Analytic Topology: Problem sheet 3

1. If $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces. Show that $\prod_{\lambda\in\Lambda} X_\lambda$ is Hausdorff in the Tychonoff topology if and only if each X_λ is Hausdorff.

2. Suppose that $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces and that $\prod_{\lambda\in\Lambda} X_\lambda$ is given the Tychonoff topology. Fix a point $f \in \prod_{\lambda\in\Lambda} X_\lambda$ and, for any fixed $\mu \in \Lambda$, let Y_μ be that subset defined by

$$Y_\mu = \{g : g(\lambda) = f(\lambda) \text{ whenever } \lambda \neq \mu\}.$$

Prove that the restriction of the projection mapping π_μ to Y_μ is a homeomorphism from Y_μ to X_μ.

3. Suppose that for each $\lambda \in \Lambda$, X_λ is non-empty and connected. For any given $f \in \prod_{\lambda\in\Lambda} X_\lambda$, show that

$$D = \{g : g(\lambda) = f(\lambda) \text{ for all but finitely many } \lambda\}$$

is connected. Show that $\prod_{\lambda\in\Lambda} X_\lambda$ is connected, in the Tychonoff topology.

4. Find the generalization of Tychonoff's Theorem to locally compact spaces, and prove it.

5. Suppose that $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces. We define the *box topology* on the cartesian product $\prod_{\lambda\in\Lambda} X_\lambda$ to be the topology with basis consisting of *all* products $\prod_{\lambda\in\Lambda} T_\lambda$, where each T_λ is open in X_λ.

Show that, in the box topology, a product of infinitely many Hausdorff spaces, each of which has at least two points, is not compact.

[Hint: consider the case where each X_λ has exactly two points. What is the box topology like then?]

6. (Optional) For each natural number n, let $X_n = [0,1]$. In the cartesian product $\prod_{n\in\mathbb{N}} X_n$, let

$$U = \left\{g : (\exists r > 0)(\forall n)\left(g(n) < \frac{r}{n}\right)\right\}.$$

Show that, in the box topology, U is clopen (ie. closed and open). Deduce that a box product of connected spaces need not be connected.

7. If X is a locally compact Hausdorff space, show that X has a basis of open sets with compact closure.

8. Prove that a locally compact subset A of a Hausdorff space X is of the form $V \cap F$, where V is open and F is closed in X. [Hint: what might F be?]

9. Prove that the following properties of a locally compact Hausdorff space X are equivalent.

(i) X is σ-compact (that is, X is a union of a countable family of compact subsets),

(ii) X can be represented as $X = \bigcup_{i=1}^{\infty} U_i$, where each U_i is an open set with compact closure, and $\overline{U_i} \subseteq U_{i+1}$ for each $i \in \mathbb{N}$,

(iii) X is Lindelöf.

10. Suppose that f is a proper mapping from X onto Y. Suppose $U \subseteq X$ is open. Defining $f^*(U) = Y \setminus f(X \setminus U)$, show that $f^*(U)$ is open, that $f^*(U) \subseteq f(U)$; and that $f^{-1}(A) \subseteq U$ implies $A \subseteq f^*(U)$. Prove that:

(i) if X is Hausdorff (respectively regular), then Y is Hausdorff (respectively regular).

(ii) if Y is Lindelöf (respectively countably compact), then X is Lindelöf (respectively countably compact).

(iii) Assuming X to be Hausdorff, X is locally compact iff Y is.

11. Suppose $\mathscr{U}$ is an ultrafilter on a (non-empty) set X. We say that $\mathscr{U}$ is *fixed*, or is a *principal ultrafilter*, if $\bigcap_{U \in \mathscr{U}} U \neq \varnothing$; otherwise it is *free*.

(i) Show that if $\mathscr{U}$ is fixed, then it has the form $\{U \subseteq X : x \in U\}$, for some $x \in X$. Deduce that it has a basis consisting of one set.

(ii) Show that if $\mathscr{U}$ is free, then it does not have a *countable* basis.