Analytic Topology: Problem sheet 3

1. *Suppose that $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces. Show that $\prod_{\lambda\in\Lambda} X_\lambda$ is Hausdorff in the Tychonoff topology if and only if each X_λ is Hausdorff.*

$\Rightarrow$: Suppose that a and b are different points of X_λ.

Choose points x and y of $\prod_{\mu\in\Lambda} X_\mu$ such that for all $\mu \neq \lambda$, $x(\mu) = y(\mu)$, and $x(\lambda) = a$ and $y(\lambda) = b$.

Then there exist disjoint open sets U and V in the product such that $x \in U$ and $y \in V$.

Without loss of generality U and V are basic open sets $\prod_{\mu\in\Lambda} U_\lambda$ and $\prod_{\mu\in\Lambda} V_\lambda$ respectively.

But then, $U \cap V = \varnothing$ implies that $U\lambda \cap V_\lambda = \varnothing$, since for $\mu \neq \lambda$, $x(\mu) = y(\mu) \in U_\mu \cap V_\mu$.

So U_λ and V_λ are disjoint open sets in X_λ containing a and b respectively.

$\Leftarrow$: Suppose that x and y are distinct points of the Tychonoff product.

Then for some λ, $x(\lambda) \neq y(\lambda)$.

Now X_λ is Hausdorff. So let U and V be disjoint open sets in X_λ containing $x(\lambda)$ and $y(\lambda)$ respectively.

Then $U \times \prod_{\mu\in\Lambda,\, \mu\neq\lambda} X_\mu$ and $V \times \prod_{\mu\in\Lambda,\, \mu\neq\lambda} X_\mu$ are disjoint sets, open in the Tychonoff topology, containing x and y respectively.

2. *Suppose that $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces and that $\prod_{\lambda\in\Lambda} X_\lambda$ is given the Tychonoff topology. Fix a point $f \in \prod_{\lambda\in\Lambda} X_\lambda$ and, for any fixed $\mu \in \Lambda$, let Y_μ be that subset defined by*

$$Y_\mu = \{g : g(\lambda) = f(\lambda) \text{ whenever } \lambda \neq \mu\}.$$

Prove that the restriction of the projection mapping π_μ to Y_μ is a homeomorphism from Y_μ to X_μ.

If U is open in X_λ, then ${\pi_\mu}^{-1}(U) = \left(U \times \prod_{\lambda\neq\mu} X_\lambda\right) \cap Y_\mu$ is open in Y_μ.

If V is open in Y_μ, then for all $g \in Y_\mu$, there exists a basic open set $\prod_\lambda U_\lambda$ such that $g \in (\prod_\lambda U_\lambda) \subseteq V$. Then $h \in (\prod_\lambda U_\lambda)$ if and only if $h(\mu) \in U_\mu$. Now $U_\mu \subseteq \pi_\mu(V)$, so we see that $\pi_\mu(V)$ is open.

3. *Suppose that for each $\lambda \in \Lambda$, X_λ is non-empty and connected. For any given $f \in \prod_{\lambda\in\Lambda} X_\lambda$, show that*

$$D = \{g : g(\lambda) = f(\lambda) \text{ for all but finitely many } \lambda\}$$

is connected. Show that $\prod_{\lambda\in\Lambda} X_\lambda$ is connected, in the Tychonoff topology.

If F is a finite subset of Λ, define Y_F to be $\{g : \forall\mu \notin F\, g(\mu) = f(\mu)\}$. Then, as in the previous question, Y_F is homeomorphic to the finite product $\prod_{\mu\in F} X_\mu$, and so is connected.

Also, D is the union of all the Y_F, and all the Y_F contain the point f.

So D is connected.

Now $\prod_{\lambda\in\Lambda} X_\lambda = \overline{D}$, and so $\prod_{\lambda\in\Lambda} X_\lambda$ is connected.

To see that $\prod_{\lambda\in\Lambda} X_\lambda = \overline{D}$, let $\prod_{\lambda\in\Lambda} U_\lambda$ be a basic open set; suppose that $U_\lambda = X_\lambda$ for all λ not belonging to some finite set F.

Define g so that $g(\lambda) = f(\lambda)$ if $\lambda \notin F$, and if $\lambda \in F$, then $g(\lambda) \in U_\lambda$.

Then $g \in D \cap \prod_{\lambda\in\Lambda} U_\lambda$.

