Analytic Topology: Problem sheet 4

The beginning of this sheet concerns the Stone-Čech compactification βX of a Tychonoff space X.

When using the Stone-Čech compactification, it's usually best to use the Stone-Čech property, rather than using the details of the construction.

Intuitively, it's probably correct to think of the embedding h of a Tychonoff space in its Stone-Čech compactification as if it were the identity. But when writing out formal answers in which h occurs, drawing appropriate commutative diagrams really helps.

1. *Suppose X is a T_4 space and that $h : X \to \beta X$ is the "canonical embedding" of X into its Stone-Čech compactification. If A and B are disjoint closed subsets of X, prove that $\overline{h(A)}^{\beta X} \cap \overline{h(B)}^{\beta X} = \varnothing$.*

Using Urysohn's Lemma, define a continuous function $f : X \to [0, 1]$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$.

Extend to a continuous function $\beta f : \beta X \to [0, 1]$.

Then $\overline{h(A)}^{\beta X} \subseteq (\beta f)^{-1}\{0\}$, and $\overline{h(B)}^{\beta X} \subseteq (\beta f)^{-1}\{1\}$, so the two sets are disjoint.

2. *Let E denote the set of even natural numbers. Show that $\overline{h(E)}^{\beta\mathbb{N}}$ is homeomorphic to βE, and hence that $\beta\mathbb{N}$ may be represented as the disjoint union of two homeomorphs of itself. (Here $\mathbb{N}$ has its usual discrete topology.)*

Showing that $\overline{h(E)}^{\beta\mathbb{N}}$ is homeomorphic to βE amounts to showing that any continuous function from E to any compact Hausdorff space (or, equivalently, to $[0, 1]$) may be extended to a continuous function on $\overline{h(E)}^{\beta\mathbb{N}}$. We do this by first extending a function $f : E \to [0, 1]$ to a function $g : \mathbb{N} \to [0, 1]$, noting that since $\mathbb{N}$ is discrete, g is continuous, then extending g to a function $\beta g : \beta\mathbb{N} \to [0, 1]$, and then noting that $\beta g \restriction \overline{h(E)}^{\beta\mathbb{N}}$ is a function from $\overline{h(E)}^{\beta\mathbb{N}}$ to $[0, 1]$ extending f, as required.

$\overline{h(E)}^{\beta\mathbb{N}}$ and $\overline{h(\mathbb{N} \setminus E)}^{\beta\mathbb{N}}$ are disjoint by the previous question, and their union is $\overline{h(E) \cup h(\mathbb{N} \setminus E)}$, namely $\beta\mathbb{N}$ itself.

Also βE and $\beta(\mathbb{N} \setminus E)$ are clearly homeomorphic to $\beta\mathbb{N}$.

3. *If $\mathscr{N}_p$ is the neighbourhood filter in $\beta\mathbb{N}$ of a point $p \in \beta\mathbb{N} \setminus h(\mathbb{N})$, and $\mathscr{U} = \{N \cap \mathbb{N} : N \in \mathscr{N}_p\}$, show that $\mathscr{U}$ is a free ultrafilter on $\mathbb{N}$, and deduce that $\mathscr{N}_p$ does not have a countable basis.*

That $\mathscr{U}$ is a filter is pretty much trivial, and follows from the observations that $\mathscr{N}_p$ is a filter and that $h(\mathbb{N})$ is dense in $\beta\mathbb{N}$. To show that it is an ultrafilter, let $A \subseteq \mathbb{N}$. Let $f : \mathbb{N} \to \{0, 1\}$ be defined so that $f(n) = 1$ if and only if $n \in A$. Extend to a continuous function $\beta f : \beta\mathbb{N} \to \{0, 1\}$. Then $\beta f(p)$ is either 1 or 0; accordingly either A or $\mathbb{N} \setminus A$ is an element of $\mathscr{U}$.

That $\mathscr{U}$ is a free ultrafilter follows from Hausdorffness of $\beta\mathbb{N}$ (p can be separated from every $h(n)$).

From the fact that $\mathscr{U}$ does not have a countable basis it follows that $\mathscr{N}_p$ does not either.

4. *Show that every compact metric space is a continuous image of $\beta\mathbb{N}$.*

Let D be a countable dense set of the compact metric space K, which exists because every Lindelöf metric space is separable; write $D = \{d_n : n \in \mathbb{N}\}$. Define $f : \mathbb{N} \to D$ by $f(n) = d_n$. This is continuous because $\mathbb{N}$ is discrete.

Now extend to $\beta f : \beta\mathbb{N} \to K$. Now βf is continuous and $\beta\mathbb{N}$ is compact, so the image $\beta f(\beta\mathbb{N})$ is compact. K is Hausdorff, so $\beta f(\beta\mathbb{N})$ is closed. $D \subseteq \beta f(\beta\mathbb{N})$ is dense, so in fact $\beta f(\beta\mathbb{N}) = K$, as required.

This actually proves that every compact Hausdorff separable space is a continuous image of $\beta\mathbb{N}$. Non-metrisable such spaces include $\beta\mathbb{N}$ itself, $\beta\mathbb{R}$, (for those familiar with set theory) the Tychonoff product of $2^{\aleph_0}$ copies of $[0, 1]$, and so on. (The fact that the last-mentioned is separable, is non-trivial and is an interesting exercise.)

5. *Is $\beta\mathbb{N}$ metrisable? (If so, prove it; if not, show why not.)*

$\beta\mathbb{N}$ has points with no countable local basis, so it is not metrisable.

$\beta\mathbb{N}$ is in fact very badly not metrisable. As an example of this, for those who know some set theory, there's a theorem that every separable metric space has cardinality no more than $2^{\aleph_0}$. $\beta\mathbb{N}$ has cardinality $2^{2^{\aleph_0}}$ ($\geq$ because $\beta\mathbb{N}$ can be mapped onto the Tychonoff product of $2^{\aleph_0}$ copies of $[0, 1]$, and $\leq$ because there are only $2^{2^{\aleph_0}}$ possible values that the ultrafilter $\mathscr{U}$ in question 3. can take).

6. *Let E be that subset of $\mathbb{R}^2$ defined by*

$$E = \left[\bigcup_{n=1}^{\infty} \left([0, 1] \times \left\{\frac{1}{n}\right\}\right)\right] \cup \{(0, 0)\} \cup \{(1, 0)\}.$$

Identify all the components and all the quasi-components of E. (E has the topology induced by the usual topology on $\mathbb{R}^2$.)

All the lines $[0, 1] \times \{\frac{1}{n}\}$ are clopen and connected, so they are clearly both components and quasi-components.

If we eliminate these, we are left with the two-element set $\{(0, 0), (1, 0)\}$. Since this is disconnected, each element forms a separate component.

