Homological algebra (Oxford, fall 2017)

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Problem sheet 3: (hand in Monday Nov. 13th at noon, or Monday Nov. 20th at noon)

Exercise 1. Let $\mathbf{Ab}_{\text{fin}}$ denote the category of finite abelian groups.
▷ Prove that the only projective object of $\mathbf{Ab}_{\text{fin}}$ is the zero group.
▷ Prove that the only injective object of $\mathbf{Ab}_{\text{fin}}$ is the zero group.

Exercise 2. Let $R := \mathbb{Z}[\sqrt{-5}]$, and let $M \subset R$ be the ideal generated by $1 + \sqrt{-5}$ and $1 - \sqrt{-5}$. Prove that the map $R \oplus R \to M \oplus M$ given by $(1, 0) \mapsto (1 + \sqrt{-5}, 2)$ and $(0, 1) \mapsto (2, 1 - \sqrt{-5})$ is an isomorphism. Deduce that M is a projective R-module.

Exercise 3. Prove that $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$-module. (without using the Baer's criterion)

Exercise 4. An abelian group A is called *divisible* if $\forall a \in A$ and $\forall n \in \mathbb{N}$, $\exists b \in A$ such that $nb = a$. Prove that every divisible abelian group is an injective $\mathbb{Z}$-module. (without using the Baer's criterion)

Exercise 5. Let $a, b, n \in \mathbb{N}$ be such that $a|n$ and $b|n$.
Let $P_\bullet = (P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \ldots)$ be a projective resolution of $\mathbb{Z}/a$ as a $\mathbb{Z}/n$-module. Compute the homology of $P_\bullet/b := (P_0/bP_0 \leftarrow P_1/bP_1 \leftarrow P_2/bP_2 \leftarrow \ldots)$. Prove that the answer is independent of the choice of projective resolution $P_\bullet$.

Exercise 6. Let $a, b, n \in \mathbb{N}$ be such that $a|n$ and $b|n$.
Let $P_\bullet = (P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \ldots)$ be a projective resolution of $\mathbb{Z}/a$ as a $\mathbb{Z}/n$-module. Compute the cohomology of $\mathrm{Hom}(P_\bullet, \mathbb{Z}/b\mathbb{Z}) := \big[\mathrm{Hom}(P_0, \mathbb{Z}/b) \to \mathrm{Hom}(P_1, \mathbb{Z}/b) \to \mathrm{Hom}(P_2, \mathbb{Z}/b) \to \ldots\big]$. Prove that the answer is independent of the choice of projective resolution $P_\bullet$.

Exercise 7. Let $a, b, n \in \mathbb{N}$ be such that $a|n$ and $b|n$.
Let $I^\bullet = (I^0 \to I^1 \to I^2 \to \ldots)$ be an injective resolution of $\mathbb{Z}/b$ as a $\mathbb{Z}/n$-module. Compute the cohomology of $\mathrm{Hom}(\mathbb{Z}/a\mathbb{Z}, I^\bullet\mathbb{Z}) := \big[\mathrm{Hom}(\mathbb{Z}/a, I^0) \to \mathrm{Hom}(\mathbb{Z}/a, I^1) \to \mathrm{Hom}(\mathbb{Z}/a, I^2) \to \ldots\big]$. Prove that the answer is independent of the choice of injective resolution $I^\bullet$.

Exercise 8. Let $P_\bullet \to M$ be a projective resolution, let $Q_\bullet \to N$ be a projective resolution, and let $0 \to M \to E \to N \to 0$ be a short exact sequence.

Show that there exists a projective resolution $S_\bullet \to E$ that fits into a short exact sequence of aumgented chain complexes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \twoheadrightarrow & M \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & S_2 & \longrightarrow & S_1 & \longrightarrow & S_0 & \twoheadrightarrow & E \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & Q_2 & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \twoheadrightarrow & N \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}
$$

Hint: Set $S_n := P_n \oplus Q_n$.