Analytic Topology: Problem sheet 1

These solutions come with a mild health warning. They are lightly edited from the file distributed to the class tutors and TAs, so aimed at a different audience. I make no claim that the solutions given are the only solutions or even the best; you may very well have found solutions that are more economical or more illuminating than mine.

1. *Prove that the following are equivalent:*

(i) *X is Hausdorff;*

(ii) *if $p \in X$, then for each $q \neq p$, there is an open set $U \ni p$ such that $q \notin \overline{U}$;*

(iii) *for each $p \in X$, $\bigcap\{\overline{U} : U$ open and $U \ni p\} = \{p\}$.*

Clause (ii) looks more like the standard statement of the T_1 axiom, or one of the standard ways of defining regularity, or like the restatement of normality in clause (ii) of the next question. Clauses (ii) and (iii) have in common that if you replace $\overline{U}$ by U, you get something equivalent to T_1.

This question is largely set algebra.

(i) implies (ii): Suppose $p \neq q$. Find disjoint open $U \ni p$ and $V \ni q$. Since V is an open neighbourhood of q not meeting U, $q \notin \overline{U}$.

(ii) implies (iii): That $\bigcap\{\overline{U} : U$ open and $U \ni p\} \supseteq \{p\}$ is trivial. That $\bigcap\{\overline{U} : U$ open and $U \ni p\} \subseteq \{p\}$ follows directly from (ii), because any $q \neq p$ fails to belong to at least one $\overline{U}$.

(iii) implies (i): Let $p \neq q$. Find an open set U such that $p \in U$ and $q \notin \overline{U}$. Let $V = X \setminus \overline{U}$. Then U and V are disjoint open sets, $p \in U$, and $q \in V$, as required.

2. (i) *If X is regular, $C \subseteq X$, $D \subseteq X$, C compact, D closed, $C \cap D = \varnothing$, find disjoint open U, V such that $C \subseteq U$ and $D \subseteq V$, and hence show that a compact Hausdorff space is normal.*

This is very similar to question 6 on sheet 0.

Appealing to question 6 on sheet 0, for each $y \in D$ let U_y and V_y be disjoint open sets such that $C \subseteq U_y$ and $y \in V_y$.

Then the family $\{V_y : y \in D\}$ is an open cover of D.

Now D is a closed subset of a compact space, so it is compact. Hence there is a finite subcover $\{V_{y_i} : i < n\}$ (where n is some natural number).

Let $V = \bigcup_{i<n} V_{y_i}$, and let $U = \bigcap_{i<n} U_{y_i}$. These sets are disjoint and open, $C \subseteq U$, and $D \subseteq V$.

It is worth emphasising that the intersection defining U is finite, so that U can be defined as an open set. This is the point at which compactness is vital.

(ii) *Show that X is normal if and only if, for each closed C and open $U \supseteq C$, there exists open V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.*

This is really set algebra.

$\Rightarrow$: Suppose that X is normal. Suppose that C is closed and U is open and $C \subseteq U$. Then $D = X \setminus U$ is closed and disjoint from C.

Let V and W be disjoint open sets such that $C \subseteq V$ and $D \subseteq W$.

V and W are disjoint, so $V \subseteq X \setminus W$; and W is open, so $X \setminus W$ is closed. Hence $\overline{V} \subseteq X \setminus W$.

Now we can see that $C \subseteq V \subseteq \overline{V} \subseteq X \setminus W \subseteq X \setminus D = U$.

$\Leftarrow$: Let C and D be disjoint and closed.

Let $U = X \setminus D$; then U is open and $C \subseteq U$.

Find an open set V such that $C \subseteq V \subseteq \overline{V} \subseteq U$.

Let $W = X \setminus \overline{V}$.

Then V and W are disjoint and open, $C \subseteq V$, and $D \subseteq W$.

(iii) *X is said to be completely normal if, for each pair of subsets A, B such that $\overline{A} \cap B = \varnothing = A \cap \overline{B}$, there exist disjoint open U, V such that $A \subseteq U$, $B \subseteq V$. Prove that a topological space is completely normal if and only if every subspace is normal.*

$\Rightarrow$: Let X be completely normal and let Y be a subspace of X. Let C and D be disjoint closed subsets of Y.

Then there must exist closed sets C' and D' in X such that $C = Y \cap C'$ and $D = Y \cap D'$; it follows at once that $C = Y \cap \overline{C}^X$ (where $\overline{C}^X$ refers to the closure of C in the space X) and $D = Y \cap \overline{D}^X$.

