C1.1: Model Theory

January 9, 2019

1 Introduction and review

Model theory hinges on a duality between *sentences* or and *structures* for a fixed language; more generally, a duality between *formulas* (with n free variables), and *definable sets* (subsets of A^n for A an L-structure.)
For a sentence ψ, we can look at the class of models of ψ; conversely given a structure N, we can look at the *complete theory* $Th(N) := \{\psi : N \models \psi\}$.
This generalizes the duality between *algebra* and *geometry* seen in many areas of mathematics; for instance a polynomial f, or rather the equality $f(x_1, \ldots, x_n) = 0$, can be viewed as a formula; the solution set in $\mathbb{R}^n$ is a geometric object corresponding to it. The special feature of model theory is going beyond atomic formulas, in particular allowing quantifiers.
In previous mathematics classes, you have run into many *universal theories:* the theories of groups, rings, integral domains, vector spaces over some field. On the other hand you have seen specific *structures*, such as the field of real numbers.
How to bridge the gap between them?
Prelude: the compactness theorem. All of model theory is dependent on the compactness theorem. During the course, you will see various applications. We will begin with some proofs, complementing the one you have seen using completeness. A proof can be given along the lines of the completeness theorem, merely replacing the proof-theoretic notions of *consistency* with the model-theoretic one of *finite satisfiability*. When the language is countable, it can be recast in terms of Baire category. There are also algebraic proofs (using ultrapowers), where all choices are performed in advance via the choice of an *ultrafilter*.

Between a complete theory and a structure.
We will cover, roughly, section 2.3 of Chang and Keisler, on countable models of complete theories.
In particular we can ask: when is a countable structure determined entirely (up to isomorphism) by its complete theory? The Ryll-Nardweski theorem gives a simple and very satisfactory answer. The notion of *types*, and methods of realizing and omitting *types*, required for the proof, will shed light on more general theories too.
A theory with a unique model (up to isomorphism) of cardinality κ is called κ-categorical. It is called *totally categorical* if it is κ-categorical for all infinite κ. The subject becomes much deeper here, and a much finer description is achieved. We will not be able to cover much of it, but will try to give a taste of a few of the ideas.
Between a universal theory and a complete theory.
But how do you ever find the complete theory of an infinite structure? And given a universal theory, is there any reasonable way to describe a complete theory containing it? The two apparently different questions are often answered by the same general construction, the *model completion* of a universal theory. We will study it both theoretically and via some examples. The model completion of the theory of integral domains is the theory of algebraically closed fields; for ordered domains is the theory of real closed fields. Knowing this shows that these theories are complete, and thus answers the question of the complete theories of the fields $\mathbb{R}$ and $\mathbb{C}$. Beyond this, we will see that it also helps understand their definable sets (formulas in n variables, rather than sentences with 0 variables); leading to connections with geometry that are quite active in current research.

Contents

1.1 Review of basic logic: Languages and structures

Language: alphabet, variables, terms, formulas.
A *language* L is specified by its *alphabet*, which consists, by definition, of the following data: :

(i) relation symbols P_i, $(i \in I)$, function symbols f_j, $(j \in J)$, and constant symbols c_k, $(k \in K)$ with some index sets I, J, K. Further, to each $i \in I$ and $j \in J$ is assigned a positive integer ρ_i, μ_j, respectively, called the **arity** of the relation symbol P_i or the function symbol f_j.
The symbols in (i) are called **non-logical** symbols. It is sometimes convenient to view constant symbols as 0-place function symbols.
The formulas of L will be formed using the non-logical symbols, and the following. (iv,v) are called the *logical* symbols.

(ii) $\doteq$ - the equality symbol;[1]
(iii) $v_1, \ldots, v_n, \ldots$ - the variables;
(iv) $\wedge$, $\neg$ - Boolean connectives;
(v) $\exists$ - the existential quantifier;
(vi) (,) - parentheses.[2]

Words of the alphabet of L constructed in a specific way are called L-terms and L-formulas:

***L*-terms** are given by recursive definition as follows:
(i) v_i is an L-term (any $i \geq 1$);
(ii) c is an L-term (any constant symbol c of L);
(iii) if f is a function symbol of L of arity μ, and $\tau_1, \ldots \tau_\mu$ are L-terms, then $f(\tau_1, \ldots, \tau_\mu)$ is an L-term;
(iv) nothing else is an L-term.

We define the **complexity** of a term τ to be just the length of τ as a word in the alphabet of L. It is obvious from the definition that any term of complexity $l > 1$ is obtained by an application of (iii) to terms of lower complexity.

We sometimes refer to a term τ as $\tau(v_{i_1}, \ldots, v_{i_n})$ to mark the fact that the variables occurring in τ are among $v_{i_1}, \ldots, v_{i_n}$. It may happen that no variables occur in τ, such terms are called **closed.**

Atomic *L*-formulas are the words of the form
(i) $\tau_1 \doteq \tau_2$ for any L-terms τ_1 and τ_2
or
(ii) $P(\tau_1, \ldots, \tau_\rho)$ for any relational L-symbol P of arity ρ and L-terms $\tau_1, \ldots, \tau_\rho$.

[1] One sometimes considers logic with several *sorts*, say of apples and thoughts, where one does not even wish to ask whether an apple equals a thought. In this case one introduces an equality symbol for each sort, but not between distinct sorts. This poses no problem when (e.g.) there are finitely many sorts. It does becomes surprsingly complicated when there are parameterized families of sorts, but we will not go into these issues, and always allow an equality symbol.

[2] These are used only to ensure *unique readability* of formulas, and can be dispensed with in many systems. We will not fuss about them, but merely use them as necessary to clarify the construction of a given formula.

Notice, that (i) can be seen as a special case of (ii) if we view $\doteq$ as a relational symbol of arity 2.

An L**-formula** is defined by the following recursive definition:
(i) any atomic L-formula is an L-formula;
(ii) if φ is an L-formula, so is $\neg\varphi$;
(iii) if φ, ψ are L-formulas, so is $(\varphi \wedge \psi)$;
(iv) if φ is an L-formula, so is $\exists v \varphi$ for any variable v;
Nothing else is an L-formula.

Some abbreviations
Let ϕ and ψ be L-formulas.
$(\phi \vee \psi)$ is an abbreviation for the formula $\neg(\neg\phi \wedge \neg\psi)$;
$(\phi \rightarrow \psi)$ is an abbreviation for the formula $\neg(\phi \wedge \neg\psi)$;
$(\phi \leftrightarrow \psi)$ is an abbreviation for the formula $((\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi))$;
$\forall v\psi$ is an abbreviation for the formula $\neg\exists v\neg\psi$.

It is typical of logic that formulas in n-variables are discussed, and n-tuples of elements of a structure occur much more frequently than just elements. Notationally, this sometimes looks unnecessarily complicated. We will thus use 'vector notation', writing a for $(a_1, \ldots, a_n)$ and x for $(x_1, \ldots, x_n)$ when possible. For instance,
$\underline{A} \vDash (\varphi_1(\alpha_1, \ldots, \alpha_n) \wedge \varphi_2(\alpha_1, \ldots, \alpha_n))$ iff $\underline{A} \vDash \varphi_1(\alpha_1, \ldots, \alpha_n)$ and $\underline{A} \vDash \varphi_2(\alpha_1, \ldots, \tilde{\alpha}_n)$ will be written thus:
$\underline{A} \vDash (\varphi_1(\alpha) \wedge \varphi_2(\alpha))$ iff $\underline{A} \vDash \varphi_1(\alpha)$ and $\underline{A} \vDash \varphi_2(\alpha)$, $\quad \alpha = (\alpha_1, \ldots, \alpha_n)$.

