Analytic Topology: Problem sheet 2

1. *Suppose that X and Y are topological spaces, $A \subseteq X$, $B \subseteq Y$. Prove that $\overline{A} \times \overline{B} = \overline{A \times B}$. Prove that if X and Y are regular, then $X \times Y$ is regular.*

$\overline{A} \times \overline{B}$ is closed because its complement, which is equal to $X \times (Y \setminus \overline{B}) \cup (X \setminus \overline{A}) \times Y$, is open. Hence $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$.

As for the reverse inclusion, if $(x, y) \in \overline{A} \times \overline{B}$, if W is an open neighbourhood of (x, y), then there exist U open in X and V open in Y such that $(x, y) \in U \times V \subseteq W$; then $x \in \overline{A}$, so there exists a point z in $U \cap A$, and similarly we can find a point w in $V \cap B$; then $(z, w) \in W \cap (A \times B)$. Hence $(x, y) \in \overline{A \times B}$.

Now suppose that X and Y are regular, and that (x, y) is a point of $X \times Y$, W is an open set in $X \times Y$, and that $(x, y) \in W$.

Then there exist U open in X and V open in Y such that $(x, y) \in U \times V \subseteq W$.

Now X is regular so there exists an open set U' in X such that $x \in U' \subseteq \overline{U'} \subseteq U$.

Similarly there exists V' open in Y such that $y \in V' \subseteq \overline{V'} \subseteq V$.

Then $(x, y) \in U' \times V' \subseteq \overline{U'} \times \overline{V'} \subseteq U \times V \subseteq W$.

The proof of regularity of $X \times Y$ is complete once we observe that $\overline{U'} \times \overline{V'} = \overline{U' \times V'}$.

2. *Prove that a space X is Hausdorff if and only if the diagonal*

$$\Delta = \{(x, x) \in X \times X : x \in X\}$$

is closed in $X \times X$.

$\Rightarrow$: We show that the complement of the diagonal is open.

Suppose that $(x, y) \notin \Delta$.

Then $x \neq y$.

Hence, since X is Hausdorff, there exist disjoint open U and V such that $x \in U$, $y \in V$.

Then $(x, y) \in U \times V$.

We now show that $U \times V$ does not meet the diagonal.

For, if $(z, w) \in \Delta \cap (U \times V)$, then $(z, w) \in \Delta$, so $w = z$; and also $(z, w) \in U \times V$, so $z \in U$ and $w \in V$. So $z \in U \cap V$, contradicting disjointness of U and V.

So the complement of the diagonal is open, as required.

$\Leftarrow$: The same argument in reverse. Briefly:

If the diagonal is closed, then its complement is open.

Suppose that $x \neq y$. Then $(x, y) \notin \Delta$. Hence since the complement of the diagonal is open, there exists open W such that $(x, y) \in W \subseteq (X \times Y) \setminus \Delta$. W may be assumed to have the form $U \times V$ for U open in X and V open in Y. Then U and V are disjoint, and the proof of Hausdorffness is complete.

3. *Suppose that X is arbitrary, Y is Hausdorff, and $f : X \to Y$, $g : X \to Y$ are both continuous. Prove that:*

(i) $\{x \in X : f(x) = g(x)\}$ *is closed in X,*

Define $h : X \to Y \times Y$ by $h(x) = (f(x), g(x))$.

Then h is continuous.

The diagonal Δ_Y in Y is closed, because Y is Hausdorff. So, the inverse image under h of Δ_Y is closed.

Thus $\{x \in X : f(x) = g(x)\}$ is closed.

(ii) *if $D \subseteq X$ is dense (that is, $\overline{D} = X$), and $f|_D = g|_D$ (that is, if $f(x) = g(x)$ for each $x \in D$), then $f = g$,*

If $f|_D = g|_D$, then $D \subseteq \{x \in X : f(x) = g(x)\}$. But $\{x \in X : f(x) = g(x)\}$ is closed, and D is dense. Hence $\{x \in X : f(x) = g(x)\} = X$. So $f = g$.

(iii) *the set $G_f = \{(x, f(x)) : x \in X\}$ is closed in $X \times Y$,*

Define $k : X \times Y \to Y \times Y$ by $k(x, y) = (f(x), y)$.

Now Y is Hausdorff, so the diagonal Δ_Y is closed.

Thus $k^{-1}[\Delta_y]$ is closed.