However, any clopen set U containing $(0, 0)$ meets infinitely many of the lines $[0, 1] \times \{\frac{1}{n}\}$, because U is open. These lines are connected so must be entirely contained in U. Now because U is closed, it must contain $(1, 0)$.

So the two-element set $\{(0, 0), (1, 0)\}$ is a quasi-component.

7. *Let F be that subset of $\mathbb{R}^2$ defined by*

$$F = \left\{(x, y) : y = \sin \frac{\pi}{x},\ x \in (0, 1]\right\} \cup A, \text{ where } A = \{(0, y) : -1 \leq y \leq 1\},$$

with the topology induced by the usual topology on $\mathbb{R}^2$. Prove that F is a connected, but not locally connected, topological space.

The set $\{(x, y) : y = \sin \frac{\pi}{x},\ x \in (0, 1]\}$ is the continuous image of the interval $(0, 1]$, so it is connected. F is the closure of this set, so it is connected also.

F is not locally connected, since any basis of neighbourhoods at $(0, \frac{1}{2})$ must, for some n, contain a set U which contains $B_{\frac{1}{n}}(0, \frac{1}{2})$ and which is contained in $B_{\frac{1}{2}}(0, \frac{1}{2})$. No such set U can be connected.

8. *Suppose that A and B are closed subsets of a compact Hausdorff space X such that no component of X meets both A and B. Show that there exists a clopen subset C of X such that $A \subseteq C \subseteq X \setminus B$.*

This is very similar to the proof that a compact Hausdorff space is normal.

By the Šura-Bura Lemma, no quasi-component of X meets both A and B. So, for each point x of A and each point y of B, x and y belong to different quasi-components, so there exists a clopen set $U_{x,y}$ such that $x \in U_{x,y}$ and $y \notin U_{x,y}$. Write $V_{x,y}$ for the complement of $U_{x,y}$, which is also clopen.

Then for each point x of A, the family $\{V_{x,y} : y \in B\}$ is an open cover of B, which is compact, so there is a finite subcover $\{V_{x,y_{x,i}} : i < n_x\}$. Let $V_x = \bigcup_{i<n_x} V_{x,y_{x,i}}$. Then V_x is a clopen set containing B and not containing x. Let U_x be the complement of V_x. Then U_x is a clopen set containing x and not meeting B.

Now $\{U_x : x \in A\}$ is an open cover of A, which is compact, so let $\{U_{x_i} : i < n\}$ be a finite subcover. Let $C = \bigcup_{i<n} U_{x_i}$. Then C is a clopen set containing A and not meeting B, as required.

9. *Suppose that D is a non-empty compact proper subset of a connected Hausdorff space Y. Show that every component of D meets the boundary of D.*

This is the so-called "boundary-bumping theorem".

Suppose K is a component of D which does not meet the boundary of D.

Now K and ∂D are closed subsets of D, and no component of D meets both.

So there exists C clopen in D such that $K \subseteq C$ but $C \cap \partial D = \varnothing$, from which it follows that $C \subseteq D^\circ$.

Now C is closed in D, and D is closed in Y, so C is closed in Y.

Also C is open in D, and hence in D°, and D° is open in Y. Hence C is open in Y.

So C is clopen in Y.

Now C contains a component of D, so C is non-empty. Also C is contained in D, so C is not the whole of Y.

Hence Y is not connected, giving a contradiction.

10. *For $f, g \in \prod_{n\in\mathbb{N}}[0,1]$, define $D(f,g) = \sum_{n=0}^{\infty} 2^{-n}|f(n) - g(n)|$. Show that the Tychonoff topology on the product $\prod_{n\in\mathbb{N}}[0,1]$, and the topology generated by the metric D, are the same.*

Firstly, we observe that D is a metric on the Tychonoff product.

Next we prove that D generates the Tychonoff topology. To do this it is sufficient to show that every D-open ball is Tychonoff open. Then the identity, considered as a function from the product with the Tychonoff topology to the product with the D-topology, is a continuous bijection from a compact space to a Hausdorff space, and is therefore a homeomorphism.

In more detail: We check that D is a metric. That $D(f,g) = 0$ if and only if $f = g$ is obvious. D is clearly symmetric. We check the triangle inequality. For each n,

$|f(n) - h(n)| \leq |f(n) - g(n)| + |g(n) - h(n)|$ by the triangle inequality on $\mathbb{R}$. The triangle inequality for D now follows by summing this inequality over n.

We now check that each D-open ball is Tychonoff open. Suppose that g belongs to the D-ball of radius r around f. Let $s = r - D(g, f)$; then by the triangle inequality, the ball of radius s around g is contained in the ball of radius r around f.

Now let $n > 1/s$, and also $n > 8$. We find a Tychonoff-open set U around g contained in the ball of radius s around g.

For each $m < n - 1$, let $U_m = (g(m) - \frac{1}{n^2}, g(m) + \frac{1}{n^2}) \cap [0, 1]$. For $m \geq n - 1$, let $U_m = [0, 1]$.

Then $U = \prod_{m \in \mathbb{N}} U_m$ is Tychonoff-open and contains g.

Now suppose that $h \in U$. We estimate $D(g, h)$.

If $m < n - 1$, then $|g(m) - h(m)| < 1/n^2$, so certainly $2^{-m} |g(m) - h(m)| < 1/n^2$.

If $m \geq n - 1$, then $|g(m) - h(m)| < 1$. Hence $2^{-m} |g(m) - h(m)| < 2^{-m}$, so $\sum_{m=n-1}^{\infty} |g(m) - h(m)| \leq \sum_{m=n-1}^{\infty} 2^{-m} = 2^{-(n-2)} < \frac{1}{n^2}$, since $n > 8$.

Now $D(g, h) < n \times \frac{1}{n^2} = 1/n < s$, as required.

Now we know that every metric-open ball is Tychonoff-open. Let ϕ be the identity on $\prod_{n \in \mathbb{N}} [0, 1]$, considered as a function from the space with the Tychonoff topology to the space with the D-metric topology.

Then ϕ is a continuous bijection from a compact space to a Hausdorff space, and so is a homeomorphism.

11. *Prove that a paracompact regular space is normal.*

This is of course similar to the proof that a compact regular space is normal, but the point of the question is what modifications you need to make it work.

Suppose that X is paracompact and regular, and that C and D are disjoint open sets.

Then for each $x \in C$, there exist disjoint open U_x and V_x such that $x \in U_x$ and $D \subseteq V_x$.

Then $\mathscr{U} = \{U_x : x \in C\} \cup \{X \setminus C\}$ is an open cover of X.

Let $\mathscr{V}$ be a locally finite open refinement of $\mathscr{U}$.

Let $\mathscr{W} = \{V \in \mathscr{V} : V \cap C \neq \varnothing\}$.

Let $W = \bigcup_{V \in \mathscr{W}} V$.