Hence since C and D are disjoint, $\overline{C}^X \cap D = \varnothing = C \cap \overline{D}^X$.

Since X is completely normal, there exist disjoint sets U' and V' which are open in X such that $C \subseteq U'$ and $D \subseteq V'$.

Let $U = Y \cap U'$ and $V = Y \cap V'$; then U and V are disjoint open subsets of Y, $C \subseteq U$ and $D \subseteq V$.

Hence the subspace Y is normal.

$\Leftarrow$: The difficult part of this is choosing the correct subspace of X.

Suppose that every subspace of X is normal. Suppose that A and B are subsets of X such that $\overline{A} \cap B = \varnothing = A \cap \overline{B}$.

Let $Y = X \setminus (\overline{A} \cap \overline{B})$.

Notice that $A, B \subseteq Y$, because of the condition that $\overline{A} \cap B = \varnothing = A \cap \overline{B}$.

Then if $C = Y \cap \overline{A}$ and $D = Y \cap \overline{B}$, then C and D are closed in Y and, by the careful choice of Y, they are disjoint.

Now Y is normal, so there exist disjoint sets U and V which are open in Y such $C \subseteq U$ and $D \subseteq V$.

Now U and V are open in Y and Y is open in X, so U and V are open in X. Also $A \subseteq U$ and $B \subseteq V$, as required.

3. *Let (X, d) be a metric space with its usual topology, $\varnothing \neq A \subseteq X$. Define, for $x \in X$, $D(x, A) = \inf\{d(x, y) : y \in A\}$. Prove that:*

(i) $D(x, A) : X \to \mathbb{R}$ *is continuous (x varies, A is fixed),*

Given $x \in X$ and $\epsilon > 0$, we seek $\delta > 0$ such that if $d(x, y) < \delta$, then it follows that $|D(x, A) - D(y, A)| < \epsilon$. We do this by the usual method of deciding what value of δ makes the calculations come out right.

If $d(x, y) < \delta$, pick $z \in A$ such that $d(x, z) < D(x, A) + \delta$. Then $d(y, z) < D(x, A) + 2\delta$, so it follows that $D(y, A) < D(x, A) + 2\delta$.

Similarly $D(x, A) < D(y, A) + 2\delta$.

So, assuming we chose δ to be $\epsilon/2$, we have the required result.

(ii) $D(x, A) = 0$ *if and only if* $x \in \overline{A}$,

If $D(x, A) > 0$, then let $\delta = \frac{1}{2}D(x, A)$; then the ball of radius δ around x does not meet A, and so $x \notin \overline{A}$.

If on the other hand $D(x, A) = 0$, then find $z_n \in A$ witnessing the fact that $D(x, A) < \frac{1}{n}$; that is, so that $d(x, z_n) < \frac{1}{n}$.

Then the sequence $(z_n)_{n\in\mathbb{N}}$ converges to x, and so $x \in \overline{A}$.

(iii) *if* C *is closed in* X, *there exists an infinite sequence* (V_n) *of sets* V_n *open in* X *with* $C = \bigcap_{n\in\mathbb{N}} V_n$,

Here we simply let $V_n = \{y \in X : D(y, C) < \frac{1}{n}\}$. This is open, because $D(\cdot, C)$ is continuous and V_n is the inverse image of the set $(-\infty, 1/n)$ under it, and the intersection $\bigcap_{n\in\mathbb{N}} V_n$ is equal to $\overline{A}$ by part (ii).

(iv) X *is completely normal.*

Suppose that A and B are subsets of X such that $\overline{A} \cap B = \varnothing = A \cap \overline{B}$. Define U to be $\{x \in X : D(x, A) < D(x, B)\}$, and V to be $\{x \in X : D(x, B) < D(x, A)\}$.

U and V are clearly disjoint. They are also open: U is open because it is the inverse image of the set $(0, \infty)$ under the continuous function $D(\cdot, B) - D(\cdot, A)$, and V is open for a similar reason.

Also $A \subseteq U$, because if $x \in A$, then $D(x, A) = 0$, and since $x \notin \overline{B}$, $D(x, B) > 0$, so $D(x, A) < D(x, B)$. $B \subseteq V$ similarly.