In practice, we will permit ourselves the use of additional standard defined symbols such as $\vee, \rightarrow, \leftrightarrow, \forall$. We will also write certain function symbols as $x+y$, x^{-1} according to standard usage.

Types of formulas Formulas that can be formed using (i)-(iii) alone are called *quantifier-free.*
A formula is *universal* if it has the form $(\forall x_1)\cdots(\forall x_n)\psi$, where ψ is quantifier-free. Similarly one of the form $(\exists x_1)\cdots(\exists x_n)\psi$ is called *existential.*

Notation. *We will write $\varphi(x_1,\ldots,x_n)$ for the pair $(\varphi,(x_1,\ldots,x_n))$ when φ is a formula and $(x_1,\ldots,x_n)$ is a tuple of variables,* including all the free variables of φ.

You will often see the expression 'let $\phi(x_1,\ldots,x_n)$ be a formula'; it is used to indicate that the free variables are among those indicated. But strictly speaking, we have specified a tuple of variables and not only a formula. See e.g. Exercise 1.6 (4), for a place where this matters.
We define the **complexity of an L-formula** φ to be just the number of occurrences of $\wedge$, $\neg$ and $\exists$ in φ. Thus an atomic formula is of complexity 0 and that any formula of complexity $l>0$ is obtained by an application of (ii),(iii) or (iv) to formulas of lower complexity.

Free variables For an atomic formula $\varphi(v_{i_1},\ldots,v_{i_n})$, all variables occurring in (the terms of) φ are said to be free. For more complex formulas, the set of free variables is defined recursively. The variables which are free in φ and ψ in (ii) and (iii) are, by definition, also free in $\neg\varphi$ and $(\varphi\wedge\psi)$. The variable v in (iv) is called **bounded** in $\exists v\varphi$ and the list of free variables for this formula is given by the free variables of φ except v.
An L-formula with no free variables is called an **L-sentence**.

We write $|L|$ for the cardinality of the set of L-formulas.

Exercise 1.2. Show that

$$|L| = \max\{\aleph_0, \text{card }(I), \text{card }(J), \text{card }(K)\}.$$

1.3 Structures.

An L-structure $\underline{A}$ consists of

(i) a set A called the *domain* or *universe* of the L-structure; [3]
(ii) an assignment of an r-ary relation (subset) $P^{\underline{A}} \subseteq A^r$ to each relation symbol P of L of arity r;
(iii) an assignment of an m-ary function $f^{\underline{A}} : A^m \to A$ to any function symbol f of L of arity m;
(iv) an assignment of an element $c^{\underline{A}} \in A$ to any constant symbol c of L.

Thus an L-structure is an object of the form

$$\underline{A} = \left\langle A; \{P_i^{\underline{A}}\}_{i\in I}; \{f_j^{\underline{A}}\}_{j\in J}; \{c_k^{\underline{A}}\}_{k\in K} \right\rangle .$$

$\{P_i^{\underline{A}}\}_{i\in I}$, $\{f_j^{\underline{A}}\}_{j\in J}$ and $\{c_k^{\underline{A}}\}_{k\in K}$ are called the *interpretations in* $\underline{A}$ of the predicate, function and constant symbols correspondingly.
We write $A = \mathrm{dom}\ (\underline{A})$.

Note that writing $\left\langle A; \{P_i^{\underline{A}}\}_{i\in I}; \{f_j^{\underline{A}}\}_{j\in J}; \{c_k^{\underline{A}}\}_{k\in K} \right\rangle$ implicitly specifies the language L.
For instance, $(\mathbb{R}, 0, -, +)$ is a structure for the *language of groups*, a language with a constant symbol, a unary function symbol and a binary function symbol. Similarly, $(\mathbb{R}, 0, 1, -, +, \cdot)$ is a structure for the *language of rings*; they have the same domain, but are structures for different languages.

1.4 Interpretation of formulas in a structure

Let $\underline{A}$ be an L-structure with domain A.
We begin with the interpretation of terms.
We assign to each L-term $\tau(v_1, \ldots, v_n)$ a function

$$\tau^{\underline{A}} : A^n \to A$$

by the following rule:
(i) if $\tau(v_1, \ldots, v_n)$ is just a variable v_j then $\tau^{\underline{A}}$ is the corresponding coordinate function $\langle a_1, \ldots a_n\rangle \mapsto a_j$;

[3] It is sometimes assumed that $A \neq \emptyset$, notably since this slightly simplifies the proof systems. As we are not concerned with syntactic proofs in this course, we do not need that assumption; of course, the empty structure itself is not of much interest, but many general statements are nicer when it is allowed. Sometimes people assume L has at least one constant symbol in order to avoid the need to pay attention to this degenerate case.

(ii) if $\tau(v_1, \ldots, v_n)$ is a constant symbol c then $\tau^{\underline{A}}(a_1, \ldots, a_n) = c^{\underline{A}}$;
(iii) if $\tau(v_1, \ldots, v_n)$ is $f(\tau_1(v_1, \ldots, v_n), \ldots, \tau_m(v_1, \ldots, v_n))$ then
$\tau^{\underline{A}}(a_1, \ldots, a_n) = f^{\underline{A}}(\tau_1^{\underline{A}}(a_1, \ldots, a_n), \ldots, \tau_m^{\underline{A}}(a_1, \ldots, a_n))$.

The interpretation of formulas. Let $\varphi(v_1, \ldots, v_n)$ an L-formula with free variables $v_1, \ldots, v_n$ and $\bar{a} = \langle a_1, \ldots, a_n \rangle \in A^n$. Given these data we assign a truth value **true**, written $\underline{A} \models \varphi(\bar{a})$, or **false**, $\underline{A} \not\models \varphi(\bar{a})$, by the following rules:
(i) $\underline{A} \models \tau_1(\bar{a}) \doteq \tau_2(\bar{a})$ iff $\tau_1^{\underline{A}}(\bar{a}) = \tau_2^{\underline{A}}(\bar{a})$;
(ii) $\underline{A} \models P(\tau_1(\bar{a}), \ldots, \tau_r(\bar{a}))$ iff $\langle \tau_1^{\underline{A}}(\bar{a}), \ldots, \tau_r^{\underline{A}}(\bar{a}) \rangle \in P_i^{\underline{A}}$;
(iii) $\underline{A} \models \varphi_1(\bar{a}) \wedge \varphi_2(\bar{a})$ iff $\underline{A} \models \varphi_1(\bar{a})$ and $\underline{A} \models \varphi_2(\bar{a})$;
(iv) $\underline{A} \models \neg\varphi(\bar{a})$ iff $\underline{A} \not\models \varphi(\bar{a})$;
(v) $\underline{A} \models \exists v_n \varphi(a_1, \ldots, a_{n-1}, v_n)$ iff there is an $a_n \in A$ such that $\underline{A} \models \varphi(a_1, \ldots, a_n)$.

In case φ is a sentence, no assignment is needed. We have thus defined the truth value of φ in $\underline{A}$. If this value is **true**, we say that φ holds in $\underline{A}$, or that $\underline{A}$ is a model of φ.

Exercise 1.5. Describe a language L_{grp} appropriate to discuss groups (with multiplication, inversion and a unit element), and write a sentence whose models are (precisely) groups.

Consider an L-structure $\underline{A}$ and an L-formula $\varphi(v_1, \ldots, v_n)$. Write

$$\varphi^{\underline{A}} = \{\bar{a} \in A^n : \underline{A} \models \varphi(\bar{a})\}.$$

The notation $\varphi(A)$ is also used. This is called a *definable set*, namely the set defined by ϕ. It is a subset of A^n, not of A! If we want to emphasize this, we refer to it as a *definable relation.*

Exercise 1.6.