Then because locally finite collections are closure-preserving, $\overline{W} = \bigcup_{V \in \mathscr{W}} \overline{V}$.

Now if $V \in \mathscr{W}$, then $V \in \mathscr{V}$, so there exists $U \in \mathscr{U}$ such that $V \subseteq U$; and also $V \cap C \neq \varnothing$ so $U \neq X \setminus C$; so $U = U_x$ for some $x \in C$.

Then $\overline{U_x} \subseteq X \setminus V_x$, so $\overline{U_x}$ does not meet D.

Hence $\overline{V}$ does not meet D either.

So $\overline{W}$ does not meet D.

This completes the proof of normality.

12. *Prove that a closed subspace of a paracompact space is paracompact.*

This closely imitates the proof that a closed subspace of a compact space is compact. The only wrinkle (which, as it happens, does not make life any more difficult) is that local finiteness is not absolute: if $Y \subseteq X$ and $\mathscr{W}$ is a locally finite family of subsets of Y, then it may not be locally finite in X. However, the reverse implication, fortunately, does work: if $\mathscr{W}$ is locally finite in X, then it is locally finite in Y.

Let C be a closed subspace of a paracompact space X.

Let $\mathscr{U}$ be an open cover of C.

Then for each $U \in \mathscr{U}$, there exists an open subset $\hat{U}$ of X such that $U = \hat{U} \cap C$. Let $\hat{\mathscr{U}} = \{\hat{U} : U \in \mathscr{U}\}$.

Then $\hat{\mathscr{U}} \cup \{X \setminus C\}$ is an open cover of X.

Let $\mathscr{V}$ be a locally finite open refinement.

Then $\{V \in \mathscr{V} : V \cap X \neq \varnothing\}$ is a refinement of $\hat{\mathscr{U}}$, and is locally finite with respect to X.

Then $\{V \cap C : V \in \mathscr{V}, V \cap X \neq \varnothing\}$ is a refinement of $\mathscr{U}$, and is locally finite with respect to C.

13. *Prove that the following three conditions on a regular space X are equivalent:*

(i) *Every open covering of X has a locally finite open refinement (that is, X is paracompact);*

(ii) *Every open covering of X has a locally finite refinement (the elements of the refinement not being necessarily open or closed);*

(iii) *Every open covering of X has a locally finite closed refinement (the elements of the refinement being closed sets).*

(i)$\Rightarrow$(ii): Trivial.

(ii)$\Rightarrow$(iii): I first give a plausible, but wrong, proof, and then give a correct proof.

Wrong proof: Let $\mathscr{U}$ be an open covering of X. Let $\mathscr{V}$ be a locally finite refinement of $\mathscr{U}$. Then $\{\overline{V} : V \in \mathscr{V}\}$ is a locally finite closed refinement of $\mathscr{U}$: one can easily check that it is locally finite, and...it isn't actually a refinement of $\mathscr{U}$, because it's quite possible that $V \subseteq U$ but $\overline{V} \not\subseteq U$. Oh dear.

The solution: make the elements of $\mathscr{V}$ a bit smaller.

Because X is regular, for each element x of X, there exists open W_x such that, for some $U \in \mathscr{U}$, $x \in W_x \subseteq \overline{W_x} \subseteq U$.

Now $\mathscr{W} = \{W_x : x \in X\}$ is an open cover of X.

Let $\mathscr{V}$ be a locally finite refinement of $\mathscr{W}$.

Let $\hat{\mathscr{V}} = \{\overline{V} : V \in \mathscr{V}\}$.

Then we argue that $\hat{\mathscr{V}}$ is a locally finite closed refinement of $\mathscr{U}$.

That $\hat{\mathscr{V}}$ is a cover, follows from the fact that $\mathscr{V}$ is a cover.

$\hat{\mathscr{V}}$ is locally finite: let $x \in X$, and let U be an open neighbourhood of x witnessing local finiteness of $\mathscr{V}$, that is, so that $U \cap V \neq \varnothing$ for only finitely many members V of $\mathscr{V}$.

Now since U is open, $U \cap V \neq \varnothing$ if and only if $U \cap \overline{V} \neq \varnothing$. Hence U witnesses local finiteness of $\hat{\mathscr{V}}$ at x.

$\hat{\mathscr{V}}$ refines $\mathscr{U}$: suppose that $V \in \mathscr{V}$. Then for some $x \in X$, $V \subseteq W_x$, and for some $U \in \mathscr{U}$, $\overline{W_x} \subseteq U$.

Then $\overline{V} \subseteq \overline{W_x} \subseteq U$, as required.

(iii)$\Rightarrow$(i) is hard.

Let $\mathscr{U}$ be an open cover of X. Let $\mathscr{W}$ be a locally finite closed refinement of $\mathscr{U}$.

Now because $\mathscr{W}$ is locally finite, for all $x \in X$, there exists open $V_x \ni x$ such that V_x meets only finitely many members of $\mathscr{W}$.

Then $\mathscr{V} = \{V_x : x \in X\}$ is an open cover of X.

Let $\mathscr{T}$ be a locally finite closed refinement of $\mathscr{V}$.

Now for each $W \in \mathscr{W}$, there exist $U_W \in \mathscr{U}$ such that $W \subseteq U_W$.

We now define an open set S_W as follows:

$$S_W = U_W \setminus \bigcup\{T \in \mathscr{T} : T \cap W = \varnothing\}.$$

Now it is clear that $S_W \subseteq U_W$.

Also $W \subseteq S_W$.

Hence $\mathscr{S} = \{S_W : W \in \mathscr{W}\}$ is a cover, and is a refinement of $\mathscr{U}$.

Now $\mathscr{T}$ is locally finite, and locally finite collections are closure-preserving, so $\bigcup\{T \in \mathscr{T} : T \cap W = \varnothing\}$ is closed, so S_W is open.

So $\mathscr{S}$ is a locally finite open refinement of $\mathscr{U}$.

We now need to check whether it is locally finite.

Let $x \in X$. Let R be an open neighbourhood of x witnessing that $\mathscr{T}$ is locally finite.

Then $R \cap T \neq \varnothing$ for just finitely many $T \in \mathscr{T}$.

Now $\mathscr{T}$ is a refinement of $\mathscr{V}$. So for each $T \in \mathscr{T}$, there exists y such that $T \subseteq V_y$. Now V_y meets only finitely many elements of $\mathscr{W}$.

Hence each T meets only finitely many elements of $\mathscr{W}$.

That means that for each $T \in \mathscr{T}$, there are only finitely many elements W' of $\mathscr{W}$ such that T is not removed in the construction of $S_{W'}$; that is, for all $T \in \mathscr{T}$, there are only finitely many $W' \in \mathscr{W}$ such that $T \cap W' \neq \varnothing$.