4. *Find the generalization of Tychonoff's Theorem to locally compact spaces, and prove it.*

The theorem is: a product $\prod_{\lambda\in\Lambda} X_\lambda$ is locally compact if all the spaces X_λ are locally compact, and all but finitely many of them are compact.

The proof: Suppose that X_λ is compact for all λ not belonging to some finite set F.

We show that each point has a compact neighbourhood.

Let f be any point of $\prod_{\lambda\in\Lambda} X_\lambda$. For $\lambda \in F$, define U_λ to be an open subset of X_λ, and K_λ to be a compact subset of X_λ, such that $f(\lambda) \in U_\lambda \subseteq K_\lambda$.

For $\lambda \notin F$, define $U_\lambda = K_\lambda = X_\lambda$, observing that this set is both open and compact.

Then if we let $U = \prod_{\lambda\in\Lambda} U_\lambda$, then U is open, and if we let $K = \prod_{\lambda\in\Lambda} K_\lambda$, then K is compact, and $f \in U \subseteq K$, as required.

5. *Suppose that $\{X_\lambda : \lambda \in \Lambda\}$ is a family of non-empty topological spaces. Show that, in the box topology, a product of infinitely many Hausdorff spaces, each of which has at least two points, is not compact.*

Following the hint, we note that a box product of discrete two-point spaces is an infinite (in fact uncountable) discrete space, which is definitely not compact.

In more generality, let $x_{\lambda,0}$ and $x_{\lambda,1}$ be distinct elements of X_λ. Let $U_{\lambda,0} = X_\lambda \setminus \{x_{\lambda,1}\}$, and let $U_{\lambda,1} = X_\lambda \setminus \{x_{\lambda,0}\}$.

For any function $f : \Lambda \to \{0,1\}$, define x_f to be the function on Λ taking λ to $x_{\lambda,f(\lambda)}$, and define U_f to be $\prod_{\lambda\in\Lambda} U_{\lambda,f(\lambda)}$.

Then x_g belongs to U_f if and only if $g = f$.

Also the set $\{U_f | f : \Lambda \to \{0,1\}\}$ is an open cover of $\prod_{\lambda\in\Lambda} X_\lambda$.

However it has no proper subcover at all, and hence *a fortiori* it has no finite subcover.

6. *(Optional) For each natural number n, let $X_n = [0,1]$. In the cartesian product $\prod_{n\in\mathbb{N}} X_n$, let*

$$U = \left\{ g : (\exists r > 0)(\forall n) \left(g(n) < \frac{r}{n} \right) \right\}.$$

Show that, in the box topology, U is clopen (ie. closed and open). Deduce that a box product of connected spaces need not be connected.

We first show that U is open. Suppose that $g \in U$. Let $r > 0$ be such that for all n, $g(n) < \frac{r}{n}$.

For each n, let $V_n = [0,1] \cap (-\infty, \frac{r}{n})$, and let $V = \prod_{n\in\mathbb{N}} V_n$.

Then V is open, $g \in V$, and $V \subseteq U$.

So U is open.

Now we argue that U is closed.

Suppose that $f \in \overline{U}$.

For each $n \in \mathbb{N}$, let $W_n = [0,1] \cap (f(n) - \frac{1}{n}, f(n) + \frac{1}{n})$.

Let $W = \prod_{n\in\mathbb{N}} W_n$.

Then $f \in W$ and W is open.
So $W \cap U \neq \varnothing$.
Let $g \in W \cap U$.
Then there exists $r > 0$ such that for all n, $g(n) < \frac{r}{n}$.
Then for all n, $f(n) < \frac{r+1}{n}$.
Hence $f \in U$.
Thus U is closed.

Now U is both closed and open in the box product, and is also a proper subset of $\prod_{n\in\mathbb{N}} X_n$ (since the function $n \mapsto 1$ is not in U), so the box product $\prod_{n\in\mathbb{N}} X_n$ is not connnected.

7. *If X is a locally compact Hausdorff space, show that X has a basis of open sets with compact closure.*

For each $x \in X$, let U_x be open, and K_x be compact, such that $x \in U_x \subseteq K_x$.

Let $\mathscr{B}$ be the set of all open subsets V of X that for some x, $V \subseteq U_x$.

Firstly $\mathscr{B}$ is a basis, for if x is any point of X and $U \ni x$ is open, then $U \cap U_x$ is en element of $\mathscr{B}$ containing x and contained in U.

Now if $V \in \mathscr{B}$, then for some x, $V \subseteq U_x$, and $U_x \subseteq K_x$. Now X is Hausdorff, so K_x is closed. Hence $\overline{V} \subseteq K_x$. But now $\overline{V}$ is a closed subset of a compact set, so $\overline{V}$ is compact.