1. Write a formula ϕ' such that $\phi'(\underline{A})$ is the *complement* of $\phi(A)$.

2. Let $p : A^n \to A^{n-1}$ be the projection map, $p(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1})$. Then $p(Y)$ is called the *projection* of Y. Explain why this is also called the *shadow* of Y (Take $n = 3$, $A = \mathbb{R}$, and a light source from above.) Write a formula ϕ'' such that $\phi''(A)$ is the projection of $\phi(A)$ under p.

3. If also given $\psi = \psi(v_1, \ldots, v_n)$ write a formula θ such that $\theta(A)$ is the *intersection* of $\phi(A), \psi(A)$.

4. Write a formula whose interpretation is $\phi(A) \times A$.

1.7 Embeddings and isomorphisms

[4]

Fix a language L.
Let $\underline{A}, \underline{B}$ be L-structures, with universes A, B respectively.
An *embedding* (or L-embedding) of $\underline{A}$ in $\underline{B}$ is a one-to-one function $\pi : A \to B$ which preserves corresponding relation, function and constant symbols, i.e. for any relation symbol P, function symbol F, constant symbol c of L we have:
(i) $\bar{a} \in P^{\underline{A}}$ iff $\pi(\bar{a}) \in P^{\underline{B}}$;
(ii) $\pi(F^{\underline{A}}(\bar{a})) = F^{\underline{B}}(\pi(\bar{a}))$;
(iii) $\pi(c^{\underline{A}}) = c^{\underline{B}}$.
We write in this case $\pi : \underline{A} \to \underline{B}$.
An important case occurs when $A \subseteq B$, and π is the inclusion map, i.e. $\pi(a) = a$ for $a \in A$. In this case we write $\underline{A} \leq \underline{B}$, and say $\underline{A}$ is a *substructure* of $\underline{B}$. The definition of an embedding can be rewritten as follows:
(i) $P^{\underline{A}} = P^{\underline{B}} \bigcap A^k$ where P is a k-place relation symbol.
(ii) $F^{\underline{A}} = F^{\underline{B}}|A^k$ where F is a k-place function symbol.
(iii) $c^{\underline{A}} = c^{\underline{B}}$ where c is a constant symbol.
Given $\underline{B}$, note that to specify $\underline{A}$ it suffices to give the universe A; the interpretation of the relation and function symbols is then completely determined by being a substructure. Moreover, a subset of B is the universe of a substructure of $\underline{B}$ if and only if it is closed under the basic functions, including the 0-place ones; more precisely:

Exercise 1.8. *A is the universe of a substructure of $\underline{B}$ if and only if $c^{\underline{B}} \in A$ for each constant symbol c, and $F^{\underline{B}}(A^k) \subset A$ for each k-place function symbol of L, $k \geq 1$.*

An *isomorphism* $\underline{A} \to \underline{B}$ is an embedding $\pi : \underline{A} \to \underline{B}$ such that $\pi : A \to B$ is bijective. In this case the inverse map $\pi^{-1} : B \to A$ is also an isomorphism from $\underline{B}$ to $\underline{A}$.
An isomorphism $\pi : \underline{A} \to \underline{A}$ of the structure onto itself is called an **automorphism** of $\underline{A}$.

[4] In case you have not have seen the notion defined in this generality, it will be an easy generalization from the various cases you have seen in algebra, such as isomorphism of groups, rings, or ordered fields.

Exercise 1.9. Let $\pi : \underline{A} \to \underline{B}$ be an embedding.

1. Show that π preserves L-terms, that is for any term $\tau(\bar{v})$

$$\pi(\tau^{\underline{A}}(\bar{a})) = \tau^{\underline{B}}(\pi(\bar{a})).$$

2. Show that π preserves atomic L-formulas, i.e. for any atomic $\varphi(v_1, \ldots, v_n)$ for any $\bar{a} \in A^n$

$$(*) \qquad \underline{A} \models \varphi(\bar{a}) \text{ iff } \underline{B} \models \varphi(\pi(\bar{a})).$$

3. If π is an isomorphism, show that (*) holds for *any* formula φ.

(You will need induction for (1) and (3).)

Definition 1.10. *An embedding of L structures $\pi : \underline{A} \to \underline{B}$ is called* **elementary** *if π preserves any L-formula $\varphi(v_1, \ldots, v_n)$, i.e. for any $a_1, \ldots, a_n \in$* dom $\underline{A}$

$$\underline{A} \models \varphi(a_1, \ldots, a_n) \textit{ iff } \underline{B} \models \varphi(\pi(a_1), \ldots, \pi(a_n)).$$

When $A \subseteq B$ and the inclusion map is elementary, we write:

$$\underline{A} \preccurlyeq \underline{B}.$$

Thus $\underline{A} \preccurlyeq \underline{B}$ iff

$$\underline{A} \models \varphi(a_1, \ldots, a_n) \text{ iff } \underline{B} \models \varphi(a_1, \ldots, a_n).$$

Example Let $\mathcal{Z} = \langle \mathbb{Z}; +, 0 \rangle$ be the additive group of integers. Then, given an integer $m > 1$, the embedding

$$[m] : \mathcal{Z} \to \mathcal{Z},$$

defined as $[m](z) = m \cdot z$, is not elementary.

1.11 The compactness theorem

Note that all mention of syntactical proofs was omitted from this review, though they probably formed a substantial part of your logic class. It will be good to keep in mind that logical implication has a syntactic counterpart, but this will only play a silent role in the background; specific proof systems play no part in model theory. In particular, we will not require the *completeness theorem* as such.

However, one corollary of the completeness theorem will be very important; this is the *compactness theorem.* A set S of sentences is called *satisfiable* if it has a model, i.e. a structure $\underline{A}$ such that the truth value of each sentence $\sigma \in S$ is **true**. A set S is *finitely satisfiable* if every finite subset of S is satisfiable. The completeness theorem for first order logic asserts that S is finitely satisfiable if and only if it is consistent (with respect to a certain set of logical axioms and deduction rules.)
The compactness theorem asserts that a set of sentences is satisfiable iff it is finitely satisfiable. It is a good idea to review the (short) proof of the compactness theorem from the completeness theorem.
However, we will recall the proof in a form that yields compactness directly, abstracting away any mention of syntactical proof. The interplay now is between *finite satisfiability* and existence of a model.
Fix a language L.
Let Σ be a set of L-sentences. We write $\underline{A} \models \Sigma$ (A models Σ, or A is a model of Σ) if, for any $\sigma \in \Sigma$, $\underline{A} \models \sigma$.
An L-sentence σ is said to be *a logical consequence of* Σ if every L-structure $\underline{A}$ satisfying $\underline{A} \models \Sigma$ also has: $\underline{A} \models \sigma$.
Notationally, it will be convenient to write $\Sigma \models \sigma$ *when σ is a logical consequence of some finite subset of* Σ. We will see towards the end of this chapter that in fact $\Sigma \models \sigma$ iff σ is a logical consequence of Σ.
A sentence σ is called **logically valid**, written $\models \sigma$, if $\emptyset \models \sigma$, i.e. $\underline{A} \models \sigma$ for every L-structure $\underline{A}$.

A set Σ of L-sentences is said to be **satisfiable** if it has a model, i.e. there is an L-structure $\underline{A}$ such that $\underline{A} \models \Sigma$. Σ is said to be **finitely satisfiable (finitely satisfiable)** if any finite subset of Σ is satisfiable.

Σ is said to be **complete** if, for any L-sentence σ, $\Sigma \vdash \sigma$ or $\Sigma \vdash \neg\sigma$. When

Σ is a theory, this is equivalent to: $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$.

