C2.1a Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 0 (not for handing in)

1. Let V be a finite dimensional vector space and let $A \subset \mathrm{End}(V)$ denote a subspace consisting of commuting diagonalisable endomorphisms. Show that we may find a basis of V in which each element of A is represented by a diagonal matrix.

Solution: We induct on the dimension of A, with the case $\dim A = 1$ being clear. Choose $0 \neq x \in T$ and write $T = kx \oplus T'$ for some subspace $T' \subset T$. Now, for all eigenvalues λ of x we set

$$V_\lambda = \ker(x - \lambda \, \mathrm{id}_V)$$

to be the λ-eigenspace of x. Then, given any $v \in V_\lambda$ and $t \in T$ we have

$$(x - \lambda \, \mathrm{id}_V)(tv) = t(x - \lambda \, \mathrm{id}_V)v = 0$$

because t commutes with x and $\lambda \, \mathrm{id}_V$. Hence T preserves the decomposition of V into eigenspaces. By assumption x acts diagonally on each V_λ and we are done, because we can apply induction to T' and each V_λ. □

2. Let k be a field and let V be a k-vector space. If $x \in \mathrm{End}(V)$, and $V = \bigoplus_\lambda V_\lambda$ is the decomposition of V into a direct sum of generalised eigenspaces of x, we define $x_s \in \mathrm{End}(V)$ to be the linear map given by $x_s(v) = \lambda.v$ for $v \in V_\lambda$. It is called the *semisimple* part of x. Clearly it is diagonalisable.

i) Show that the element $x_n = x - x_s$ is nilpotent, and check that x_s and x_n commute.

ii) Show that if $x, y \in \mathrm{End}(V)$ commute, and y is nilpotent, then the generalised eigenspaces of x and $x + y$ coincide.

Solution: For the first part, note that on each V_λ, $x - x_s$ is nilpotent by the definition of a generalised eigenspace, *i.e.* x_n is certainly nilpotent each V_λ and hence is nilpotent on V.

For the second part, consider the generalised eigenspace decomposition $V = \bigoplus_\lambda V_\lambda(x)$ for x. On each V_λ we may write $x = \lambda.1 + n$ where n is nilpotent. Since x and y commute, it is clear that y preserves each $V_\lambda(x)$, and on $V_\lambda(x)$ we have n and y commute. Thus $(n + y)$ is nilpotent on $V_\lambda(x)$, and hence $V = \bigoplus_\lambda V_\lambda(x)$ is the generalised eigenspace decomposition of $x + y$.

To elaborate on this: it's clear (with the obvious notation) that $V_\lambda(x) \subseteq V_\lambda(x + y)$ from the above, but then the fact that the $V_\lambda(x)$s and the $V_\lambda(x+y)$s both form a direct sum decomposition of V ensures by a dimension count that the containments must all actually be equalities.

□

3. Let k be an infinite field (not necessarily algebraically closed or of characteristic zero), and suppose that V is a finite dimensional k-vector space. If $U_1, U_2, \ldots, U_r$ are proper subspaces of V, show that $V \neq U_1 \cup U_2 \cup \ldots \cup U_r$.

Solution: We use induction on r. For $r = 1$ the result is immediate. Now suppose that $V = \bigcup_{i=1}^{r+1} U_i$. Then by induction we can assume that $V \neq \bigcup_{i=1}^{r} U_i$, so we may pick $u \in V$ such that $u \notin \bigcup_{i=1}^{r} U_i$. By assumption, we must then have $u \in U_{r+1}$. Similarly, since U_{r+1} is a proper subspace of V we may pick $w \in V$ such that $w \notin U_{r+1}$. But then consider the vectors

$$v_\lambda = w + \lambda.u \in V, \quad \lambda \in \mathsf{k}^\times$$

Clearly $v_\lambda \notin U_{r+1}$, since that would imply $w \in U_{r+1}$, thus each v_λ lies in some U_i for $1 \leq i \leq r$. But then if $\mathsf{k}^\times$ contains more than n elements there must be two distinct elements λ, μ such that $v_\lambda, v_\mu \in U_j$ for the same j $(1 \leq j \leq r)$. But then it is clear that both w and u lie in U_j which contradicts our assumption. □

There is an algebraic way to think about the idea of "infinitesimals". The next two questions of the sheet explore this idea a little. Let k be a field and let $D_{\mathsf{k}} = \mathsf{k}[t]/(t^2)$. Write ε for the image of t in D_{k}, so that $\varepsilon^2 = 0$. We want to consider $\mathrm{Mat}_n(D_{\mathsf{k}})$ the space of $n \times n$ matrices over D_{k}.

