

Analytic Topology: Problem sheet 0

1. (i) *Prove that every compact subset of a Hausdorff space is closed.*

Let X be a Hausdorff space, and let K be a compact subset of X .

We show that K is closed by showing that its complement is open.

Suppose that x is not an element of K .

For each $y \in K$, we use Hausdorffness of X to find disjoint open subsets U_y and V_y of X such that $x \in U_y$ and $y \in V_y$.

Now the family $\{V_y : y \in K\}$ is an open cover of K .

K is compact, so there exists a finite subcover $\{V_{y_i} : i < n\}$ (where n is some natural number).

Let $U = \bigcap_{i < n} U_{y_i}$.

Then for all $i < n$, $U \subseteq U_{y_i}$. So $U \cap V_{y_i} = \emptyset$. Also notice that U is a finite intersection of open sets, so it is open.

Let $V = \bigcup_{i < n} V_{y_i}$.

Then because $U \cap V_{y_i} = \emptyset$ for all i , $U \cap V = \emptyset$, and because $\{V_{y_i} : i < n\}$ is a cover of K , $K \subseteq V$. Hence $U \cap K = \emptyset$.

Hence for any $x \notin K$, we have an open set U such that $x \in U$ and $U \cap K = \emptyset$.

So the complement of K is open, and hence K itself is closed, as required.

- (ii) *Give an example of a space X with a compact subset K which is not closed.*

There are many examples. The simplest is probably the two-point indiscrete space $X = \{0, 1\}$ (recall that a space X is indiscrete if and only if the only open sets are \emptyset and the the whole space X), with $K = \{0\}$: K is neither the empty set nor the whole of X , so it is not closed, but it is finite, so it is compact.

2. (i) *Prove that every closed subset of a compact space is compact.*

Let X be a compact space, and let C be a closed subset.

Let \mathcal{U} be an open cover of C .

Then because C is closed, $X \setminus C$ is open.

Thus $\mathcal{U} \cup \{X \setminus C\}$ is an open cover of X .

Now X is compact, so there exists \mathcal{V} which is a finite subset of $\mathcal{U} \cup \{X \setminus C\}$ and is a cover for X .

The extra set $X \setminus C$ may or may not be a member of \mathcal{V} . If it is not, then \mathcal{V} is already the finite subcover of \mathcal{U} that we are seeking. If it is, we eliminate it, to obtain $\mathcal{V} \setminus \{X \setminus C\}$, which is a finite subset of \mathcal{U} , and is a cover of C .

Thus C is compact.

- (ii) *Give an example of a space X with a closed subspace A which is not compact.*

There are many examples of this. One of the most familiar is $X = \mathbb{R}$ with the usual topology, and $A = [0, \infty)$.

3. *Prove that the image of any compact space under a continuous function is compact.*

Suppose that X is a compact space, and that $f : X \rightarrow Y$ is a continuous surjection.

Let \mathcal{U} be an open cover of Y .

Then for each element U of \mathcal{U} , $f^{-1}(U)$ is an open set, because f is continuous.

Also the set $\{f^{-1}(U) : U \in \mathcal{U}\}$ is a cover of X .

Because X is compact, there is a finite subset $\{f^{-1}(U_i) : i < n\}$ which is a cover of X .

Then $\{U_i : i < n\}$ is a finite subset of \mathcal{U} which is a cover for Y .

Hence Y is compact.

4. (i) *Prove that if X is a compact space, Y is a Hausdorff space, and $f : X \rightarrow Y$ is bijective and continuous, then it is a homeomorphism.*

It is sufficient to prove that $f^{-1} : Y \rightarrow X$ is continuous.

Recall that a function is continuous if and only if the inverse image of every closed set is closed.

So let C be a closed subset of X . We consider its inverse image under the function f^{-1} , namely $(f^{-1})^{-1}[C]$.

This is equal to the forward image $f[C]$ of C under f .

Now C is a closed subset of X , and X is compact.

Hence C is compact.

The image of any compact space under a continuous function is compact.

Hence $f[C]$ is compact.