Exercise 1.12. Let $\alpha, \alpha_1, \ldots, \alpha_n, \beta, \beta_1, \ldots, \beta_n, \gamma$ be closed L-terms, P, f L-symbols for n-ary predicate and n-ary function, correspondingly, and $\psi(v_0, v_1, \ldots, v_n)$ an L-formula with free variables $v_0, v_1, \ldots, v_n$. Prove that

1. $\alpha \doteq \beta \models \beta \doteq \alpha$;
2. $\alpha \doteq \beta, \beta \doteq \gamma \models \alpha \doteq \gamma$;
3. $\models \alpha \doteq \alpha$;
4. $\alpha_1 \doteq \beta_1, \ldots, \alpha_n \doteq \beta_n, P(\alpha_1, \ldots, \alpha_n) \models P(\beta_1, \ldots, \beta_n)$;
5. $\alpha \doteq \beta, \alpha_1 \doteq \beta_1, \ldots, \alpha_n \doteq \beta_n, f(\alpha_1, \ldots, \alpha_n) \doteq \alpha \models \ f(\beta_1, \ldots, \beta_n) \doteq \beta$;
6. $\psi(\beta, \alpha_1, \ldots, \alpha_n) \models \ \exists v_0 \psi(v_0, \alpha_1, \ldots, \alpha_n)$.

Definition 1.13. *(1)-(5) are the axioms of equality. A binary relation satisfying these laws is called a* congruence*.*

A *weak L-structure* $\underline{A}$ is a set A, an assignment of a subset $R^{\underline{A}} \subset A^n$ for each binary relation R of L, and of a function $F^{\underline{A}} : A^n \to A$ for each n-place function symbol, such that the interpretation of $\doteq$ is a *congruence*.
Given a weak L-structure $\underline{A}$, we can form a *quotient structure* $\underline{B} = \underline{A}/\doteq$ as follows. Let $\sim = \doteq^{\underline{A}}$. The universe is $B = A/\doteq^{\underline{A}}$. Let $p : A \to B$ be the quotient map, i.e. $p(a) = [a]_\sim$ is the $\sim$-equivalence class of A. Also write $p(a_1, \ldots, a_n) := (p(a_1), \ldots, p(a_n))$. For a subset Y of A^n, let $p(Y) = \{p(a) : a \in Y\}$. We pose:

$$R^{\underline{B}} = p(R^{\underline{A}})$$

$$F^{\underline{B}}(p(a)) = p(F^{\underline{A}}(a))$$

Exercise 1.14. Let $\underline{A}$ be a weak L- structure.

1. Check that the quotient structure $\underline{B}$ is well-defined (the issue is with the definition of the interpretation of function symbols.).
2. Check that for any term t, we have

$$t^{\underline{B}}(p(a)) = p(t^{\underline{A}}(a))$$

3. For any formula $\phi = \phi(x_1, \ldots, x_n)$ and any $a = (a_1, \ldots, a_n)$,

$$\underline{B} \models \phi(pa) \leftrightarrow \underline{A} \models \phi(a)$$

(Prove this for atomic formulas, and then by induction on the complexity of ϕ.)

A set of L-sentences Σ is said to be **deductively closed** if

$$\Sigma \models \sigma \text{ implies } \sigma \in \Sigma.$$

Proposition 1.15. *For any finitely satisfiable set of L-sentences Σ there is a complete finitely satisfiable set of L-sentences $\Sigma^{\#}$ such that $\Sigma \subseteq \Sigma^{\#}$.*

Proof Let

$$\mathcal{S} = \{\Sigma' : \Sigma \subseteq \Sigma' \text{ a finitely satisfiable set of } L\text{-sentences } \}.$$

Clearly $\mathcal{S}$ satisfies the hypothesis of Zorn's Lemma, so it contains a maximal element $\Sigma^{\#}$ say. This is complete for otherwise, say $\sigma \notin \Sigma^{\#}$ and $\neg\sigma \notin \Sigma^{\#}$. By maximality neither $\{\sigma\} \cup \Sigma^{\#}$ nor $\{\neg\sigma\} \cup \Sigma^{\#}$ is finitely satisfiable. Hence there exist finite $S_1 \subseteq \Sigma^{\#}$ and $S_2 \subseteq \Sigma^{\#}$ such that neither $\{\sigma\} \cup S_1$ nor $\{\neg\sigma\} \cup S_2$ is satisfiable. However, $S_1 \cup S_2 \subseteq \Sigma^{\#}$, finite, so has a model, $\underline{A}$ say. But either $\underline{A} \models \sigma$, so $\underline{A} \models \{\sigma\} \cup S_1$, or $\underline{A} \models \neg\sigma$, so $\underline{A} \models \{\neg\sigma\} \cup S_2$, a contradiction. □

Alternative proof. Here is proof of the same result by a different construction, assuming the language is countable. Let $\{\psi_n\}$ enumerate all sentences of L. We define σ_n recursively. We assume σ_m has been defined for $m < n$, and let $\Sigma_{<n} = \Sigma \bigcup \{\sigma_m : m < n\}$. Define:
$\sigma_n = \psi_n$ if $\Sigma_{<n} \bigcup \{\psi_n\}$ is finitely satisfiable;
$\sigma_n = \neg\psi_n$ if not.
In either case, show as above that $\Sigma_{<n} \bigcup \{\sigma_n\}$ is finitely satisfiable. Hence by induction, Σ_n is finitely satisfiable for each n. One verifies as above that $\bigcup_n \Sigma_n = \Sigma \bigcup \{\sigma_n : n = 0, 1, 2, \ldots\}$ is complete and finitely satisfiable.
Remark Indeed the Henkin proof of compactness (or completeness) does not require the axiom of choice, provided the symbols of the language itself are well-ordered; in this case the sentences can be enumerated as $\{\psi_n : n < \kappa\}$ for some ordinal κ; the above 'alternative proof' continues to work.
Exercise Fill in the details of the above remark (take care of the case that n is a limit ordinal.

Exercise 1.16. Let us allow 0-place relation symbols $(R_i : i \in I)$; they are also called *propositional symbols*. Assume there are no other symbols; so that a structure consists just of assigning a truth value to each R_i. Thus a structure is just an element of the I-fold product $\{0,1\}^I$.

1. In this case, show that a complete finitely satisfiable set of sentences Σ determines a model $M(\Sigma)$ of Σ, simply by assigning 1 to R_i if $R_i \in \Sigma$ and 0 otherwise.

2. Define a topology on $\{0,1\}^I$ by letting a basic open set have the form

$$B(i_0, \ldots, i_k; \nu_0, \ldots, \nu_k) = \{f : f(i_0) = \nu_0, \ldots, f(i_k) = \nu_k\}$$

where $i_0, \ldots, i_k \in I$ and $\nu_0, \ldots, \nu_k \in \{0,1\}$. Tychonoff's theorem asserts that this topology is compact. Prove this using this case of the compactness theorem.

A set Σ of L-sentences is said to be **witnessing** if for any sentence in Σ of the form $\exists v \varphi(v)$ there is a closed L-term λ such that $\varphi(\lambda) \in \Sigma$.

Exercise 1.17. Any complete finitely satisfiable witnessing set of L-sentences has a closed L-term.
Hint: Consider the L-sentence $\exists v\ v \doteq v$.

Definition 1.18. *An L-structure $\underline{A}$ is called* **minimal** *if it has no proper substructure.*

We will sometimes say *L-minimal* for clarity. Notably, *an L-minimal model of T* is just a minimal L-structure, which is a model of T. (This is not the same as 'a minimal model of T', in the sense of Definition 7. !)