Thus R meets only finitely many elements of $\mathscr{T}$, and each of these meets only finitely many elements of $\mathscr{S}$.

Hence R meets only finitely many elements of $\mathscr{S}$.

So $\mathscr{S}$ is locally finite.

14. *Let $\mathbb{B}$ be a Boolean algebra. Prove that the following hold, for all $a, b, c \in \mathbb{B}$:*

(i) $a \leq b$ *if and only if* $a \vee b = b$,

Using the definition of $\vee$.

(ii) $b = \neg a$ *if and only if* $b \wedge a = \mathbb{0}$ *and* $b \vee a = \mathbb{1}$,

The forward direction is spelt out in the definition of a Boolean algebra.

As for the reverse direction, $\neg a = \neg a \vee \mathbb{0}$ since $\mathbb{0} \leq \neg a$, which equal to $\neg a \vee (b \wedge a) = (\neg a \wedge b) \vee (\neg a \wedge a) = (\neg a \wedge b) \vee \mathbb{0} = \neg a \wedge b$, so $\neg a \leq b$ by an argument similar to part (i). Also $\neg a = \neg a \wedge \mathbb{1}$ since $\mathbb{1} \geq \neg a$, which equal to $\neg a \wedge (b \wedge a) = (\neg a \vee b) \wedge (\neg a \vee a) = (\neg a \vee b) \wedge \mathbb{1} = \neg a \vee b$, so $\neg a \geq b$.

(iii) $\neg(a \wedge b) = (\neg a) \vee (\neg b)$.

We show that $(a \wedge b) \vee ((\neg a) \vee (\neg b)) = \mathbb{1}$ and $(a \wedge b) \wedge ((\neg a) \vee (\neg b)) = \mathbb{0}$, and use the previous part of the question.

We use the distributive law to show that $(a \wedge b) \vee ((\neg a) \vee (\neg b)) = (a \vee \neg a \vee \neg b) \wedge (b \vee \neg a \vee \neg b)$, which is equal to $\mathbb{1}$.

The other argument is similar.

15. *Let $\mathbb{A}$, $\mathbb{B}$ be Boolean algebras, and let $\phi : \mathbb{A} \to \mathbb{B}$ be a homomorphism. Prove that $\mathscr{S}\phi$ is one-to-one if and only if ϕ is onto, and is onto if and only if ϕ is one-to-one.*

In this solution, and subsequent solutions as well, we presume that the reader has no acquaintance with category theory.

Suppose that $\mathscr{S}\phi$ is not one-to-one. Suppose that q and r are distinct elements of $\mathscr{S}\mathbb{B}$ such that $\mathscr{S}\phi(q) = \mathscr{S}\phi(r)$.

Then $\{a \in \mathbb{A} : \phi(a) \in q\} = \{a \in \mathbb{A} : \phi(a) \in r\}$.

Thus either $q \neq \{a \in \mathbb{A} : \phi(a) \in q\}$, or $r \neq \{a \in \mathbb{A} : \phi(a) \in r\}$, or more likely both.

Now suppose that $\mathscr{S}\phi$ is one-to-one. Let b be an element of $\mathbb{B}$.

Then, letting $[\![b]\!]$ be the subset of $\mathscr{S}\mathbb{B}$ consisting of all q such that $b \in q$, $\mathscr{S}\phi([\![b]\!])$ is disjoint from $\mathscr{S}\phi(\mathscr{S}\mathbb{B} \setminus [\![b]\!])$. Since $\mathscr{S}\phi$ is continuous and both sets $[\![b]\!]$ and $\mathscr{S}\mathbb{B} \setminus [\![b]\!]$ are closed and therefore compact, their images $\mathscr{S}\phi([\![b]\!])$ and $\mathscr{S}\phi(\mathscr{S}\mathbb{B} \setminus [\![b]\!])$ and therefore closed. Since all components of $\mathscr{S}\mathbb{A}$ are single points, we can use question 8. to find a clopen subset C of $\mathscr{S}\mathbb{A}$ which contains $\mathscr{S}\phi([\![b]\!])$ and is disjoint from $\mathscr{S}\phi(\mathscr{S}\mathbb{B} \setminus [\![b]\!])$.

Then C has the form $[\![a]\!]$, for some $a \in \mathbb{A}$.

We now argue that $\phi(a) = b$.

For any ultrafilter q on $\mathbb{B}$, we ask when $\phi(a)$ belongs to q.

$\phi(a)$ belongs to q if and only if $a \in \mathscr{S}\phi(q)$, if and only if $\mathscr{S}\phi(q) \in [\![a]\!]$, if and only if $q \in [\![b]\!]$, if and only if $b \in q$.

Thus for all ultrafilters q, $\phi(a) \in q$ if and only if $b \in q$.

If $\phi(a)$ and b were different, we would be able to find an ultrafilter containing one and not the other.

Thus $\phi(a) = b$, and ϕ is onto.

Suppose that $\mathscr{S}\phi$ is onto. Suppose that a and b are distinct elements of $\mathbb{A}$, and that p is some ultrafilter on $\mathbb{A}$ which contains a and not b.

Using the fact that $\mathscr{S}\phi$ is onto, let q be some element of $\mathscr{S}\mathbb{B}$ such that $\mathscr{S}\phi(q) = p$.

Now $p = \{c \in \mathbb{A} : \phi(c) \in q\}$. Then $\phi(a) \in q$, while $\phi(b) \notin q$; so $\phi(a) \neq \phi(b)$; so ϕ is one-to-one.

Now suppose that ϕ is one-to-one. Let p be any element of $\mathscr{S}\mathbb{A}$.

We consider the set $\hat{q} = \{\phi(a) : a \in p\}$.

Now $\hat{q}$ is certainly closed under the operation $\wedge$, because if $\phi(a)$ and $\phi(b)$ belong to $\hat{q}$, where $a, b \in p$, then $\phi(a) \wedge \phi(b) = \phi(a \wedge b)$, and $a \wedge b$ belongs to p because p is a filter, so $\phi(a \wedge b) \in \hat{q}$.

Also $\mathbb{O}_\mathbb{B} \notin \hat{q}$, because $\mathbb{O}_\mathbb{B} = \phi(\mathbb{O}_\mathbb{A})$, $\mathbb{O}_\mathbb{A} \notin p$, and ϕ is one-to-one.

It now follows that $\tilde{q} = \{a \in \mathbb{A} : \exists b \leq a\, b \in \hat{q}\}$ is a filter, which can be extended to an ultrafilter q.

Then $\mathscr{S}\phi(q) = p$; so $\mathscr{S}\phi$ is onto.

16. *Let $\mathbb{B}$ be a Boolean algebra. Prove that $\eta_\mathbb{B} : \mathbb{B} \to \mathscr{B}\mathscr{S}\mathbb{B}$ is an isomorphism.*

Recall that $\eta_\mathbb{B}$ takes an element a of $\mathbb{B}$ to the subset $[\![a]\!]$ of $\mathscr{B}\mathscr{S}\mathbb{B}$.