Now Y is Hausdorff, so $f[C]$ is closed.

Thus the inverse image of any closed set under the map f^{-1} is closed, so f^{-1} is continuous.

Thus f is a homeomorphism.

(ii) *Give examples to show that the hypotheses that X is compact and that Y is Hausdorff cannot be omitted.*

We note that any function whose domain is discrete is continuous, and any function whose range is indiscrete is continuous.

Let X and Y be spaces of the same infinite size, such that X is discrete and Y is not, and let $f : X \rightarrow Y$ be a bijection. Then f is automatically continuous, but is not a homeomorphism.

Note that because X is infinite and discrete, it is not compact.

Now let X and Y be spaces of the same size, which must be at least two, such that Y is indiscrete and X is not, and let $f : X \rightarrow Y$ be a bijection. Then again f is automatically continuous, but is not a homeomorphism.

Note that because Y is indiscrete and has at least two distinct points, it is not Hausdorff.

5. *Let (X, d) be a metric space.*

(i) *Show that a subset A of X is closed if and only if every accumulation point a of a sequence $(a_n)_{n \in \mathbb{N}}$ of elements of A , is itself an element of A .*

First, suppose that A is closed. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of elements of A , and let a be an accumulation point of this sequence.

Recall that a belongs to the closure of A if and only if every open set containing a meets A .

But every open set containing a also contains a_n for infinitely many values of n , and thus contains at least one point of A , as required.

So a belongs to \overline{A} .

But A is closed, so $\overline{A} = A$, so a belongs to A .

Now suppose that whenever a is an accumulation point of a sequence $(a_n)_{n \in \mathbb{N}}$ on A , then a is an element of A .

We argue that A is closed, by showing that $\overline{A} \subseteq A$.

For suppose that a is an element of \overline{A} .

Then every open set containing a meets A .

Let a_n be a point of A contained in the ball of radius $1/n$ around a .

Then the sequence $(a_n)_{n \in \mathbb{N}}$ converges to a .

A fortiori, a is an accumulation point of the sequence.

By our hypothesis, a now belongs to A .

So we have shown that $\overline{A} \subseteq A$, and so A is closed.

(ii) Show that if a is an accumulation point of a sequence $(a_n)_{n \in \mathbb{N}}$, then there is a subsequence of $(a_n)_{n \in \mathbb{N}}$ which converges to a .

Since a is an accumulation point of the sequence $(a_n)_{n \in \mathbb{N}}$, every open set containing a contains a_n for infinitely many values of n .

For each natural number $k \geq 1$, let n_k be such that a_{n_k} is contained in the ball of radius $1/k$ about a , and such that for all k , $n_k < n_{k+1}$.

Then the sequence $(a_{n_k})_{k \in \mathbb{N}}$ converges to a , as required.

(iii) Deduce that in a metric space, the topology can be completely described in terms of convergent sequences.

This part just consists of restating what we've already proved.

A set A in a metric space is closed if and only if every accumulation point of any sequence on A is contained in A , if and only if the limit of every convergent sequence of elements of A is contained in A .

That is, we can tell whether A is closed purely by examining the convergent sequences of X .

Thus we can tell whether a set is *open* purely by looking at the convergent sequences of X ; that is, the topology can be completely described in terms of convergent sequences.

6. Let X be a Hausdorff space, let x be an element of X , and let C be a compact subset of X such that $x \notin C$. Prove that there exist disjoint open sets U and V such that $x \in U$ and $C \subseteq V$.

Following the hint, we use Hausdorffness of X to show that for each $y \in C$ there exist disjoint open sets $U_y \ni x$ and $V_y \ni y$.

Now the family $\{V_y : y \in C\}$ is an open cover of C .

Let $\{V_{y_i} : i < n\}$ be a finite subcover.

Let $V = \bigcup_{i < n} V_{y_i}$ and let $U = \bigcap_{i < n} U_{y_i}$. U is a finite intersection of open sets, so it's open. (This is the point at which compactness is crucial.) Also, $U \cap V = \emptyset$; and $x \in U$ and $C \subseteq V$.