But this set is equal to $\{(x, y) : f(x) = y\}$, that is, to G_f.

(iv) *if $Z \subseteq Y$ and the continuous function $h : Y \to Z$ is such that $h(y) = y$ for each $y \in Z$, then Z is closed in Y. (Such an h is called a retraction.)*

In this case, $Z = \{y \in Y : y = h(y)\}$, and we can apply part (i).

4. *X is said to be countably compact if every countable open covering has a finite subcovering. Prove that a T_1 space X is countably compact if and only if every infinite subset has a limit point in X.*

$\Rightarrow$: Let A be an infinite subset of X with no limit point. Since any subset of a set with no limit points also has no limit points, we may assume that A is countable; write A as $\{a_n : n \in \mathbb{N}\}$, where all points a_n are distinct. (The statement that every infinite set has a countably infinite subset depends on the Axiom of Choice. However, for the purposes of this course, we are assuming that the Axiom of Choice is true.)

Since A has no limit points, for every point x, x has an open neighbourhood containing no points of A apart from possibly x itself; since $x = a_n$ for at most one value of n, there exists n for which $x \notin \overline{\{a_m : m \geq n\}}$.

Let $U_n = X \setminus \overline{\{a_m : m \geq n\}}$.

Then each U_n is open; and by the above observation the sets U_n cover X. However there is no finite subcover, since for all n, $a_n \notin \bigcup_{i \leq n} U_i$.

So there is a countable open cover with no finite subcover, so X is not countably compact.

$\Leftarrow$: Suppose that X is not countably compact.

Let $\{U_n : n \in \mathbb{N}\}$ be a countable open cover with no finite subcover.

Since this cover has no finite subcover, we can find, for each natural number n, a point a_n such that $a_n \notin \bigcup_{i \leq n} U_i$.

Now if $x \in X$, then for some n, $x \in U_n$.

For all $m \geq n$, $a_m \notin U_n$.

Also, the set $\{a_i : i \leq n,\, a_i \neq x\}$ is finite, and therefore closed (remember X is T_1).

The open neighbourhood $U_n \setminus \{a_i : i \leq n,\, a_i \neq x\}$ of x now witnesses that x is not a limit point of $\{a_n : n \in \mathbb{N}\}$.

So $\{a_n : n \in \mathbb{N}\}$ is an infinite subset of X with no limit point.

One can simplify the above argument by noting that in a T_1 space, x is a limit point of A if and only if every neighbourhood of x meets infinitely many points of A.

5. *Prove that a countably compact, first countable, Hausdorff space is regular.*

Suppose that x is a point of a countably compact, first countable, Hausdorff space X. Let $\{U_n : n \in \mathbb{N}\}$ be a countable basis at x. We may assume without loss of generality (since finite intersections of open sets are open) that U_{n+1} is a strict subset of U_n for all n.

We argue that for all n, there exists m such that $\overline{U_m} \subseteq U_n$. If we can do this, then we will have proved that the space X is regular.

Suppose that this is not true. Then there exists n such that for all m, $\overline{U_m} \not\subseteq U_n$.

We argue, first or all, that $\bigcap_{m\in\mathbb{N}} \overline{U_m} = \{x\}$. For, for each point $y \neq x$, there exists by Hausdorffness an open set V and a natural number m such that $y \in V$ and $U_m \cap V = \varnothing$; so V witnesses that $y \notin \overline{U_m}$.

Now for each m, let a_m be a point of $\overline{U_m} \setminus U_n$.

Then because $\bigcap_{m\in\mathbb{N}} \overline{U_m} \setminus V = \varnothing$, the set $\{a_m : m \in \mathbb{N}\}$ is infinite.

Because X is countably compact, this infinite set must have a limit point a.

For each m, this limit point must belong to $\overline{U_m}$, or else the complement of $\overline{U_m}$ is an open neighbourhood of a meeting only finitely many of the a_n.

Hence $a \in \bigcap_{m\in\mathbb{N}} \overline{U_m} \setminus U_n$, so a is an element of $\bigcap_{m\in\mathbb{N}} U_m$ different from x, giving a contradiction.

This problem can also be solved by adapting the proof that a compact Hausdorff space is compact.

6. *Show that a metric space is Lindelöf if and only if it is separable.*

$\Rightarrow$: Suppose that X is Lindelöf and metric.