Every element of $\mathscr{S}\mathbb{B}$ contains $\mathbb{1}_\mathbb{B}$, so every element of $\mathscr{S}\mathbb{B}$ belongs to $[\![\mathbb{1}_\mathbb{B}]\!]$, so $[\![\mathbb{1}_\mathbb{B}]\!] = \mathscr{S}\mathbb{B} = \mathbb{1}_{\mathscr{B}\mathscr{S}\mathbb{B}}$.

An element p of $\mathscr{S}\mathbb{B}$ contains both a and b if and only if it contains $a \wedge b$.

So, p belongs to both $[\![a]\!]$ and $[\![b]\!]$ if and only if it belongs to $[\![a \wedge b]\!]$.

So $[\![a]\!] \cap [\![b]\!] = [\![a \wedge b]\!]$.

Now any element p of $\mathscr{S}\mathbb{B}$ contains a if and only if it does not contain $\neg a$. Hence p belongs to $[\![a]\!]$ if and only if it does not belong to $[\![\neg a]\!]$.

Thus $[\![\neg a]\!]$ is the complement of $[\![a]\!]$.

This is enough to establish that $\eta_{\mathbb{B}}$ is a homomorphism.

If a and b are distinct elements of $\mathbb{B}$, then there is an ultrafilter containing one and not the other. Thus $[\![a]\!]$ and $[\![b]\!]$ are different. So $\eta_{\mathbb{B}}$ is one-to-one.

Now let U belong to $\mathscr{B}\mathscr{S}\mathbb{B}$.

Then U is a clopen subset of $\mathscr{S}\mathbb{B}$.

Since it is open, it is a union of basic open sets $[\![a]\!]$. So the basic open sets $[\![a]\!]$, for which $[\![a]\!] \subseteq U$, cover U.

Since U is closed, it is compact, so there is a finite subcover by sets $[\![a_1]\!], \ldots, [\![a_n]\!]$.

Then $U = [\![a_1]\!] \cup \cdots \cup [\![a_n]\!] = [\![a_1 \vee \cdots \vee a_n]\!]$, and so U is in the range of $\eta_{\mathbb{B}}$.

So $\eta_{\mathbb{B}}$ is onto.

17. (i) *What is the Stone dual of the one-point topological space?*

The one point topological space $*$ has two different clopen sets, namely $\varnothing$ and $*$ itself, and $\varnothing$ is a strict subset of $*$. Thus the Stone dual of $*$ is a two-element Boolean algebra, and all such are isomorphic.

(ii) *Let $*$ be the one-point topological space. Let X be a compact Hausdorff zero-dimensional space. One can think of a point of X as being the range of a function from $*$ to X. Let $f : * \to X$, and let p be the unique point in the range of f. (f is automatically continuous. Why?) Describe $\mathscr{S}f$ completely in terms of p.*

In the previous sentence $\mathscr{S}f$ is an error and should be $\mathscr{B}f$.

f is continuous because $*$ is discrete.

$\mathscr{B}f : \mathscr{B}X \to \mathscr{B}*$. Let U be a clopen subset of X.

Then $\mathscr{B}f(U) = *$ if and only if $f^{-1}(U) = *$ if an only if U contains p.

Thus $(\mathscr{B}f)^{-1}(*) = p$, and this fact determines $\mathscr{B}f$.

(iii) *A product of two Boolean algebras $\mathbb{A}$ and $\mathbb{B}$ is the cartesian product with pointwise operations, that is, $\mathbb{1}_{\mathbb{A}\times\mathbb{B}} = (\mathbb{1}_{\mathbb{A}}, \mathbb{1}_{\mathbb{B}})$, $\mathbb{O}_{\mathbb{A}\times\mathbb{B}} = (\mathbb{O}_{\mathbb{A}}, \mathbb{1}_{\mathbb{B}})$, $\neg(a,b) = (\neg a, \neg b)$, $(a_1,b_1) \wedge (a_2,b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$, $(a_1,b_1) \vee (a_2,b_2) = (a_1 \vee a_2, b_1 \vee b_2)$, and $(a_1,b_1) \leq (a_2,b_2)$ if and only if $a_1 \leq b_1$ and $a_2 \leq b_2$. Suppose that Z is the disjoint union of two compact zero-dimensional Hausdorff spaces X and Y. Show that $\mathscr{B}Z$ is isomorphic to $\mathscr{B}X \times \mathscr{B}Y$.*

The clopen subsets of Z are precisely the unions $W = U \cup V$, where U is a clopen subset of X and V is a clopen subset of Y. Under these circumstances, $U = W \cap X$ and $V = W \cap Y$.

Thus the map $W \mapsto (W \cap X, W \cap Y)$ is an isomorphism between $\mathscr{B}Z$ and $\mathscr{B}X \times \mathscr{B}Y$.

Optional questions

18. *One statement of the Baire Category Theorem is that any intersection of countably many dense open subsets of $\mathbb{R}$ is non-empty.*

(i) *Deduce from the above statement that any intersection of countably many dense open subsets of $\mathbb{R}$ is dense.*

Let $\{U_n : n \in \mathbb{N}\}$ be a countable family of dense open sets.

Let (a, b) be any non-empty open interval.

Then for each n, $U_n \cap (a, b)$ is dense in U_n.

Also, (a, b) is homeomorphic to $\mathbb{R}$, and so the Baire Category Theorem applies.

Hence the intersection of the sets $U_n \cap (a, b)$ is non-empty.

Hence the intersection of the sets U_n is dense.

(ii) *We say that a subset of $\mathbb{R}$ is of first category if it is contained in the complement of an intersection of countably many dense sets. Intuitively, one can regard first category sets as being small, and the Baire Category Theorem says that $\mathbb{R}$ is not of first category. Prove that any countable union of first category sets is of first category and so is not small.*

The word "not" at the end of the preceding sentence is an error, which I've only just found.

For each natural number n, let A_n be a set of first category, and let $\{U_{n,m} : m \in \mathbb{N}\}$ be a family of dense open sets witnessing this, that is, such that

$$A_n \cap \bigcap_{m \in \mathbb{N}} U_{n,m} = \varnothing.$$

Then $\{U_{n,m} : n,\ m \in \mathbb{N}\}$ is a family of dense open sets, and

$$\bigcup_{n \in \mathbb{N}} A_n \cap \bigcap_{n,m \in \mathbb{N}} U_{n,m} = \varnothing.$$

Hence $\bigcup_{n \in \mathbb{N}} A_n$ is of first category.