Proposition 1.19. *For any complete, witnessing, finitely satisfiable set Σ of L-sentences there exists a L-minimal model $\underline{A}$ of Σ.*

Proof Let Λ be the set of closed terms of L. This is nonempty by 1.17. For $\alpha, \beta \in \Lambda$ define $\alpha \backsim \beta$ iff $\alpha \doteq \beta \in \Sigma$.
This is an equivalence relation by 1.12.1- 1.12.3.
For $\alpha \in \Lambda$, let $\tilde{\alpha}$ denote the $\backsim$-equivalence class containing α. Let

$$A = \{\tilde{\alpha} :\ \alpha \in \Lambda\}.$$

This will be the domain of our model $\underline{A}$. We want to define relations, functions and constants of L on A.
Let P be an n-ary relation symbol of L and $\alpha_1, \ldots, \alpha_n \in \Lambda$. Define

$$\langle \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n \rangle \in P^{\underline{A}} \text{ iff } P(\alpha_1, \ldots, \alpha_n) \in \Sigma.$$

By 1.12.4 the definition does not depend on the choice of representatives in the $\backsim$-classes.
For a unary function symbol f of L of arity m and $\alpha_1, \ldots, \alpha_m \in \Lambda$ define

$$f^{\underline{A}}(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m) = \tilde{\tau}, \text{ where } \tau = f(\alpha_1, \ldots, \alpha_m).$$

By 1.12.5 this is well-defined.
Finally, for a constant symbol, $c^{\underline{A}}$ is just $\tilde{c}$.
We now prove by induction on complexity of an L-formula $\varphi(v_1, \ldots, v_n)$ that

$$(*) \quad \underline{A} \vDash \varphi(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \text{ iff } \varphi(\alpha_1, \ldots, \alpha_n) \in \Sigma.$$

For atomic formulas we have this by definition.
If $\varphi = (\varphi_1 \wedge \varphi_2)$ then
$\underline{A} \vDash (\varphi_1(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n) \wedge \varphi_2(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n))$ iff $\underline{A} \vDash \varphi_1(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ and $\underline{A} \vDash \varphi_2(\tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ iff (by induction hypothesis) $\varphi_1(\alpha_1, \ldots, \alpha_n), \varphi_2(\alpha_1, \ldots, \alpha_n) \in \Sigma$ iff $(\varphi_1(\alpha_1, \ldots, \alpha_n) \wedge \varphi_2(\alpha_1, \ldots, \alpha_n)) \in \Sigma$. Which proves (*) in this case.
The case $\varphi = \neg\psi$ is proved similarly.
In case $\varphi = \exists v \psi$
$\underline{A} \vDash \exists v \psi(v, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ iff there is $\beta \in \Lambda$ such that $\underline{A} \vDash \psi(\tilde{\beta}, \tilde{\alpha}_1, \ldots, \tilde{\alpha}_n)$ iff there is $\beta \in \Lambda$ such that $\psi(\beta, \alpha_1, \ldots, \alpha_n) \in \Sigma$. The latter implies, by 1.12.6, that $\exists v \psi(v, \alpha_1, \ldots, \alpha_n) \in \Sigma$, and the converse holds because Σ is witnessing. This proves (*) for the formula and finishes the proof of (*) for all formulas.
Finally notice that (*) implies that $\underline{A} \vDash \Sigma$. □
We sometimes need to expand or reduce our language.
Let L be a language with non-logical symbols $\{P_i\}_{i \in I} \cup \{f_j\}_{j \in J} \cup \{c_k\}_{k \in K}$ and $L' \subseteq L$ with non-logical symbols $\{P_i\}_{i \in I'} \cup \{f_j\}_{j \in J'} \cup \{c_k\}_{k \in K'}$ ($I' \subseteq I$, $J' \subseteq J$, $K' \subseteq K$). Let

$$\underline{A} = \langle A; \{P_i^{\underline{A}}\}_{i \in I}; \{f_j^{\underline{A}}\}_{j \in J}; \{c_k^{\underline{A}}\}_{k \in K} \rangle$$

and

$$\underline{A}' = \langle A; \{P_i^{\underline{A}}\}_{i \in I'}; \{f_j^{\underline{A}}\}_{j \in J'}; \{c_k^{\underline{A}}\}_{k \in K'} \rangle.$$

Under these conditions we call $\underline{A}'$ **the L'-reduct of** $\underline{A}$ and, correspondingly, $\underline{A}$ is **an L-expansion of** $\underline{A}'$.

Remark Obviously, under the notations above for an L'-formula $\varphi(v_1,\ldots,v_n)$ and $a_1,\ldots,a_n \in A$

$$\underline{A}' \vDash \varphi(a_1,\ldots,a_n) \text{ iff } \underline{A} \vDash \varphi(a_1,\ldots,a_n).$$

Exercise 1.20. Let, for each $i \in \mathbb{N}$, Σ_i denote a set of L sentences. Suppose

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \Sigma_i \ldots$$

and each Σ_i is finitely satisfiable.
Then the union of the chain, $\bigcup_{i\in\mathbb{N}} \Sigma_i$, is finitely satisfiable.

Theorem 1.21 (Compactness Theorem)**.** *Any finitely satisfiable set of L-sentences Σ is satisfiable. Moreover, Σ has a model of cardinality less or equal to $|L|$.*

Proof We introduce new languages L_i and complete set of L_i-sentences Σ_i $(i = 0, 1, \ldots)$. Let $L_0 = L$. By Propsition 1.15 there exists $\Sigma_0 \supseteq \Sigma$, a complete set of L_0-sentences.
Given finitely satisfiable Σ_i in language L_i, introduce the new language

$$L_{i+1} = L_i \cup \{c_\phi : \ \phi \ \text{ a one variable } L_i\text{-formula}\}$$

and the new set of L_{i+1} sentences

$$\Sigma_i^* = \Sigma_i \cup \{(\exists v \phi(v) \to \phi(c_\phi)) \ : \ \phi \text{ a one variable } L_i\text{-formula}\}.$$

Claim. Σ_i^* is finitely satisfiable Indeed, for any finite $S \subseteq \Sigma_i^*$ let $S_1 = S \cap \Sigma_i$ and take a model $\underline{A}$ of S_1 with domain A, which we assume well-ordered. Assign constants to symbols c_ϕ as follows:

$$c_\phi = \begin{cases} \text{the first element in } \phi(\underline{A}) & \text{if } \phi(\underline{A}) \neq \emptyset \\ \text{the first element in } A & \text{if } \phi(\underline{A}) = \emptyset \end{cases}.$$

Denote the expanded structure $\underline{A}^*$. By the definition, for all $\phi(v)$,
$\underline{A}^* \vDash \exists v \phi(v) \to \phi(c_\phi)$. So $\underline{A}^* \vDash S$. This proves the claim.

Let Σ_{i+1} be a complete finitely satisfiable set of L_{i+1}-sentences containing Σ_i^*.

Take $\Sigma^* = \bigcup_{i\in\mathbb{N}} \Sigma_i$. This is finitely satisfiable by 1.20. By construction one sees immediately that Σ^* is also witnessing and complete set of sentences in the language $\bigcup L_i = L + \{$ new constants$\}$. Proposition 1.19 gives us a model $\underline{A}^*$, of Σ^*. The reduct of $\underline{A}^*$ to language L is a model of Σ.
The cardinality of the model we constructed is less or equal to $|L|$ (see also Exercise 1.2). □
As noted in the logic class, the contrapositive of the compactness theorem is the statement that logical consequence is intrinsically finitary:

Exercise 1.22. Show that a sentence σ is a logical consequence of some finite subset of a set Σ of sentences, if and only if it is a logical consequence of Σ.