For each n, let $\mathscr{U}_n$ be the set of all balls of radius $1/n$.

Let $\{B_{\frac{1}{n}}(x_{n,m}) : m \in \mathbb{N}\}$ be a countable subcover.

We argue that the set $\{x_{n,m} : n, m \in \mathbb{N}\}$ is dense.

For, let U be any non-empty open set. Let x be a point and n a natural number such that $B_{\frac{1}{n}}(x) \subseteq U$.

Now $\{B_{\frac{1}{n}}(x_{n,m}) : m \in \mathbb{N}\}$ covers X.

So, find m such that $x \in B_{\frac{1}{n}}(x_{n,m})$.

Then $x_{n,m} \in B_{\frac{1}{n}}(x)$ by symmetry of the metric.

So $x_{n,m} \in U$, as required.

$\Leftarrow$: Suppose that X is Lindelöf and separable. Let D be a countable dense set.

Let $\mathscr{U}$ be an open cover.

For each $d \in D$ and $n \in \mathbb{N}$, find an element $U_{d,n}$, if there is one, such that $B_{\frac{1}{n}}(d) \subseteq U_{d,n}$. (This is a use of the Axiom of Choice.)

We argue that $\{U_{d,n} : d \in D, n \in \mathbb{N}\}$ is a countable subcover of $\mathscr{U}$.

For, let $x \in X$.

Then $x \in U$ for some $U \in \mathscr{U}$.

Also U is open, so for some N, $x \in B_{\frac{1}{N}}(x) \subseteq U$.

Let $n = 2N$. Find $d \in D$ such that $d \in B_{\frac{1}{n}}(x)$.

Then $x \in B_{\frac{1}{n}}(d) \subseteq B_{\frac{1}{N}}(x) \subseteq U$, using symmetry and the triangle inequality.

It follows that $U_{d,n}$ is defined; and since $x \in B_{\frac{1}{n}}(d)$, $x \in U_{d,n}$.

The hypothesis that the space is metric is necessary: there are Lindelöf spaces that are not separable, and separable spaces that are not Lindelöf.

7. (i) *Prove that a family of subsets of a set X is a filter-basis if and only if it is the basis of some filter on X.*

$\Rightarrow$: If $\mathscr{B}$ is a filter-basis, then let $\mathscr{F} = \{F \subseteq X : \exists B \in \mathscr{B}\, B \subseteq F\}$.

We prove that $\mathscr{F}$ is a filter, for which $\mathscr{B}$ is a basis, by checking the conditions. $\mathscr{F} \neq \varnothing$ because $\mathscr{B} \neq \varnothing$. $\varnothing \notin \mathscr{F}$ because $\varnothing \notin \mathscr{B}$. If F_1 and F_2 and belong to $\mathscr{F}$, then there exist elements B_1 and B_2 of $\mathscr{B}$ such that $B_1 \subseteq F_1$ and $B_2 \subseteq F_2$. Now there exists $B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$. Then $B_3 \subseteq F_1 \cap F_2$, so $F_1 \cap F_2 \in \mathscr{F}$. If $F \in \mathscr{F}$ and $F \subseteq G$, then there exists $B \in \mathscr{B}$ such that $B \subseteq F$; then $B \subseteq G$ so $G \in \mathscr{F}$.

$\Leftarrow$: Suppose $\mathscr{B}$ is a basis for a filter $\mathscr{F}$.

Since $\varnothing \notin \mathscr{F}$, it follows that $\varnothing \notin \mathscr{B}$. Also, since $\mathscr{F} \neq \varnothing$, it follows that $\mathscr{B} \neq \varnothing$.

Now suppose that $B_1, B_2 \in \mathscr{B}$. Then $B_1, B_2 \in \mathscr{F}$. Now $\mathscr{F}$ is closed under pairwise intersections. Hence $B_1 \cap B_2 \in \mathscr{F}$. Hence there exists $B_3 \in \mathscr{B}$ such that $B_3 \subseteq B_1 \cap B_2$.

This is worth knowing because filter-bases are much easier to deal with than filters.

(ii) *Suppose that $\mathscr{N}_x$ is the filter of all neighbourhoods of a point x. A filter-basis $\mathscr{D}$ converges to y if U open, $U \ni y$ implies $\exists D \in \mathscr{D}$ with $D \subseteq U$. Prove that $f : X \to Y$ is continuous if and only if, for every $x \in X$, $f(\mathscr{N}_x)$ converges to $f(x)$.*

Note that if f is not onto, then $Y \notin f(\mathscr{F})$ so $f(\mathscr{F})$ is not a filter.