(iii) *Prove the Baire Category Theorem.*

[*Hint: Remember that any intersection of a strictly decreasing sequence of closed bounded intervals is non-empty.*]

Let $\{U_n : n \in \mathbb{N}\}$ be a countable family of dense open sets in $\mathbb{R}$.

We construct intervals I_n, for $n \in \mathbb{N}$, by recursion, so that I_n has length less than or equal to $\frac{1}{2^n}$, such that $I_n \subseteq U_n$, and $\overline{I_{n+1}} \subseteq I_n$. The inductive step uses the fact that U_{n+1} is dense, allowing us to conclude that $U_{n+1} \cap I_n$ is non-empty.

Now the sets $\overline{I_n}$ are a decreasing sequence of compact, closed subsets of $\mathbb{R}$, and so have non-empty intersection. Thus so do the sets I_n, and therefore so do the sets U_n.

(iv) *(For those who know some measure theory.) Find a subset Q of $\mathbb{R}$ such that $\mathbb{R} \setminus Q$ is of first category, and Q is null.*

The set $\mathbb{Q}$ of rationals is countable and therefore null. For each n, find a countable family $\{I_{n,m} : m \in \mathbb{N}\}$ of total length less than or equal to $\frac{1}{2^n}$ such that $\mathbb{Q} \subseteq \bigcup_{m \in \mathbb{N}} I_{n,m}$.

Let $U_n = \bigcup_{m \in \mathbb{N}} I_{n,m}$. Then U_n is dense and open, and has measure less than or equal to $\frac{1}{2^n}$.

Let $Q = \bigcap_{n \in \mathbb{N}} U_n$. Then Q is dense and null, and $\mathbb{R} \setminus Q$ is of first category.

19. (i) *Let A be a subset of $\mathbb{R}$ having the property that for all points x of $\mathbb{R}$ except at most one, there exists an open neighbourhood U of x such that $A \cap U$ is countable. Prove that A is countable.*

Let a be a point of $\mathbb{R}$ such that if $x \neq a$, then there is an open neighbourhood U_x of x such that $A \cap U_x$ is countable.

Let $\mathscr{U}$ be the set of all the U_x for $x \neq a$.

Now $\mathbb{R} \setminus \{a\}$ is a separable metric space and therefore is Lindelöf. Let $\mathscr{V}$ be a countable subcover of $\mathscr{U}$.

Then $A \subseteq \{a\} \cup \bigcup_{V \in \mathscr{V}} A \cap V$, which is a countable union of countable sets and therefore countable.

So A is countable.

(ii) *Deduce that if A is an uncountable subset of $\mathbb{R}$, there exist (at least) two distinct points x and y such that for every open neighbourhood U of either x or y, $U \cap A$ is uncountable.*

This follows immediately from the previous part.

(iii) *Prove that every uncountable closed subset of $\mathbb{R}$ contains a homeomorphic copy of the Cantor set.*

Let A be an uncountable closed subset of $\mathbb{R}$.

For any finite sequence s of zero's and ones, we construct an open interval I_s by recursion on the length of s, using the above fact, with the following properties.

*) The length of the interval I_s is less than or equal to 2^s.

*) If the sequence s is an initial part of the sequence t, then $\overline{I_t} \subseteq I_s$.

*) If s and t are different sequences of the same length, then $\overline{I_s} \cap \overline{I_t} = \varnothing$.

*) For all s, $I_s \cap A$ is uncountable.

The initial step of the recursion, for s equal to the empty sequence, simply requires us to find an interval of length 1 whose intersection with A is uncountable. But

$$\{(\tfrac{n}{2} - 1, \tfrac{n}{2} - 1) : n \in \mathbb{Z}\}$$

is a countable cover of $\mathbb{R}$ by intervals of length 1, so at least one of these must have uncountable intersection with A.

For the inductive step, suppose I_s has been chosen. Then we need to define I_{s0} and I_{s1}.

Now $A \cap I_s$ is uncountable by the inductive hypothesis.

Therefore there exist distinct points x and y in I_s such that every neighbourhood of either one of x and y meets A in an uncountable set.

So, we find intervals I_{s0} and I_{s1} containing x and y respectively, such that the length of both is less than or equal to $\frac{1}{2^m}$, where m is the length of the sequences $s0$ and $s1$, and such that the closures of these two intervals are disjoint and contained in I_s.

Now if f is any infinitely long sequence of zeroes and ones, the intervals $\overline{I_s}$, for s a finite initial part of f, are a nested sequence of closed compact subsets of $\mathbb{R}$, so have non-empty intersection. That intersection is a single point, because of the restriction on the lengths of the intervals I_s. Let x_f be that single point.

Now if $f \neq g$, then x_f and x_g must be different; also each x_f is, for each n, within distance $\frac{1}{2^n}$ of some element of A; so because A is closed, x_f belongs to A.

So K, the set of all the x_f, is an uncountable subset of A. It is also reasonably clear that it is homeomorphic to the Cantor set.

(iv) *(Harder) Prove that any intersection of countably many dense open subsets of* $\mathbb{R}$ *contains a homeomorphic copy of the Cantor set.*

Let $\{U_n : n \in \mathbb{N}\}$ be a countable family of dense open subsets of $\mathbb{R}$.

Repeat the above construction, only drop the requirement that $I_s \cap A$ should be uncountable, and add the requirement that for n less than or equal to the length of the sequence s, I_s should be a subset of U_n.

20. (i) *Let X be a compact zero-dimensional metric space. Note that X is second countable, by Urysohn's Metrisation Theorem. Prove that every clopen subset of X is a finite union of basic open sets. Prove that $\mathscr{B}X$ is countable.*

The proof that X is second countable goes like this. X is Lindelöf and metrisable, so must be separable. But X is a separable metric space, so must be second countable.

Let U be a clopen subset of X. Then U is a union of basic open sets. Therefore there is a cover $\mathscr{U}$ of U by basic open sets. But U is closed, therefore compact. So $\mathscr{U}$ has a finite subcover, $\mathscr{V}$. Now $\mathscr{V}$ is a finite subset of the original basis, and $U = \bigcup \mathscr{V}$.

Now let B be a countable basis for X. Then B has just countably many finite subsets. Thus X has just countably many clopen subsets altogether.

Therefore $\mathscr{B}X$ is countable.

(ii) *Let $\mathbb{P}$ be $\wp\mathbb{N}$, considered as a Boolean algebra. Define $h : \mathbb{N} \to \mathscr{S}\mathbb{P}$ so that $h(n) = \{A \subseteq \mathbb{N} : n \in A\}$. Then h is continuous. (Why?) Let $f : \mathbb{N} \to [0, 1]$ be any function. Define a continuous function $g : \mathscr{S}\mathbb{P} \to [0, 1]$ so that for all $n \in \mathbb{N}$, $g(h(n)) = f(n)$, and deduce that $\mathscr{S}\mathbb{P}$ is homeomorpic to $\beta\mathbb{N}$.*

[*Hint: Suppose that $p \in \mathscr{S}\mathbb{P}$. What is p? What value might you choose for $g(p)$?*]

h is continuous because $\mathbb{N}$ is discrete.