Exercise 1.23. Let $T = Th(\mathbb{N}, +, \cdot, 0, 1)$. Let $L' = \{+, \cdot, c\}$ be the language obtained by adjoining a new constant symbol, $T' = T \bigcup \{c \neq 0, c \neq 1, c \neq 1+1, \cdots\}$. Show that T' has a model $\underline{A}'$, and let $\underline{A}$ be the L-reduct of $\underline{A}'$. Show that $\underline{A}$ is a model of T, and is not a minimal L-structure. Conclude that $\underline{A}, \mathbb{N}$ are not isomorphic. This proves Skolem's theorem, that the natural numbers are not characterized by their first-order theory.

Solution. We first prove that T' is finitely satisfiable. Let us write $\underline{m}$ for the term $1 + 1 + \cdots + 1$ (m times). Any finite subset of T' is contained in $T \bigcup \{c \neq 0, c \neq 1, \cdots, c \neq \underline{m}\}$ for some $m \in \mathbb{N}$. It has a model, namely $\mathbb{N}$ with the usual interpretation of $+, \cdot, 0, 1$ and with c interpreted as $m + 1$. Thus T' is finitely satisfiable, so it has a model $\underline{A}'$.
Let $\underline{A}$ be the L-reduct of $\underline{A}'$. It is clear that $\underline{A}$ is a model of T, since all L-formulas have the same interpretation in $\underline{A}$ as they do in $\underline{A}'$. Moreover $\underline{A}$ is not minimal since $\mathbb{N}$ is a proper substructure of $\underline{A}$. Indeed let $a = c^{\underline{A}'}$. Then $a \neq \underline{m}$ for any $m \in \mathbb{N}$, so $a \notin \mathbb{N}$ and thus $\mathbb{N} \neq A$.
Finally, it is clear that if two L-structures $\underline{A}$,$\underline{B}$ are isomorphic, and one is minimal, then so is the other. (If $f : \underline{A} \to \underline{B}$ is an isomorphism, and C is the universe of a proper substructure of $\underline{A}$, then $f(C)$ is the universe of a proper substructure of $\underline{B}$.) Further $\mathbb{N}$ is minimal, since any substructure must include 0 and 1 and must be closed under addition. Thus $\underline{A}, \mathbb{N}$ are not isomorphic.

Exercise 1.24. Show that $\underline{A}$ is a minimal L-structure iff every element of A is named by a term; i.e. for every $a \in A$ there is a closed L-term λ such that $\lambda^{\underline{A}} = a$.

Solution. First suppose that $\underline{A}$ ia an L-structure, such that for every $a \in A$ there is a closed L-term λ such that $\lambda^{\underline{A}} = a$. Let $\underline{B} \leq \underline{A}$ be any substructure, and $a \in A$ as above; then $a = \lambda^{\underline{A}} = \lambda^{\underline{B}} \in B$. As this holds for every $a \in A$, we have $A = B$. Thus $\underline{A}$ is L-minimal. Conversely, suppose $\underline{A}$ is L-minimal. Let $\mathcal{T}$ be the set of closed terms of L. Let $B = \{\tau^{\underline{A}} : \tau \in \mathcal{T}$ be the set of elements of A named by a term. Then B is the universe of a substructure $\underline{B}$ of $\underline{A}$. By L-minimality, $A = B$; so indeed every element of $\underline{A}$ is named by a term

Exercise 1.25. Let Σ be a set of quantifier-free sentences. Assume Σ is satisfiable and that for any atomic sentence σ, either $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$. Show that there exists a unique minimal L-structure, up to isomorphism, which is a model of Σ.

Solution. We first show existence of an L-mimimal model. Let $\underline{A}_1$ be any model of Σ, with universe A_1. Let $A = \{t^{\underline{A}_1} : t\}$ be the set of interpretations of closed terms in $\underline{A}_1$. It is clear that A is closed under the basic functions of $\underline{A}$, hence is the universe of a substructure $\underline{A}$ of $\underline{A}_1$. By Exercise 1.24, $\underline{A}$ is L-minimal. If $\sigma \in \Sigma$ then σ is quantifier-free, so it takes the same truth value in $\underline{A}$ as in $\underline{A}_1$; as $\underline{A}_1 \models \Sigma$ we have $\underline{A} \models \Sigma$.
Now let $\underline{A}, \underline{B}$ be two L-minimal models of Σ. Let $\mathcal{T}$ be the set of closed terms of L. Let

$$F = \{(t^{\underline{A}}, t^{\underline{B}}) : t \in \mathcal{T}\}$$

So F is a subset of $A \times B$. Let us show that F is the graph of a bijection. Any element of A can be written $t^{\underline{A}}$ for some $t \in \mathcal{T}$ (Exercise 1.24); moreover if the same element also equals $s^{\underline{A}}$, then the atomic sentence $s \sim t$ is true in $\underline{A}$, hence by completeness of Σ must be in Σ; but then , it must be true in $\underline{B}$, so $s^{\underline{B}} = t^{\underline{B}}$. This shows that F is the graph of a function $f : A \to B$; dually, one sees that f is 1-1 and onto.
Next we check that f is a homomorphism. Let R be an n-place relation symbol; let $a = (a_1, \ldots, a_n)$ be an n-tuple from A, $b = (fa_1, \ldots, fa_n)$. We have to show that $a \in R^{\underline{A}}$ iff $b \in R^{\underline{B}}$. Using again the minimality of $\underline{A}$, there exist $t_i \in \mathcal{T}$ with $t_i^{\underline{A}} = a_i$; and by the definition of f we have $t_i^{\underline{B}} = b_i$. Now by the atomic completeness of Σ, we have $\sigma \in \Sigma$ or $\neg\sigma \in \Sigma$; in the former case we have $a \in R^{\underline{A}}$ and $b \in R^{\underline{B}}$, and in the latter we have $a \notin R^{\underline{A}}$ and $b \notin R^{\underline{B}}$. So in any case $a \in R^{\underline{A}}$ iff $b \in R^{\underline{B}}$.
Finally if H is an n-place function symbol, we must show that $f(H^{\underline{A}}(a)) = H^{\underline{B}}(b)$. Let s be the term $H(t_1, \ldots, t_n)$. Then $f(H^{\underline{A}}(a)) = f(s^{\underline{A}}) = s^{\underline{B}} =$

$H^{\underline{B}}(b)$. This proves that any two L-minimal models of Σ are isomorphic. We note that the proof would work for a set of universal sentences, too.

Exercise 1.26. Assume, for each $n \in \mathbb{N}$, that T has a model with at least n elements. Let λ be any set. Show that T has a model $\underline{A}$ whose universe A satisfies $|A| \geq |\lambda|$. (Hint: introduce new constant symbols c_i for $i \in \lambda$, and sentences $c_i \neq c_j$; use compactness.)

Compare to the upward Löwenheim-Skolem theorem, below.

Terminology With the completeness theorem in mind, will use the term *consistent* synonymously with *finitely satisfiable.* (and hence, by the compactness theorem, with *satisfiable.*) This is a matter of terminology, and does not require fixing a proof system.

2 Compactness via ultraproducts (optional chapter)

Let L be a language, I an index set, and assume given an L-structure $\underline{A}_i$ for each $i \in I$.

2.1 Products

We first define the full product structure $\underline{B} = \Pi_{i \in I} \underline{A}_i$. The universe is the product set $\Pi_{i \in I} A_i$. As usual in algebra we define the interpretation of function symbols coordinatewise: $F^{\underline{B}}(a_1, \ldots, a_n) = b$ where $b(i) = F^{\underline{A}_i}(a_1(i), \ldots, a_n(i))$ For a basic relation symbol R, define the interpretation of R by

$$R^{\underline{B}}(b_1, \ldots, b_n) \leftrightarrow (\forall i \in I)(b_1(i), \ldots, b_n(i)) \in R^{\underline{A}_i}$$

Exercise 2.2. Let $\underline{B} = \Pi_{i \in I} \underline{A}_i$.