8. *Prove that a locally compact subset A of a Hausdorff space X is of the form $V \cap F$, where V is open and F is closed in X.*

Let $F = \overline{A}$.

For each $x \in A$, let U_x and K_x be subsets of A witnessing local compactness, that is, such that $x \in U_x \subseteq K_x$, U_x is open in A, and K_x is compact.

Now U_x is open in A, so there exists V_x open in X such that $U_x = A \cap V_x$.

Also K_x is compact, so because X is Hausdorff, K_x is closed. So in the topology of X, K_x is closed, contains U_x, and is contained in A.

Let $V = \bigcup_{x\in A} V_x$. We argue that $V \cap F = A$.

Clearly $A \subseteq V \cap F$.

As for the reverse inclusion, suppose that $y \in V \cap F$. Then $y \in V$, so $y \in V_x$ for some $x \in A$. Hence $y \in V_x \cap F = V_x \cap \overline{A}$.

We argue that in fact, $y \in \overline{V_x \cap A}$. For, let W be any open set containing y. Then $W \cap V_x$ is also an open set containing y. Since $y \in \overline{A}$, $W \cap V_x$ must meet A; that is, $(W \cap V_x) \cap A \neq \varnothing$. Rebracketing, $W \cap (V_x \cap A) \neq \varnothing$, as required.

Hence $y \in \overline{V_x \cap A} = \overline{U_x}$. But K_x is closed and $U_x \subseteq K_x$, so $\overline{U_x} \subseteq K_x$.
Hence $y \in K_x$.
Hence $y \in A$, as required.

9. *Prove that the following properties of a locally compact Hausdorff space X are equivalent.*

(i) *X is σ-compact (that is, X is a union of a countable family of compact subsets),*

(ii) *X can be represented as $X = \bigcup_{i=1}^{\infty} U_i$, where each U_i is an open set with compact closure, and $\overline{U_i} \subseteq U_{i+1}$ for each $i \in \mathbb{N}$;*

(iii) *X is Lindelöf.*

(i)⇒(ii): Suppose K_i is compact for each natural number i, and that $X = \bigcup_{i\in\mathbb{N}} K_i$.

We define the open sets U_i by recursion, so $\overline{U_i}$ is compact, $\overline{U_i} \subseteq U_{i+1}$ and $K_i \subseteq U_i$; this last condition ensures that $X = \bigcup_{i\in\mathbb{N}} U_i$.

If $i = 1$, let $L_i = \varnothing$, and otherwise let $L_i = \overline{U_{i-1}}$.

Then L_i is compact; hence so is $L_i \cup K_i$.

For each $x \in L_i \cup K_i$, let V_x be an open set with compact closure. Let $\{V_{x_j} : j < n\}$ be a finite subcover. Let $U_i = \bigcup_{j<n} V_{x_j}$. Certainly U_i is open and contains both $\overline{U_{i-1}}$ and K_i. Also, $\bigcup_{j<n} \overline{V_{x_j}}$ is a finite union of closed compact sets containing all of U_i, so is closed and compact. Hence U_i has compact closure.

(ii)⇒(iii): Let $\mathscr{V}$ be an open cover of X. Let $\mathscr{V}_i$ be a finite subset of $\mathscr{V}$ which covers $\overline{U_i}$.

Then $\bigcup_{i\in\mathbb{N}} \mathscr{V}_i$ is a countable subset of $\mathscr{V}$ which covers all of X.

(iii)⇒(i): For each $x \in X$, let U_x be an open set such that $x \in U_x$ and $\overline{U_x}$ is compact.

Let $\{U_{x_i} : i \in \mathbb{N}\}$ be a countable subcover.

Then $\{\overline{U_{x_i}} : i \in \mathbb{N}\}$ is a countable cover of X by compact sets.

10. *Suppose that f is a proper mapping from X onto Y. Suppose $U \subseteq X$ is open. Defining $f^*(U) = Y \setminus f(X \setminus U)$, show that $f^*(U)$ is open, that $f^*(U) \subseteq f(U)$; and that $f^{-1}(A) \subseteq U$ implies $A \subseteq f^*(U)$.*

$f^*(U)$ is known as the *small image* of U under f.

To show that $f^*(U)$ is open, observe that $X\setminus U$ is closed; f is a closed map, so $f(X\setminus U)$ is closed; now $f^*(U)$ is open.