We apply ourselves to defining g.

Let p be an element of $\mathscr{S}\mathbb{P}$.

Then p is an ultrafilter on the Boolean algebra $\mathbb{P}$.

Therefore p is an ultrafilter on $\wp\mathbb{N}$.

For each n, we can cover $[0, 1]$ with 2^n disjoint intervals of length 2^n. (For example, for $n = 2$, we can cover $[0, 1]$ with $[0, \frac{1}{4})$, $[\frac{1}{4}, \frac{1}{2})$, $[\frac{1}{2}, \frac{3}{4})$, $[\frac{3}{4}, 1]$.) The inverse images of these intervals under f partition $\mathbb{N}$ into finitely many parts, of which exactly one must belong to the ultrafilter p. So let I_n be the chosen interval.

Since the sets $f^{-1}(I_n)$ belong to p, every finite set of them must have non-empty intersection, so the same must be true of the intervals I_n.

Then there is a unique point x such that x belongs to the closures of all the intervals I_n.

Define $g(p)$ to be x.

We now check that the inverse image of any open set around x, is an open neighbourhood of p.

Let U be an open neighbourhood of x, and let q be an element of $g^{-1}(U)$.

Let $\{J_n : n \in \mathbb{N}\}$ be the sequence of intervals used in defining $g(q)$. Let n be large enough that $\overline{J_n} \subseteq U$. Then $g(q) \in \overline{J_n}$, and $f^{-1}(J_n) \in q$. If $r \in \mathscr{S}\mathbb{P}$ is such that $f^{-1}(J_n) \in r$, then by definition of g, $g(r) \in \overline{J_n}$ also.

But the set $\{r \in \mathscr{S}\mathbb{P} : f^{-1}(J_n) \in r\}$ is open in $\mathscr{S}\mathbb{P}$; and its image under g is contained in U.

So $g^{-1}(U)$ is open.

Now let p be any element of $g^{-1}(I_n)$.

Then, considering p as an ultrafilter on $\mathbb{N}$, $f^{-1}(I_n) \in p$.

Now $(h, \mathscr{S}\mathbb{P})$ is a compactification of $\mathbb{N}$ with the Stone-Čech property.

So $\mathscr{S}\mathbb{P}$ must be homeomorphic to the Stone-Čech compactification of $\mathbb{N}$.

(iii) *(Much easier.) Now deduce that $\beta\mathbb{N}$ is not metrisable.*

$\mathscr{S}\mathbb{P}$ is a Stone space whose dual Boolean algebra is uncountable. Therefore it cannot be metrisable.

21. *(Quite hard.) The Sorgenfrey Line $\mathbb{S}$ is the real line with the topology generated by sets of the form $(a, b]$, for real numbers a and b. The Michael Line $\mathbb{M}$ is the real line equipped with basic open sets of the form $\{r\}$ for irrational r and $(r - \frac{1}{n}, r + \frac{1}{n})$ for r rational and n a natural number.*

(i) *Prove that the Michael Line is T_4.*

Let C and D be disjoint and closed.

Let x be any element of C. If x is irrational, then let $U_x = \{x\}$. If on the other hand x is rational, then let $U_x = (x - \frac{1}{2n_x}, x + \frac{1}{2n_x})$, where n_x is such that $(x - \frac{1}{n_x}, x + \frac{1}{n_x})$ does not meet D.

Similarly, let y be any element of C. If y is irrational, then let $V_y = \{y\}$. If on the other hand y is rational, then let $V_y = (y - \frac{1}{2n_y}, y + \frac{1}{2n_y})$, where n_y is such that $(y - \frac{1}{n_y}, y + \frac{1}{n_y})$ does not meet C.

Let $U = \bigcup_{x \in C} U_x$, and let $V = \bigcup_{y \in D} V_y$.

Then U and V are open, $C \subseteq U$, and $D \subseteq V$.

Suppose that U and V meet.

Then for some $x \in \mathbb{Q} \cap C$ and $y \in \mathbb{Q} \cap D$, $U_x \cap V_y \neq \varnothing$.

Then the distance between x and y is less than $\frac{1}{2n_x} + \frac{1}{2n_y}$, which is less than or equal to the greater of $\frac{1}{n_x}$ and $\frac{1}{n_y}$. But this contradicts the definition of either n_x or n_y.

So U and V are disjoint, as required.

(ii) *In the product $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{M}$, let C be the set of points (x, y) such that y is rational, and let D be the set of points (x, x), for x irrational. Prove that C and D are disjoint and closed, but that if U and V are any open sets such that $C \subseteq U$ and $D \subseteq V$, then $U \cap V \neq \varnothing$. Deduce that this product is not normal.*

[*Intuitively, this is because, letting $V_{(x,x)}$ be a neighbourhood of the point (x, x) in D such that $V_{(x,x)}$ is contained in V, then too many of the sets $V_{(x,x)}$ are too big and crowd too close together so as to force U and V to intersect. To make this argument formal, you will need the Baire Category Theorem, from further up the sheet.*]

The complement of C is the product of the irrationals with the irrational points of the Michael line. Since $\mathbb{R} \setminus \mathbb{Q}$ is an open subset of $\mathbb{M}$, the complement of C is open; hence C is closed.

Every subset of $\mathbb{M}$ which is open in the usual topology on $\mathbb{R}$, is open in $\mathbb{M}$. So every set which is open in the usual product topology on $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$, is open in $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{M}$.

Hence every set which is closed in the usual product topology on $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$, is closed in $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{M}$. Now D is the intersection with $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$ of the diagonal, so is closed in $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$. Hence D is closed in $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{M}$.

C and D are clearly disjoint.

Now suppose that U and V are open sets such that $C \subseteq U$ and $D \subseteq V$. We show that U and V must meet.

A basic open neighbourhood of an element (x, x) of D has the form $(x-\frac{1}{n}, x+\frac{1}{n}) \times \{x\}$.

For each irrational x, let n_x be such that $(x - \frac{1}{n_x}, x + \frac{1}{n_x}) \times \{x\} \subseteq V$.

For each n, let D_n be the set of all elements x of D such that $n_x = n$.

Then $\mathbb{R}$ can be expressed as a countable union thus:

$$\mathbb{R} = \bigcup_{q \in \mathbb{Q}} \{q\} \cup \bigcup_{n \in \mathbb{N}} D_n.$$

By the Baire Category Theorem, it cannot be the case that every one of these sets is disjoint from a dense open set.

So there must exist n such that whenever U is a dense open set, then $D_n \cap U \neq \varnothing$.

