Analytic Topology: Problem sheet 2

1. Suppose that X and Y are topological spaces, $A \subseteq X$, $B \subseteq Y$. Prove that $\overline{A} \times \overline{B} = \overline{A \times B}$. Prove that if X and Y are regular, then $X \times Y$ is regular.

2. Prove that a space X is Hausdorff if and only if the diagonal

$$\Delta = \{(x, x) \in X \times X : x \in X\}$$

is closed in $X \times X$.

3. Suppose that X is arbitrary, Y is Hausdorff, and $f : X \to Y$, $g : X \to Y$ are both continuous. Prove that:

(i) $\{x \in X : f(x) = g(x)\}$ is closed in X,

(ii) if $D \subseteq X$ is dense (that is, $\overline{D} = X$), and $f|_D = g|_D$ (that is, if $f(x) = g(x)$ for each $x \in D$), then $f = g$,

(iii) the set $G_f = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$ (G_f is known as the *graph* of f. People who have done set theory will note that it has become conventional to identify a function with its graph),

(iv) if $Z \subseteq Y$ and the continuous function $h : Y \to Z$ is such that $h(y) = y$ for each $y \in Z$, then Z is closed in Y. (Such an h is called a *retraction*.)

4. X is said to be *countably compact* if every countable open covering has a finite subcovering. Prove that a T_1 space X is countably compact if and only if every infinite subset has a limit point in X.

5. X has a *countable basis at* x if there is a sequence $\{U_n : n \in \mathbb{N}\}$ of open subsets, each containing x, such that, for each open $V \ni x$, there exists n such that $x \in U_n \subseteq V$, and X is *first countable* if it has a countable basis at every point $x \in X$. (So, for example, metric spaces are first countable.) Prove that a countably compact, first countable, Hausdorff space is regular.

6. Show that a metric space is Lindelöf if and only if it is separable.

7. (i) $\mathscr{B}$ is a *basis* for a filter $\mathscr{F}$ on a set X if and only if

$$\mathscr{F} = \{F \subseteq X : (\exists B \in \mathscr{B})(B \subseteq F)\}.$$

$\mathscr{B}$ is a *filter-basis* on X if and only if

(a) $\varnothing \notin \mathscr{B}$ and $\mathscr{B} \neq \varnothing$,

(b) if B_1, $B_2 \in \mathscr{B}$, then $\exists B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

Prove that a family of subsets of a set X is a filter-basis if and only if it is the basis of some filter on X.

(ii) Suppose that $\mathscr{N}_x$ is the filter of all neighbourhoods of a point x. A filter-basis $\mathscr{D}$ *converges to* y if U open, $U \ni y$ implies $\exists D \in \mathscr{D}$ with $D \subseteq U$. Prove that $f : X \to Y$ is continuous if and only if, for every $x \in X$, $f(\mathscr{N}_x)$ converges to $f(x)$.

(iii) Prove that the following are equivalent:
(a) X is Hausdorff,
(b) no filter on X converges to more than one point,
(c) if a filter $\mathscr{F}$ on X converges to x, then x is the only cluster point of $\mathscr{F}$.

8. Suppose that f is a function from X onto Y, and that $x \in X$. Prove that f is continuous at x if and only if, for every ultrafilter $\mathscr{U}$ on X which converges to x, the ultrafilter $f(\mathscr{U})$ converges to $f(x)$.

9. Suppose M, N, X, Y are topological spaces, $\pi_X : X \times Y \to X$ is the usual projection.

(i) Prove that $f : M \to N$ is closed (ie. $f(C)$ is closed in N, for each C closed in M) if and only if, for each $n \in N$ and each open $U \supseteq f^{-1}(n)$, there is an open $V \ni n$ such that $f^{-1}(V) \subseteq U$.

(ii) If Y is compact, prove that π_X is closed.

(iii) If Y is compact Hausdorff, prove that $g : X \to Y$ is continuous if and only if its graph is closed in $X \times Y$.