That $f^*(U) \subseteq f(U)$, and that $f^{-1}(A) \subseteq U$ implies $A \subseteq f^*(U)$, are easy set algebra.

Prove that:

(i) *if X is Hausdorff (respectively regular), then Y is Hausdorff (respectively regular).*

Suppose x and y are distinct points of Y.

Then $f^{-1}(x)$ and $f^{-1}(y)$ are disjoint compact subsets of X, which, in a Hausdorff space, may be separated by disjoint open sets U and V.

Then $f^*(U)$ and $f^*(V)$ are disjoint open sets containing x and y respectively.

The proof for regularity is similar.

(ii) *if Y is Lindelöf (respectively countably compact), then X is Lindelöf (respectively countably compact).*

We do the argument for Lindelöfness; the argument for countable compactness is very similar.

Let $\mathscr{U}$ be an open cover of X. For each point y of Y, $f^{-1}(y)$ is compact, so there is a finite subset $\mathscr{U}_y$ of $\mathscr{U}$ covering it. Let $U_y = f^*(\bigcup \mathscr{U}_y)$; then U_y is an open set containing y.

Thus $\{U_y : y \in Y\}$ is an open cover of Y.

So there is a countable subcover $\{U_{y_n} : n \in \mathbb{N}\}$ of Y.

Then $\bigcup_{n\in\mathbb{N}} \mathscr{U}_{y_n}$ is a countable subcover of $\mathscr{U}$, completing the argument.

(iii) *Assuming X to be Hausdorff, X is locally compact iff Y is.*

⇒: Suppose that $y \in Y$. Then $f^{-1}(y)$ is compact because f is proper. For each point x of $f^{-1}(y)$, let U_x and K_x be neighbourhoods of x such that $x \in U_x \subseteq K_x$, U_x is open, and K_x is compact. Let $\{U_{x_i} : i < n\}$ be a finite cover of $f^{-1}(y)$. Let $U = \bigcup_{i<n} U_{x_i}$ and $K = \bigcup_{i<n} K_{x_i}$. Then $f^{-1}(y) \subseteq U \subseteq K$, U is open, and K is compact. Then $y \in f^*(U) \subseteq f(K)$, $f^*(U)$ is open, and $f(K)$ is compact.

⇐: Suppose that $x \in X$. Then $f(x) \in Y$. Find U open and K compact such that $f(x) \in U \subseteq K$. Then $x \in f^{-1}(U) \subseteq f^{-1}(K)$, and $f^{-1}(U)$ is open and $f^{-1}(K)$ is compact.

11. *Suppose $\mathscr{U}$ is an ultrafilter on a (non-empty) set X. We say that $\mathscr{U}$ is fixed, or is a principal ultrafilter, if $\bigcap_{U \in \mathscr{U}} U \neq \varnothing$; otherwise it is free.*

(i) *Show that if $\mathscr{U}$ is fixed, then it has the form $\{U \subseteq X : x \in U\}$, for some $x \in X$. Deduce that it has a basis consisting of one set.*

Suppose that $x \in \bigcap_{U \in \mathscr{U}} U$.

Because $\mathscr{U}$ is an ultrafilter, exactly one of $\{x\}$ and $X \setminus \{x\}$ belongs to $\mathscr{U}$. $X \setminus \{x\}$ cannot, since $x \in \bigcap_{U \in \mathscr{U}} U$, so $\{x\} \in \mathscr{U}$.

But then $\{x\}$ is a subset of every element of $\mathscr{U}$, and since $\mathscr{U}$ is closed under $\supseteq$, $\mathscr{U} = \{U : U \ni x\}$. Now $\{\{x\}\}$ is a basis for $\mathscr{U}$ consisting of exactly one element.

(ii) *Show that if $\mathscr{U}$ is free, then it does not have a countable bas*

Suppose that $\mathscr{U}$ is an free ultrafilter, and that $\{U_n : n \in \mathbb{N}\}$ is a countable basis for it.

Because ultrafilters are closed under finite intersections, we may assume that $U_{n+1} \subseteq U_n$ for all n, and because $\mathscr{U}$ is free, we may assume that the inclusions are strict.

Let $A = \bigcup_{k \in \mathbb{N}} U_{2k} \setminus U_{2k+1}$.

Then neither A nor $X \setminus A$ contains any of the sets U_n. Since $\{U_n : n \in \mathbb{N}\}$ is a basis for $\mathscr{U}$, this means that neither A nor $X \setminus A$ belongs to $\mathscr{U}$. But then $\mathscr{U}$ cannot be an ultrafilter.