It follows that there exists a non-trivial interval I such that D_n is dense in I. For if not, for every non-empty open interval I, let U_I be some open subset of I not meeting D_n. Then let $U = \bigcup_I U_I$. Then U is a dense open set not meeting D_n.

Now let y be a rational number in the interior of I, and let x be an irrational element of I at a distance of less than $\frac{1}{n}$ from y.

Then any basic open neighbourhood of (y, x) meets some set $(z - \frac{1}{n}, z + \frac{1}{n})$ for some $z \in D_n$.

Thus $(y, x) \in \overline{V}$.

Since $(y, x) \subseteq U$, $U \cap V \neq \varnothing$.

(iii) *Prove that the Sorgenfrey Line is T_4.*

Let C and D be disjoint closed sets.

For each $x \in C$, let $a_x < x$ be some point such that the open set $(a_x, x]$ does not meet D. Let $U = \bigcup_{x \in C}(a_x, x]$. Similarly, for each $y \in D$, let $b_y < y$ be some point such that the open set $(b_y, y]$ does not meet C. Let $V = \bigcup_{y \in D}(b_y, y]$.

Then U and V are open sets including C and D respectively.

Suppose that U and V overlap.

Then so must some pair of intervals $(a_x, x]$ and $(b_y, y]$, where $x \in C$ and $y \in D$.

Either $x < y$ or $y < x$. Suppose the former. Then because $(a_x, x] \cap (b_y, y] \neq \varnothing$, $b_y < x < y$. But this contradicts the definition of b_y.

So U and V are disjoint.

(iv) *Prove that in $\mathbb{S} \times \mathbb{S}$, any subset of the antidiagonal $\{(x, -x) : x \in \mathbb{R}\}$ is closed.*

Every open subset of $\mathbb{R}$ is open when regarded as a subset of $\mathbb{S}$, so every open subset of $\mathbb{R} \times \mathbb{R}$ is open when regarded as a subset of $\mathbb{S} \times \mathbb{S}$. Hence every closed subset of $\mathbb{R} \times \mathbb{R}$ is closed when regarded as a subset of $\mathbb{S} \times \mathbb{S}$.

Hence the antidiagonal is closed in $\mathbb{S} \times \mathbb{S}$.

Now let $(x, -x)$ be any point of the antidiagonal.

Then the open set $(x-1, x] \times (-x-1, -x]$ meets the antidiagonal only at the point $(x, -x)$.

Hence all points of the antidiagonal in $\mathbb{S} \times \mathbb{S}$ are open in the antidiagonal.

Hence the antidiagonal is discrete.

Hence every subset of the antidiagonal in $\mathbb{S} \times \mathbb{S}$ is closed in the antidiagonal, and is therefore closed in $\mathbb{S} \times \mathbb{S}$, since the antidiagonal itself is closed.

(v) *In $\mathbb{S} \times \mathbb{S}$, let C be the set of points of the antidiagonal with rational coordinates, and let D be the set of points of the antidiagonal with irrational coordinates. Prove that if U and V are open sets such that $C \subseteq U$ and $D \subseteq V$, then $U \cap V \neq \varnothing$, and deduce that $\mathbb{S} \times \mathbb{S}$ is not normal.*

For each $(x, -x) \in D$, let n_x be such that the basic open neighbourhood $(x - \frac{1}{n}, x] \times (-x - \frac{1}{n}, -x]$ of (x, x) is contained in V.

Let D_n be the set of all x such that $n_x = n$.

Then by the Baire Category Theorem, there exists n and there exists a non-empty open interval I such that D_n is dense in I.

It is now possible to check that if x is a rational element of I, then the element $(x, -x)$ of C is in the closure of V. Thus U and V must meet.

22. *(For those who know some set theory.) Assume ZFC.*

(i) *Prove that every uncountable closed subset of $\mathbb{R}$ has cardinality $2^{\aleph_0}$.*

Earlier on this sheet, it was proved that every uncountable closed subset of $\mathbb{R}$ contains a copy of the Cantor Set.

Thus every uncountable closed subset of $\mathbb{R}$ has size at least $2^{\aleph_0}$.

But any such set is a subset of $\mathbb{R}$, so has size at most $2^{\aleph_0}$.

Now, by the Schröder-Bernstein Theorem, every uncountable closed subset of $\mathbb{R}$ has size exactly $2^{\aleph_0}$.

(ii) *Prove that there are $2^{\aleph_0}$ closed subsets of $\mathbb{R}$. [Hint: how many open subsets are there?]*

Let $\mathscr{B}$ be a countable basis for $\mathbb{R}$.

Let U be an open subset of $\mathbb{R}$. Then U is a union of elements of $\mathscr{B}$. So let $\mathscr{U}_U$ be the set of all elements of $\mathscr{B}$ which are subsets of U. Then $U = \bigcap \mathscr{U}_U$.

Then the map $U \mapsto \mathscr{U}_U$ is a one-to-one map from the set of open subsets of $\mathbb{R}$ to the powerset of $\mathscr{B}$. The powerset of a countably infinite set has size $2^{\aleph_0}$, so there are at most $2^{\aleph_0}$ open sets in $\mathbb{R}$.

There are certainly at least $2^{\aleph_0}$ open sets in $\mathbb{R}$—consider the intervals (a, ∞) for $a \in \mathbb{R}$—so by the Schröder-Bernstein Theorem, there are exactly $2^{\aleph_0}$ open subsets of $\mathbb{R}$.

Hence there are the same number of closed subsets.

(iii) *(Hard, and requiring some form of Choice.) A Bernstein Set is a subset B of $\mathbb{R}$ such that both B and its complement meet every uncountable closed set. Prove that a Bernstein set exists.*

List all the uncountable closed subsets of $\mathbb{R}$ in order-type $\mathfrak{c} = 2^{\aleph_0}$ as $(C_\alpha : \alpha < \mathfrak{c})$.

Using transfinite recursion, we define points x_α, y_α of $\mathbb{R}$, for $\alpha < \mathfrak{c}$, such that

*) all the points x_α and y_α, for all the different values of α, are different,

*) $x_\alpha, y_\alpha \in C_\alpha$.

This is possible, because for any $\alpha < \mathfrak{c}$, there are fewer than $\mathfrak{c}$ values of $\beta < \alpha$, so the set $\{x_\beta : \beta < \alpha\} \cup \{y_\beta : \beta < \alpha\}$ has size less than $\mathfrak{c}$.

But C_α, as an uncountable closed subset of $\mathbb{R}$, has size exactly $\mathfrak{c}$.

So $C_\alpha \setminus \{x_\beta : \beta < \alpha\} \cup \{y_\beta : \beta < \alpha\}$ has size at least two.

So x_α and y_α can be defined.

When the recursion is complete, let $B = \{x_\alpha : \alpha < \mathfrak{c}\}$.

Then B is a Bernstein set.