4. Show that $\mathrm{GL}_n(D_{\mathsf{k}})$, the group of invertible matrices over D_{k} is exactly the set:

$$\{A + \varepsilon B : A \in \mathrm{GL}_n(\mathsf{k}), B \in \mathrm{Mat}_n(\mathsf{k})\}.$$

The natural homomorphism $\mathrm{e}\colon D_{\mathsf{k}} \to \mathsf{k}$ given by $\epsilon \mapsto 0$ induces a homomorphism of groups $\mathrm{e}_n : \mathrm{GL}_n(D_{\mathsf{k}}) \to \mathrm{GL}_n(\mathsf{k})$. Deduce that the kernel can be identified with $\mathrm{Mat}_n(\mathsf{k})$, *i.e.* $\mathfrak{gl}_n(\mathsf{k})$.

Solution: Clearly any matrix $X \in \mathrm{Mat}_n(D_{\mathsf{k}})$ can be written uniquely as $A + \varepsilon B$ where $A, B \in \mathrm{Mat}_n(\mathsf{k})$. If we write $Y \in \mathrm{Mat}_n(D_{\mathsf{k}})$ similarly as $C + \varepsilon D$ then we see

$$XY = (A + \varepsilon B)(C + \varepsilon D) = AC + \varepsilon(BC + AD).$$

hence $XY = I_n$ if and only if $AC = I_n$ and $BC + AD = 0$. Thus it follows that X is invertible if and only if A is invertible with inverse $Y = A^{-1} - \varepsilon A^{-1}BA^{-1}$. The second part of the question is immediate, since the kernel consists of the matrices of the form $I_n + \varepsilon B$. $\square$

5. i) The determinant is defined for a matrix with entries in any commutative ring. For $X \in \mathrm{Mat}_n(D_{\mathsf{k}})$ find $\det(X)$ in terms of the column vectors of A, B where $X = A + \varepsilon B$, $A, B \in \mathrm{Mat}_n(\mathsf{k})$. In particular, show that if $X = I + \varepsilon B$ then $\det(X) = 1$ if and only if $\mathrm{tr}(B) = 0$.

ii) The special orthogonal group is defined to be

$$\mathrm{SO}_n(\mathsf{k}) = \{A \in \mathrm{GL}_n(\mathsf{k}) : \det(A) = 1, A.A^t = I\}.$$

Show that the kernel of the map $\mathrm{SO}_n(D_{\mathsf{k}}) \to \mathrm{SO}_n(\mathsf{k})$ can be identified with

$$\mathfrak{so}_n(\mathsf{k}) = \{X \in \mathfrak{gl}_n(\mathsf{k}) : X + X^t = 0\}.$$

Solution: Write $A = (\mathbf{a}_1|\dots|\mathbf{a}_n)$ where $\mathbf{a}_i$ $(1 \le i \le n)$ are the column vectors of A, and similarly let $B = (\mathbf{b}_1|\dots|\mathbf{b}_n)$. Then using the multilinearity of the determinant and the fact that $\varepsilon^2 = 0$ we see that

$$\begin{aligned}\det(X) &= \det(\mathbf{a}_1 + \varepsilon\mathbf{b}_1|\dots|\mathbf{a}_n + \varepsilon\mathbf{b}_n)\\ &= \det(\mathbf{a}_1|\dots|\mathbf{a}_n) + \sum_{i=1}^{n} \varepsilon \det(\mathbf{a}_1|\dots|\mathbf{b}_i|\dots|a_n)\end{aligned}$$

If $A = I_n$, then $\det(\mathbf{a}_1|\dots|\mathbf{b}_i|\dots|\mathbf{a}_n) = B_{ii}$, so that the above formula becomes $\det(I+\varepsilon B) = 1+\varepsilon\mathrm{tr}(B)$.

For the second part, we just need to check when $A + \varepsilon B$ lies in $\mathrm{SO}_n(D_{\mathsf{k}})$, which happens when $\mathrm{tr}(B) = 0$ by the first part, and

$$(A + \varepsilon B)(A^t + \varepsilon B^t) = AA^t + \varepsilon(BA^t + aB^t),$$

which is equal to I if and only if $AA^t = I$ and $BA^t + B^tA = 0$. Thus the kernel consists of the matrices of the form $I + \varepsilon B$ where $B + B^t = 0$ and $\mathrm{tr}(B) = 0$. (This first condition implies the second if the characteristic of k is not 2).

For the last part, note that if $A, B \in \mathfrak{so}_n(\mathsf{k})$ then

$$[A, B]^t = (AB - BA)^t = B^tA^t - A^tB^t = (-B)(-A) - (-A)(-B) = BA - AB = -[A, B],$$

so that $[A, B] \in \mathfrak{so}_n(\mathsf{k})$. $\square$

6. Read Appendix 1 in the lecture notes for a review of the relevant facts about symmetric bilinear forms needed for this lecture course.