$\Rightarrow$: Suppose f is continuous. Let $U \ni f(x)$ be open. Then since f is continuous, $f^{-1}(U)$ is open. So $f^{-1}(U) \in \mathscr{N}_x$. So $U \supseteq f(f^{-1}(U)) \in f(\mathscr{N}_f)$.

$\Leftarrow$: Suppose $f(\mathscr{N}_x) \to f(x)$. Let $U \ni f(x)$ be open. Then there exists $V \in \mathscr{N}_x$ such that $f(V) \subseteq U$. Then $V \subseteq f^{-1}(U)$. So $f^{-1}(U)$ is a neighbourhood of x. This is true for all x, so $f^{-1}(U)$ is open.

(iii) *Prove that the following are equivalent:*

(a) X is Hausdorff,

(b) no filter on X converges to more than one point,

(c) if a filter $\mathscr{F}$ on X converges to x, then x is the only cluster point of $\mathscr{F}$.

(a)$\Rightarrow$(c): Suppose that $\mathscr{F}$ is a filter converging to x. Suppose that $y \neq x$.

Then because X is Hausdorff, there exist disjoint open U and V such that $x \in U$ and $y \in V$.

Since $\mathscr{F} \to x$, $U \in \mathscr{F}$.

Since V is open and $V \cap U = \varnothing$, $V \cap \overline{U} = \varnothing$.

Hence $y \notin \overline{U}$.

Hence y is not a cluster point of $\mathscr{F}$.

(c)$\Rightarrow$(b): Suppose that $\mathscr{F}$ converges to x, and $y \neq x$. Then by hypothesis, y is not a cluster point of $\mathscr{F}$. Hence there exists $U \in \mathscr{F}$ such that $y \notin \overline{U}$. Then $X \setminus \overline{U}$ is an open neighbourhood of y which does not belong to $\mathscr{F}$. Hence $\mathscr{F}$ does not converge to y.

(b)$\Rightarrow$(a): Suppose that X is not Hausdorff, and that x and y are distinct points such that whenever $U \ni x$ and $V \ni y$ are open, then $U \cap V \neq \varnothing$.

It is easy to check that the set $\mathscr{F}$ of all $U \cap V$ such that U is a neighbourhood of x and V is a neighbourhood of y, is a filter, and that it converges to both x and y.

8. *Suppose that f is a function from X onto Y, and that $x \in X$. Prove that f is continuous at x if and only if, for every ultrafilter $\mathscr{U}$ on X which converges to x, the ultrafilter $f(\mathscr{U})$ converges to $f(x)$.*

We mentioned the Axiom of Choice earlier. The construction of free ultrafilters depends on the Axiom of Choice (specifically, on the Boolean Prime Ideal Theorem). We are, for the purposes of this course, regarding the Axiom of Choice as being true.

This theorem of course parallels the one about convergence in metric spaces earlier in the previous sheet.

$\Rightarrow$: Suppose that f is continuous at x, and that $\mathscr{U}$ converges to x.

Let V be an open neighbourhood of $f(x)$. Then, because f is continuous, $f^{-1}(V)$ is an open neighbourhood of x. Since $\mathscr{U}$ converges to x, $f^{-1}(V) \in \mathscr{U}$. Hence $V = f[f^{-1}(V)]$ belongs to $f(\mathscr{U})$.

So $f(\mathscr{U})$ converges to $f(x)$.

$\Leftarrow$: Suppose that f is not continuous at x. Let V be an open neighbourhood of $f(x)$ such that $f^{-1}(V)$ is not a neighbourhood of x.

Then for every neighbourhood U of x, $U \subsetneq f^{-1}(V)$, so $U \setminus f^{-1}(V) \neq \varnothing$.

We can now check that the set of all $U \setminus f^{-1}(V)$, for U a neighbourhood of x, is a filter basis.

For, X is a neighbourhood of x, so $X \setminus f^{-1}(V) \in \mathscr{B}$, so $\mathscr{B} \neq \varnothing$.

$\varnothing \notin \mathscr{B}$ by definition.

