Homological algebra (Oxford, fall 2017)

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Problem sheet 1:
(hand in Monday Oct. 16th at noon, or Monday Oct. 23rd at noon)

Exercise 1. Construct an example of a chain complex of $\mathbb{Z}$-modules $C_\bullet$ such that:
(1) each $\mathbb{Z}$-module C_n is free,
(2) the only non-zero homology group of $C_\bullet$ is in degree zero, and it is isomorphic to $\mathbb{Z}/2$.

Exercise 2. An abelian group A is called 2-*divisible* if the map $A \to A : a \mapsto 2a$ is surjective.
Construct an example of a chain complex of abelian groups $C_\bullet$ such that:
(1) each abelian group C_n is 2-divisible,
(2) the only non-zero homology group of $C_\bullet$ is in degree zero, and it is isomorphic to $\mathbb{Z}/2$.

Exercise 3. Let k be a field, and let $R := k[x, y]$. Construct a chain complex of R-modules $C_\bullet$ s.t.:
(1) each R-module C_n is free,
(2) the only non-zero homology group of $C_\bullet$ is in degree zero, and it is isomorphic to k.
Repeat this exercise for the rings $R = k[x, y, z]$ and $R = k[x, y, z, t]$.

Exercise 4. Let k be a field. Prove that any chain complex of k-vector space is a direct sum of chain comlpexes of the following form:
(1) $\ldots \to 0 \to 0 \to V \xrightarrow{=} V \to 0 \to 0 \to \ldots$, where the two non-zero spaces are in degrees n and $n-1$,
(2) $\ldots \to 0 \to 0 \to W \to 0 \to 0 \to \ldots$, where the unique non-zero space is in degree n.

Exercise 5. Let $C_\bullet$ be the chain complex of $\mathbb{F}_2$-vector spaces associated to a polyhedral complex X, as discussed in class (see Application 1.1.3 in the book).
Describe $C_\bullet$ and compute its homology when X is the 1-skeleton of a tetrahedron.
Describe $C_\bullet$ and compute its homology when X is a 2-dimensional torus.

Exercise 6. Show that for a short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$, the following three conditions are equivalent:
(1) The sequence $0 \to A \to B \to C \to 0$ is isomorphic to one of the form $0 \to A \to A \oplus B \to B \to 0$.
(2) There exists a section $s : C \to B$ of the map g.
(2) There exists a retraction $r : B \to A$ of the map f.

Exercise 7. Provide an example of a short exact sequence of abelian groups $0 \to A \to B \to C \to 0$ such that the corresponding chain complex $0 \to A/2A \to B/2B \to C/2C \to 0$ is not exact.
Prove that the map $B/2B \to C/2C$ is always surjective.
Prove that we always have $\mathrm{im}(A/2A \to B/2B) = \ker(B/2B \to C/2C)$.

Exercise 8. Let A be a set, and suppose that we are given a chain complex $C^\alpha_\bullet$ for every $\alpha \in A$. Then we can form the chain complexes $\bigoplus_{\alpha\in A} C^\alpha_\bullet$ and $\prod_{\alpha\in A} C^\alpha_\bullet$. Prove that, for every $n \in \mathbb{Z}$, we have

$$H_n\Big(\bigoplus_{\alpha\in A} C^\alpha_\bullet\Big) = \bigoplus_{\alpha\in A} H_n(C^\alpha_\bullet) \qquad \text{and} \qquad H_n\Big(\prod_{\alpha\in A} C^\alpha_\bullet\Big) = \prod_{\alpha\in A} H_n(C^\alpha_\bullet).$$