4. *X is extremally disconnected if the closure of every open set is open. Subsets A and B are functionally separated if there is a continuous function $f : X \to [0, 1]$ such that $f[A] \subseteq \{0\}$, $f[B] \subseteq \{1\}$. ($[0, 1]$ has its subspace topology inherited from $\mathbb{R}$. We write $\subseteq$ for $=$ in case A or B is empty.) Prove that the following are equivalent:*

(i) X *is extremally disconnected,*

(ii) *every two disjoint open sets in* X *have disjoint closures,*

(iii) *every two disjoint open sets in* X *are functionally separated.*

(i)⇒(ii): Let U and V be disjoint open sets.
Then $\overline{U}$ is open.
Hence, if $\overline{U} \cap \overline{V} \neq \varnothing$, then $\overline{U} \cap V \neq \varnothing$.
But now, since V is open, $V \cap U \neq \varnothing$.
But this contradicts disjointness of U and V.
So $\overline{U}$ and $\overline{V}$ must be disjoint.

(ii)⇒(i): Let U be an open set.
Then $X \setminus \overline{U}$ is also open, and is disjoint from U.
Hence $X \setminus \overline{U}$ and U have disjoint closures.
It follows that $\overline{X \setminus \overline{U}} = X \setminus \overline{U}$.
So $X \setminus \overline{U}$ is closed.
Hence $\overline{U}$ is open, as required.

(i), (ii)$\Rightarrow$(iii): Let U and V be disjoint open sets.

Then by (i), $\overline{U}$ is open, and by (ii), $\overline{U}$ and $\overline{V}$ are disjoint.

Since $\overline{U}$ is both closed and open, the function f which is defined so that $f(x) = 0$ if $x \in \overline{U}$ and $f(x) = 1$ if $x \notin \overline{U}$, is continuous, and since $V \cap \overline{U} = \varnothing$, $f(x) = 1$ for all $x \in V$.

So f functionally separates U and V.

(iii)$\Rightarrow$(ii): Let U and V be disjoint open sets.

Let $f : X \to [0, 1]$ be a continuous function such that $f[U] \subseteq \{0\}$ and $f[V] \subseteq \{1\}$.

Then $\overline{U} \subseteq f^{-1}(0)$, and $\overline{V} \subseteq f^{-1}(1)$.

Hence U and V have disjoint closures.

5. *Suppose X is first countable and $f : X \to Y$. Prove that:*

(i) *if $A \subseteq X$, then $x \in \overline{A}$ if and only if there is a sequence on A converging to x;*

Basically the same proof as for metric spaces.

Let $\{U_n : n \in \mathbb{N}\}$ be a countable basis at x, with the property that for all n, $U_{n+1} \subseteq U_n$. Such a basis can be found because if $\{V_n : n \in \mathbb{N}\}$ is a countable basis at x, then we can define U_n to be $\bigcap_{m \leq n} V_n$. This is a finite intersection of open sets and so open, and it is clear that $U_{n+1} \subseteq U_n$.

$\Rightarrow$: If $x \in \overline{A}$, then every open neighbourhood of x meets A. Let $a_n \in A \cap U_n$. Then every U_m contains all but finitely many of the a_n. Hence every open neighbourhood of x contains all but finitely many of the a_n. Hence the sequence $(a_n)_{n \in \mathbb{N}}$ converges to x.

$\Leftarrow$: This is trivial, because if every open neighbourhood around x contains all but finitely many members of some sequence $(a_n)_{n \in \mathbb{N}}$ on A, then certainly every open neighbourhod of x meets A.

(ii) *f is continuous at x_0 if and only if $f(x_n) \to f(x_0)$ for each sequence (x_n) for which $x_n \to x_0$.*

Also the same proof as for metric spaces.

Let $\{U_n : n \in \mathbb{N}\}$ be a local basis at x_0, with the property that for all n, $U_{n+1} \subseteq U_n$.

$\Rightarrow$: Suppose that $x_n \to x_0$. Let V be an open neighbourhood of $f(x_0)$. Then $f^{-1}(V)$ is a neighbourhood of x_0, because f is continuous at x_0. Because $x_n \to x_0$, $f^{-1}(V)$ contains all but finitely many of the x_n. Hence V contains all but finitely many of the $f(x_n)$. So $f(x_n) \to f(x_0)$.

$\Leftarrow$: Suppose that V is an open neighbourhood of $f(x_0)$, and that $f^{-1}(V)$ is not a neighbourhood of x_0. Then for all n, $U_n \not\subseteq f^{-1}(V)$. Let $x_n \in U_n \setminus f^{-1}(V)$. Then $x_n \to x_0$, but $f(x_n) \notin V$, so $f(x_n)$ does not converge to $f(x_0)$.