1. (r) Show that for any term t, $t^{\underline{B}}(a_1, \ldots, a_n) = b$ where $b(i) = t^{\underline{A}_i}(a_1(i), \ldots, a_n(i))$

2. Show that any sentence formed using $\forall, \exists, \wedge$ from atomic formulas is true in $\underline{B}$ if it holds in each $\underline{A}_i$. This generalises the products of groups, rings, vector spaces, ….

3. A partial order is a transitive relation $<$; it is *dense* if whenever $a < b$, there exists c with $a < c < b$. Show that the product of dense partial orderings is a dense partial ordering. Is the analogue true for linear orderings?

4. If each $\underline{A}_i$ is an integral domain, and $|I| > 1$, show that $\underline{B}$ is never an integral domain.

Remark The precise characterization of sentences preserved under products is a little complicated, and we will omit it here. Note however that any sentence of the form $(\forall x_1) \cdots (\forall x_n)(\phi_1 \wedge \cdots \wedge \phi_k \rightarrow \phi)$, where $\phi_1, \ldots, \phi_k, \phi$ are atomic formulas, is preserved under produtcs; this includes the axioms for partial orderings.

2.3 Ultraproducts

Consider a family of L -structure $(A_i : i \in I)$; I is some nonempty index set. Let $P(I)$ be the set of all subsets of I; we view $P(I)$ as a structure with two binary operations (union, intersection), a unary operation (complement), and two constants (1 interpreted as the full set I, and 0 interpreted as $\emptyset$.)

$$P(I) = (P(I), \bigcup, \bigcap, \neg, 0, 1)$$

It forms a *boolean algebra*, but we will not need to know the axioms of a Boolean algebra here.
When I has a single element, say $I = \{0\}$, we obtain the 2-element Boolean algebra $P(\{0\})$; we denote by it by $\mathbf{2}$.
Fix also a homomorphism $u : P(I) \to \mathbf{2}$. This means that $u(b \bigcup b') = u(b) \bigcup u(b')$, $u(\emptyset) = \emptyset$, $u(I) = 1 = \{0\}$, and similarly u respects complements. Note that u must be order-preserving: if $b \subseteq b'$, then $b \bigcup b' = b'$ and it follows that $u(b) \subseteq u(b')$. [5]
Let $B = \Pi_{i \in I} A_i$, and let $p_i : B \to A_i$ be the i'th projection.
For a tuple $a = (a_1, \ldots, a_n)$, let $p_i(a) = (p_i(a_1), \ldots, p_i(a_n))$.
Given $a \in B^n$, and L-formula $\phi = \phi(x_1, \ldots, x_n)$, define

$$[\phi(a)] = \{i \in I : \underline{A}_i \models \phi(p_i(a))\}$$

So $[\phi(a)]$ is an element of $P(I)$. Thus $u([\phi(a)]) \in \mathbf{2}$. We will write $u[\phi(a)]$ for this.

Exercise 2.4. 1. Take ϕ to be the formula $x \doteq y$. Define $a \sim_u b$ iff $u[a \doteq b] = 1$. Show that $\sim_u$ is an equivalence relation.

2. If $a \sim_u a'$, then $u[\phi(a)] = u[\phi(a')]$.

We now define a weak structure $\underline{B}$.
The universe will be the product $B = \Pi_{i \in I} A_i$. Function symbols in the ultraproduct are interpreted coordinatewise, in the same way as for the product structure in §2.1. Relation symbols are interpreted in this way:

$$a \in R^{\underline{B}} \iff u(R[a]) = 1$$

[5] $u^{-1}(1)$ is called an *ultrafilter*. Of course it carries the same information as u. One can think of the elements of $u^{-1}(1)$ as being "large" in some sense, determined by u.

Lemma 2.5. *For any $b \in B^n$, and L-formula $\phi = \phi(x_1, \dots, x_n)$,*

$$\phi(b)^{\underline{B}} = u[\phi(b)]$$

Proof. For atomic formulas, this follows from the definition of $R^{\underline{B}}$.
When ϕ is a Boolean combination of ϕ', ϕ'', this follows from the fact that both u and the map

$$\phi \mapsto \phi^{\underline{B}}$$

a Boolean homomorphisms.
Let us verify the claim when $\phi = (\exists y)\psi$, $\psi = \psi(x, y)$. Assume $\underline{B} \models \phi(b)$. Then for some element c of B, $\underline{B} \models \psi(b, c)$. By induction, $u([\psi(b, c)]) = 1$. But $[\psi(b, c)] \subset [\phi(b)]$, so $u[\phi(b)] = 1$.
Conversely assume $u[\phi(b)] = 1$. For $i \in [\phi(b)]$, $\underline{A}_i \models \phi(b(i)) = (\exists y)\psi(b(i), y)$, so one can choose $c(i)$ such that $\underline{A}_i \models \psi(b(i), c(i))$. Define $c(i)$ in some arbitrary way for $i \notin [\phi(b)]$. [6] So $[\phi(b)] \subset [\psi(b, c)]$. Since $u([\phi(b)]) = 1$, we have $u([\psi(b, c)]) = 1$ so by induction, $\underline{B} \models \psi(b, c)$ and so $\underline{B} \models (\exists y)\psi(b, y)$, as required. The fact that $\doteqdot$ is a congruence follows from Exercise 2.4.

□

Now let $\underline{A} := \underline{B}/\sim_u$ be the quotient structure (see Exercise 1.14.)
$\underline{A}$ is called the ultraproduct or ultralimit of the $\underline{A}_i$ along u.

Theorem 2.6 (Łoś). *For any $a \in A^n$, and L-formula $\phi = \phi(x_1, \dots, x_n)$*

$$\phi(a)^{\underline{A}} = u[\phi(a)]$$

In particular for any sentence ϕ,

$$\underline{A} \models \phi \leftrightarrow u[\phi] = 1$$

.

Proof. We have $a = \pi(b)$ for some b from B; and: $\phi(a)^{\underline{A}} = \phi(b)^{\underline{B}} = u[\phi(b)^{\underline{B}}] = u[\phi(a)^{\underline{A}}]$. □

Exercise 2.7. Fix $\delta \in I$. Define $u = u_\delta$ by $u = 1$ iff $\delta \in s$. Check that $u : B \to \mathbf{2}$ is a homomorphism; it is called the *principal* homomorphism associated with δ. [7] Show for this u that $\underline{A}_u \cong A_\delta$.

[6] Here we assume $A_i \neq \emptyset$. A slight change in the definition, allowing partial functions, is required if one wishes to include the structure with empty universe.

[7] In the terminology of Arrow's theorem, δ is a dictator for the voting process determined by u.

Ultrapower proof of the compactness theorem; countable case. Let $\{\sigma_1, \sigma_2, \ldots\}$ be a countable set of sentences, and assume it is finitely satisfiable; thus there exists a model A_n of $\{\sigma_1, \ldots, \sigma_n\}$. Let $u : P(\mathbb{N}) \to \mathbf{2}$ be a non-principal homomorphism, and let A_u be the ultraproduct. Then $u(\{1, \ldots, n\}) = 0$ for all n; so $u(\{k, k+1, k+2, \ldots\}) = 1$ for all k. But $[\sigma_k] \supset \{k, k+1, k+2, \ldots\}$. Thus $u[\sigma_k] = 1$ so $A_u \models \sigma_k$. □

Exercise 2.8. Let I be the set of prime numbers, and let $u : P(I) \to \mathbf{2}$ be a non-principal homomorphism. Let F_p be the p-element field, and let F_u be the ultraproduct. Show that F_u is a field. Show that the set of nonzero squares in F_u forms a subgroup of F_u^* of index 2. (Optional: what can you say about the set of nonzero cubes, $\{x^3 : x \in F_u^*\}$?)

