, and analyse its properties.*

In order to show that a functor F is not right exact, it suffices to exhibit an epimorphism f such that $F(f)$ is not an epimorphism.
We consider the morphism

$$\begin{array}{ccccccccc} \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \mathbb{Z} & \xleftarrow{id} & \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathbb{Z}/2\mathbb{Z} & \twoheadleftarrow & \mathbb{Z}/4\mathbb{Z} & \twoheadleftarrow & \mathbb{Z}/8\mathbb{Z} & \twoheadleftarrow & \mathbb{Z}/16\mathbb{Z} & \twoheadleftarrow & \dots \end{array}$$

Its image under the functor $\varprojlim$ is the morphism of abelian groups $\mathbb{Z} \to \mathbb{Z}_2$ (the inclusion of the integers into the 2-adic integers). The latter is not be an epimorphism.

Consider the derived functors $\lim^i := R^i(\varprojlim)$ *of the inverse limit functor*

$$\varprojlim : (M_1 \xleftarrow{f_1} M_2 \xleftarrow{f_2} M_3 \xleftarrow{f_3} \dots) \mapsto (\varprojlim M_i).$$

[You may assume the knowledge that the inverse limit functor is left exact]
Assuming the knowledge that the functors $\lim^i$ *for* $i \geq 1$ *yield zero when evaluated on the object* $(\mathbb{Z} \xleftarrow{id} \mathbb{Z} \xleftarrow{id} \mathbb{Z} \dots)$*, compute the value of*

$$\lim{}^1(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots).$$

The short exact sequence

$$0 \to (\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots) \to (\mathbb{Z} \xleftarrow{id} \mathbb{Z} \xleftarrow{id} \mathbb{Z} \dots) \to (\mathbb{Z}/2\mathbb{Z} \twoheadleftarrow \mathbb{Z}/4\mathbb{Z} \twoheadleftarrow \mathbb{Z}/8\mathbb{Z} \twoheadleftarrow \dots) \to 0$$

yields a long exact sequence of derived functors

$$\begin{aligned} 0 \to \varprojlim(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots) &\to \varprojlim(\mathbb{Z} \xleftarrow{id} \mathbb{Z} \xleftarrow{id} \mathbb{Z} \dots) \\ &\to \varprojlim(\mathbb{Z}/2\mathbb{Z} \twoheadleftarrow \mathbb{Z}/4\mathbb{Z} \twoheadleftarrow \mathbb{Z}/8\mathbb{Z} \twoheadleftarrow \dots) \\ &\to \lim{}^1(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots) \to 0 \end{aligned}$$

which reads

$$0 \to 0 \to \mathbb{Z} \to \mathbb{Z}_2 \to ? \to 0$$

It follows that $\lim^1(\mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \mathbb{Z} \xleftarrow{\cdot 2} \dots) = \mathbb{Z}_2/\mathbb{Z}$.

Exercise 2. *Given a possibly non-abelian group G, the nth homology group of G with coefficients in an abelian group A is defined to be the nth Tor-group $\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A)$. (Here, $\mathbb{Z}[G]$ denotes the group algebra of G i.e., the free abelian group on the elements of G, equipped with the ring structure inherited from the multiplication in G).*

Here, both $\mathbb{Z}$ and A are equipped with the action of $\mathbb{Z}[G]$ in which all the generators of G act trivially.

Let G be the cyclic group of order four, so that $\mathbb{Z}[G] = \mathbb{Z}[x]/(x^4 - 1)$. Compute the group homology $H_i(G, \mathbb{Z})$ for all i.

The group algebra $\mathbb{Z}[G]$ is the same as the ring $\mathbb{Z}[x]/(x^4 - 1)$. So, by definition, $H_i(G, \mathbb{Z}) = \mathrm{Tor}_i^R(\mathbb{Z}, \mathbb{Z})$.

A free resolution of $\mathbb{Z}$ is given by

$$\dots R \xrightarrow{1 \mapsto 1+x+x^2+x^3} R \xrightarrow{1 \mapsto 1-x} R \xrightarrow{1 \mapsto 1+x+x^2+x^3} R \xrightarrow{1 \mapsto 1-x} R \to \mathbb{Z}$$

Removing the last term and tensoring by $\mathbb{Z}$, we get

$$\dots \mathbb{Z} \xrightarrow{1+x+x^2+x^3} \mathbb{Z} \xrightarrow{1-x} \mathbb{Z} \xrightarrow{1+x+x^2+x^3} \mathbb{Z} \xrightarrow{1-x} \mathbb{Z} \to 0$$

which is

$$\dots \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{4} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

So the homology is $\mathbb{Z}$ in degree zero, $\mathbb{Z}/4$ is odd degrees, and zero otherwise.

Exercise 3. *Compute the structure of the graded ring $\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$.*
Compute the structure of the graded ring $\mathrm{Ext}^*_{\mathbb{Z}/8}(\mathbb{Z}/4, \mathbb{Z}/4)$.

$\mathrm{Ext}^*_{\mathbb{Z}}(\mathbb{Z}/2, \mathbb{Z}/2)$ is $\mathbb{Z}/2$ in degree 0, $\mathbb{Z}/2$ in degree 1 and zero in all other degrees. There's only one ring with that structure, namely $(\mathbb{Z}/2)[x]/(x^2)$.

Let $R := \mathbb{Z}/8$ and let $P_\bullet := \left(R \xleftarrow{4} R \xleftarrow{2} R \xleftarrow{4} R \dots\right)$ be a resolution of $\mathbb{Z}/4$. Then the generator y of $\mathrm{Ext}^1(\mathbb{Z}/4, \mathbb{Z}/4) = \mathbb{Z}/2$ is given by

$$y := \begin{array}{ccccccccccccc}
0 & \longleftarrow & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R \;\cdots \\
 & & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} \\
 & & 0 & \longleftarrow & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R & \xleftarrow{2} & R \;\cdots
\end{array}$$

and the generator z of $\mathrm{Ext}^2(\mathbb{Z}/4, \mathbb{Z}/4) = \mathbb{Z}/2$ is given by

$$z := \begin{array}{ccccccccccccc}
0 & \longleftarrow & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R \;\cdots \\
 & & & & & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} \\
 & & & & 0 & \longleftarrow & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R \;\cdots
\end{array}$$

To check that $y^2 = 0$ in the ring $\mathrm{Ext}^*(\mathbb{Z}/4, \mathbb{Z}/4)$, one composes the chain maps as follows:

$$\begin{array}{ccccccccccccc}
0 & \longleftarrow & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R \;\cdots \\
 & & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} \\
 & & 0 & \longleftarrow & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R & \xleftarrow{2} & R \;\cdots \\
 & & & & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 1} \\
 & & & & 0 & \longleftarrow & R & \xleftarrow{4} & R & \xleftarrow{2} & R & \xleftarrow{4} & R \;\cdots
\end{array}$$

This gives $2{\cdot}z$, which is zero. So $\mathrm{Ext}^*(\mathbb{Z}/4, \mathbb{Z}/4) = (\mathbb{Z}/4)[y, z]/(2y, y^2, 2z)$, where y is in degree 1, and z is in degree 2.