

C2.1a Lie algebras

Mathematical Institute, University of Oxford
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Problem Sheet 4

Throughout this sheet we assume that all Lie algebras and all representations discussed are finite dimensional unless the contrary is explicitly stated, and we work over a field k which is algebraically closed of characteristic zero.

1. Let \mathfrak{g} be a simple Lie algebra. Show that any nonzero trace form on \mathfrak{g} is a multiple of the Killing form. (*Hint*: Show that the form can be used to identify \mathfrak{g} with \mathfrak{g}^* as a \mathfrak{g} -representation. See Problem Sheet 3.)
 2. Show that homomorphisms between semisimple Lie algebras are compatible with the Jordan decomposition, that is, if $\mathfrak{g}_1, \mathfrak{g}_2$ are semisimple Lie algebras, and $\phi: \mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ is a homomorphism, then if $x = s + n$ is the Jordan decomposition of $x \in \mathfrak{g}_1$, $\phi(x) = \phi(s) + \phi(n)$ is the Jordan decomposition of $\phi(x)$ in \mathfrak{g}_2 . (*You may assume the fact, stated in lectures, that if $x = s + n$ is the Jordan decomposition of x and $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a representation, then $\rho(s)$ is semisimple and $\rho(n)$ is nilpotent.*)
 3. Use Weyl's theorem to give an alternative proof of the fact that any derivation of a semisimple Lie algebra \mathfrak{g} is inner. (*Hint*: Suppose that δ is a derivation, show that $V = k \oplus \mathfrak{g}$ has the structure of a \mathfrak{g} representation via $x(a, y) = (0, a\delta(x) + [x, y])$, and consider a complement to the subrepresentation \mathfrak{g} .)
 4. Let $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$ be the symplectic Lie algebra. Show that \mathfrak{h} , the space of matrices in \mathfrak{g} which are diagonal, is a Cartan subalgebra, and find the roots of $\mathfrak{sp}_{2n}(\mathbb{C})$.
 5. Let \mathfrak{g} be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. If $\Phi \subset \mathfrak{h}^*$ is the corresponding root system find an expression for the dimension of \mathfrak{g} in terms of Φ . (In particular, the dimension of \mathfrak{g} is determined by the root system).
 6. Suppose that \mathfrak{g} is a Lie subalgebra of $\mathfrak{gl}(V)$. Show that if V is irreducible as a \mathfrak{g} -representation and $\text{tr}(\rho(x)) = 0$ for all $x \in \mathfrak{g}$, then \mathfrak{g} is semisimple.
 7. Let k be a field and let \mathfrak{s}_k be the 3-dimensional k -Lie algebra with basis $\{e_0, e_1, e_2\}$ and structure constants $[e_i, e_{i+1}] = e_{i+2}$ (where we read the indices modulo 3, so that we have for example $[e_2, e_0] = e_1$).
 - i) Show that \mathfrak{s}_k is a simple Lie algebra.
 - ii) Show that $\mathfrak{s}_{\mathbb{R}}$ is isomorphic to the Lie algebra (\mathbb{R}^3, \wedge) , where \wedge is the cross product.
 - iii) Show that $\mathfrak{s}_{\mathbb{R}}$ (equivalently, (\mathbb{R}^3, \wedge)) is not isomorphic to $\mathfrak{sl}_2(\mathbb{R})$.
 - iv) Show that $\mathfrak{s}_{\mathbb{C}} \cong \mathfrak{sl}_2(\mathbb{C})$.
- 8 (Optional for the vacation). Complete the details in the classification of Dynkin diagrams and root systems of simple Lie algebras began in lectures.