If $B_1, B_2 \in \mathscr{B}$, say $B_1 = U_1 \setminus f^{-1}(V)$ and $B_2 = U_2 \setminus f^{-1}(V)$, then $B_1 \cap B_2 = U_1 \cap U_2 \setminus f^{-1}(V) \in \mathscr{B}$.

Let $\mathscr{F}$ be the filter whose basis is $\mathscr{B}$.

Let $\mathscr{U}$ be some ultrafilter refining $\mathscr{F}$.

We can now check that $\mathscr{U} \to x$.

Now if U is a neighbourhood of x, then $f(U \setminus f^{-1}(V)) = f(U) \setminus V$, which is a member of $f(\mathscr{U})$.

Hence V is not a member of $f(\mathscr{U})$.

So $f(\mathscr{U})$ does not converge to $f(x)$.

9. *Suppose M, N, X, Y are topological spaces, $\pi_X : X \times Y \to X$ is the usual projection.*

It will be helpful to draw pictures for this question.

(i) *Prove that $f : M \to N$ is closed (ie. $f(C)$ is closed in N, for each C closed in M) if and only if, for each $n \in N$ and each open $U \supseteq f^{-1}(n)$, there is an open $V \ni n$ such that $f^{-1}(V) \subseteq U$.*

$\Rightarrow$: Suppose that f is closed, that U is open in M, that $n \in N$, and that $U \supseteq f^{-1}(n)$.

Then $M \setminus U$ is closed. Since f is a closed map, $f(M \setminus U)$ is closed. Since $U \supseteq f^{-1}(n)$, $n \notin f(M \setminus U)$. Hence $V = N \setminus f(M \setminus U)$ is an open set containing n. Now $f^{-1}(V) = f^{-1}(N \setminus f(M \setminus U)) \subseteq M \setminus (M \setminus U) = U$, as required.

$\Leftarrow$: Let C be a closed subset of M. We show that $f(C)$ is closed by showing that its complement is open.

Suppose that $n \in N \setminus f(C)$.
Let $U = M \setminus C$. Then U is open, and $U \supseteq f^{-1}(n)$.
Let V be an open set containing n such that $f^{-1}(V) \subseteq U$.
Then $f^{-1}(V) \cap C = \varnothing$.
Hence $V \cap f(C) = \varnothing$.
Hence V is an open neighbourhood of n missing $f(C)$.
So the complement of $f(C)$ is open, so $f(C)$ is closed.

(ii) *If Y is compact, prove that π_X is closed.*

We use the previous part.
Suppose that $x \in X$, that $U \subseteq X \times Y$ is open, and that ${\pi_X}^{-1}(x) \subseteq U$.
Then for each $y \in Y$, there exist open V_y in X and W_y in Y such that

$$(x, y) \in V_y \times W_y \subseteq U.$$

Note that $x \in V_y$ and $y \in W_y$.
Then the W_y cover Y. Since Y is compact, there is a finite subcover $\{W_{y_i} : i < n\}$.
Now let $V = \bigcap_{i<n} V_{y_i}$.
Then V is an open neighbourhood of x.
Also,

$${\pi_X}^{-1}(V) = V \times Y = \bigcup_{i<n} V \times W_{y_i} \subseteq \bigcup_{i<n} V_{y_i} \times W_{y_i} \subseteq U.$$

Hence π_X is closed.

(iii) *If Y is compact Hausdorff, prove that $g : X \to Y$ is continuous if and only if its graph is closed in $X \times Y$.*

We have already done the $\Rightarrow$ direction on a previous sheet.
$\Leftarrow$: We show that if the graph of g is a closed set, then the inverse image under g of any closed set is closed.
Let C be closed in Y.
Then $X \times C$ is closed in $X \times Y$.
Hence $G_g \cap (X \times C)$ is closed in $X \times Y$.
Now Y is compact, so π_X is a closed map.
Hence $\pi_X(G_g \cap (X \times C))$ is a closed subset of X.
Now, $x \in \pi_X(G_g \cap (X \times C))$ if and only if there exists y such that $(x, y) \in G_g \cap (X \times C)$. Now $(x, y) \in G_g$ if and only if $y = g(x)$, and $(x, y) \in X \times C$ if and only if $y \in C$.
So $x \in \pi_X(G_g \cap (X \times C))$ if and only if $f(x) \in C$.
So $\pi_X(G_g \cap (X \times C)) = f^{-1}(C)$.
Hence $f^{-1}(C)$ is closed, as required.
So f is continuous.