Homological algebra (Oxford, fall 2017)

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Problem sheet 2: (hand in Monday Oct. 30th at noon, or Monday Nov. 6th at noon)

Exercise 1. Prove that, in the category of free abelian groups, the cokerel of the map $2 : \mathbb{Z} \to \mathbb{Z}$ is the zero group.

Exercise 2. Prove that, in the category of R-modules, a morphism $f : M \to N$ is an epimorphism if and only if it is surjective, and a monomorphism if and only if it is injective.

Exercise 3. Let $f : A \to B$ be a morphism in an abelian category.
Prove that the morphism $\ker(f) \to A$ is always a monomorphism.
Prove that the morphism $B \to \mathrm{coker}(f)$ is always an epimorphism.

Exercise 4. Let $A_\bullet, B_\bullet \in \mathrm{Ch}(\mathcal{A})$ be chain complexes in an abelian category $\mathcal{A}$, and let $f_\bullet : A_\bullet \to B_\bullet$ be a morphism in $\mathrm{Ch}(\mathcal{A})$. Let $K_n := \ker(f_n)$, with structure morphism $\iota_n : K_n \to A_n$.
▷ Show that the differentials $d_n^A : A_n \to A_{n-1}$ induce morphisms $d_n^K : K_n \to K_{n-1}$, and that the latter satisfy $d_n^K \circ d_{n+1}^K = 0$.
▷ Show that the morphisms $\iota_n : K_n \to A_n$ assemble to a morphism of chain complexes $\iota_\bullet : K_\bullet \to A_\bullet$, and that the latter exhibits $K_\bullet$ as the kernel of $f_\bullet$.

Exercise 5. Compute the long exact sequence associated to the following short exact sequence of chain complexes:

$$
\begin{array}{ccccccccc}
 & & & \vdots & & \vdots & & & \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \cdot 8} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathbb{Z}/4 & \xrightarrow{\cdot 4} & \mathbb{Z}/16 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \cdot 8} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathbb{Z}/4 & \xrightarrow{\cdot 4} & \mathbb{Z}/16 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \cdot 8} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & \mathbb{Z}/4 & \xrightarrow{\cdot 4} & \mathbb{Z}/16 & \longrightarrow & \mathbb{Z}/4 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \cdot 8} & & \downarrow{\scriptstyle 0} & & \\
 & & & \vdots & & \vdots & & &
\end{array}
$$

Exercise 6. Provide an example of a short exact sequence of chain complexes
$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ such that:
i) the only non-zero homology group of $A_\bullet$ is in degree $n-1$, and is isomorphic to $\mathbb{Z}$.
ii) the only non-zero homology group of $C_\bullet$ is in degree n, and is isomorphic to $\mathbb{Z}$.
iii) The connecting homomorphism $\partial_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ is an isomorphism.

Exercise 7. Provide an example of a short exact sequence of chain complexes
$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ with the property that:
i) the only non-zero homology group of $A_\bullet$ is in degree zero, and is isomorphic to $\mathbb{Z}/2$.
ii) the only non-zero homology group of $C_\bullet$ is in degree one, and is isomorphic to $\mathbb{Z}/2$.
iii) The connecting homomorphism $\partial_1 : H_1(C_\bullet) \to H_0(A_\bullet)$ is an isomorphism.
iv) The groups A_n, B_n, and C_n are all free abelian groups.

Exercise 8. Let $0 \to M \to N \to P \to 0$ be a short exact sequence of R-modules.
Give an example of a short exact sequence of chain complexes $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ such that:
i) the only non-zero homology group of $A_\bullet$ is in degree zero, and is isomorphic to N.
ii) the only non-zero homology group of $B_\bullet$ is in degree zero, and is isomorphic to P.
ii) the only non-zero homology group of $C_\bullet$ is in degree one, and is isomorphic to M.
iii) The long exact sequence associated to the short exact sequence $0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ recovers the short exact sequence $0 \to M \to N \to P \to 0$ we started with.