C2.1a Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 1

1. Show that $\mathfrak{sl}_2(\mathbb{C})$ is a simple Lie algebra, i.e. its only ideals are zero and itself.

2. Let S be an $n \times n$ matrix with entries in a field k. Define

$$\mathfrak{gl}_S(k) = \{x \in \mathfrak{gl}_n(k) : x^t S + Sx = 0\}.$$

1. Show that $\mathfrak{gl}_S(k)$ is a Lie subalgebra of $\mathfrak{gl}_n(k)$.
2. Let J_n be the $n \times n$-matrix:

$$J_n = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}$$

Now let S be the $2n \times 2n$ matrix:

$$S = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$$

Find the conditions for a matrix to lie in $\mathfrak{gl}_S$ and hence determine the dimension of $\mathfrak{gl}_S$.

3. Classify all Lie algebras $\mathfrak{g}$ with $\dim \mathfrak{g} = 3$ and $\mathfrak{z}(\mathfrak{g}) \neq 0$.

4. *(The classical Lie algebras)* In this course a fundamental role is played by the classical Lie algebras. In this question they will be defined, we will calculate their dimensions and we will look at some small dimensional examples. Assume throughout that k is a field of characteristic $\neq 2$.

a) (*The special linear algebra* $\mathfrak{sl}_n$) Recall that $\mathfrak{sl}_n(k) \subset \mathfrak{gl}_n(k)$ denotes the subspace of traceless $n \times n$-matrices. Check that $\mathfrak{sl}_n(k)$ is a subalgebra of $\mathfrak{gl}_n(k)$. Calculate the dimension of $\mathfrak{sl}_n(k)$.

b) (*The special orthogonal Lie algebra* $\mathfrak{so}_n$) Recall the definition of $\mathfrak{gl}_S(k)$ and J_n from Question 1. Consider the matrix

$$S = J_n.$$

We define $\mathfrak{so}_n(k)$ to be $\mathfrak{gl}_S(k)$. Find conditions for a matrix to belong to $\mathfrak{so}_n(k)$ and hence calculate its dimension.

c) (*The symplectic Lie algebra* $\mathfrak{sp}_{2n}(k)$) Consider the matrix

$$S = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}.$$

We define $\mathfrak{sp}_{2n}(k)$ to be $\mathfrak{gl}_S(k)$. You already calculated its dimension. Give an explicit description of $\mathfrak{sp}_2(k)$ in terms of another Lie algebra occuring on the list above.

d) Show that $\mathfrak{so}_2(k)$ is abelian and that $\mathfrak{sl}_2(k) \cong \mathfrak{so}_3(k)$.

5. Let k be an arbitrary field. Show that the derived subalgebra of $\mathfrak{gl}_n(\mathsf{k})$ is $\mathfrak{sl}_n(\mathsf{k})$.

6. Show that $\mathfrak{sl}_n(\mathbb{C})$, $n \geq 2$, is a simple Lie algebra. (*Hint:* It might be easier show that $\mathfrak{gl}_n(\mathbb{C})$ has no non-trivial ideals contained in $\mathfrak{sl}_n(\mathbb{C})$.)