Exercise 2.9. [Stone-Cech compactification.] Recall the Tychonoff or product topology on $\mathbf{2}^B$, the set of functions from B into $\mathbf{2}$. Here $\mathbf{2}$ is taken to have the discrete topology. This is a compact space; see Exercise 1.16 Let I^* be the subset of $\mathbf{2}^B$ consisting of Boolean homomorphisms $B \to \mathbf{2}$. We endow I^* with the subspace topology.

1. Show that I^* is a closed subset of $\mathbf{2}^B$; hence it is a compact, Hausdorff topological space.

2. Show that the map $\delta \mapsto u_\delta$ embeds I as a discrete subset of I^*, and that the image is dense.

3. Conclude that if I is infinite, there exist non-principal elements of I^*.

4. For any sentence σ, show that $\{u \in I^* : \underline{A}_u \models \sigma\}$ is an open (hence clopen) subset of I^*.

Remark The ultrafilter construction extends the given family $(A_i : i \in I)$ to a family $(A_u : u \in I^*)$. The ultraproduct A_u can be viewed as a limit, in the direction of u, of the family A_i.

Exercise 2.10. Prove the compactness theorem, using ultraproducts, for any finitely satisfiable set of sentences Σ. Here are hints for two alternative proofs.

1. Let I be the set of finite subsets of Σ. By assumption, for $w \in I$, there exists a model A_w of w. What property does $u : P(I) \to \mathbf{2}$ need to have, in order to ensure that $A_u \models \Sigma$?

2. Let $(M_i : i \in I)$ be a set of structures, such that any finite subset of Σ is satisfiable by at least one M_i. Let I^* be the Stone-Cech compactification of I as above. Deduce the compactness theorem from the compactness of I^* and Ex. 2.9 (4).

Exercise 2.11. *Assume u is a nonprincipal homomorphism $P(I) \to \mathbf{2}$. Let $\underline{A}$ be an L-structure. The* ultrapower *$\underline{A}^*$ with respect to u is defined to be the ultraproduct, where the I-indexed family of structures is constant, $\underline{A}_i = \underline{A}$. Define an embedding $f : \underline{A} \to \underline{A}^*$ by mapping a to the element a^* represented by the constant function with value a.*

1. *Show that f is an elementary embedding.*

2. *Assume I is countable. Show that f is surjective if and only if A is finite.*

3 The method of diagrams

We will be interested not just in a single structure $\underline{A}$, but in embeddings $\underline{A} \to \underline{B}$. But we do not need to develop techniques from scratch; we can study embeddings of one theory, using structures of another theory constructed for the purpose.
For an L-structure $\underline{M}$ and $A \subset M$, let $L_A = L \cup \{c_a : \ a \in A\}$ be the expansion of the language L obtained by adjoining a new constant symbol c_a for each element $a \in A$. (It will sometimes be convenient to denote the new constant symbol by $\underline{a}$.)
Let $\underline{M}_A$ be the natural expansion of $\underline{M}$ to L_A assigning to c_a the element a. We define the *diagram of A in M*, $\mathrm{Diag}_{\underline{M}}(A)$, to be the set of quantifier-free sentences of L_A true in $\underline{M}_A$.
Now assume A is the universe of a substructure $\underline{A}$ of $\underline{M}$. Let $\underline{A}^+ = \underline{A}_A$. Then the diagram of A in $\underline{A}^+$ is the same as the diagram of A in $\underline{M}$, so no reference to $\underline{M}$ is needed; and we write:
$\mathrm{Diag}(\underline{A}) = Th_{qf}(\underline{A}^+) = \{\sigma : \ \sigma$ a quantifier-free L_A-sentence, such that $\underline{A}^+ \vDash \sigma\}$.
Remark. For a substructure $\underline{A}$, $Diag(\underline{A})$ is sometimes defined in the same way, but using only atomic sentences and their negations. But this restricted part of $Diag(\underline{A})$, call it $Diag_0(\underline{A})$, logically implies all of $Diag(\underline{A})$. (This can easily be seen by writing a quantifier-free sentence as a disjunction of conjunctions of atomic and negated-atomic sentences. If $\sigma \bigvee_{i=1}^{k} \bigwedge_{j=1}^{l_i} \phi_{ij} \in Diag(\underline{A})$, then for some i_0, for each j, $\phi_{i_0 j} \in Diag(\underline{A})$; so $\phi_{i_0 j} \in Diag_0(\underline{A})$; and of course $\{\phi_{i_0 j} : j \leq l_{i_0} \vdash \sigma$.)

We also define the *complete diagram* of $\underline{A}$:

$$\mathrm{CDiag}(\underline{A}) = Th(\underline{A}^+) = \{\sigma : \ \sigma \ L_A\text{-sentence such that} \quad \underline{A}^+ \models \sigma\}.$$

Theorem 3.1 (Method of Diagrams)**.**
(i) There is a natural bijection between models of $\mathrm{Diag}(\underline{A})$*, and L-structures* $\underline{B}$ along with *an embedding $j : \underline{A} \to \underline{B}$.*
(ii) there is a natural bijection between models of $\mathrm{CDiag}(\underline{A})$ *and L-structures* $\underline{B}$ along with *an elementary embedding $j : \underline{A} \to \underline{B}$.*

Proof Let $\mathcal{C}$ be the class of models of $\mathrm{Diag}(\underline{A})$, and let $\mathcal{D}$ be the class of pairs $(\underline{B}, j)$ with $\underline{B}$ an L-structure and $j : \underline{A} \to \underline{B}$ an embedding. We will describe maps $\alpha : \mathcal{C} \to \mathcal{D}$ and $\beta : \mathcal{D} \to \mathcal{C}$.

Let $\underline{C} \in \mathcal{C}$. Define $\alpha(\underline{C}) = (\underline{C}|L, j_{\underline{C}})$ where $\underline{C}|L$ is the restriction (reduct) of $\underline{C}$ to an L-structure, and $j_{\underline{C}}$ is defined by:

$$j_{\underline{C}}(a) = \underline{a}^{\underline{C}}$$

It is straightforward to verify that $\alpha(\underline{C}) \in \mathcal{D}$.
Let $(\underline{B}, j) \in \mathcal{D}$. Define $\beta(\underline{B}, j)$ to be the expansion of $\underline{B}$ to L_A obtained by interpreting $\underline{a}$ by the element $j(a)$.
Again it is straightforward to verify that $\beta(\underline{B}, j) \in \mathcal{D}$.
Clearly, $\alpha \circ \beta = Id_{\mathcal{D}}$ and $\beta \circ \alpha = Id_{\mathcal{C}}$.
This give the bijection of (i). As for (ii), it suffices to check for $\underline{C} \in \mathcal{C}$ corresponding as above to $(\underline{B}, j) \in \mathcal{D}$, that $\underline{C} \models CDiag(\underline{A})$ iff j is elementary.

Corollary 3.2. *Assume given an L-structure $\underline{A}$ and a set of L-sentences T.*

(i) the set $T \cup \mathrm{Diag}(\underline{A})$ is finitely satisfiable iff there is a model $\underline{B}$ of T such that $\underline{A} \subseteq \underline{B}$.
(ii) the set $T \cup \mathrm{CDiag}(\underline{A})$ is finitely satisfiable iff there is a model $\underline{B}$ of T such that $\underline{A} \preccurlyeq \underline{B}$.

Proof. See Lemma 3.4. □

Exercise 3.3.

1. *Let $\underline{A}$ be any structure, and B any set. Then there exists $\underline{A}' \cong \underline{A}$ with universe A' such that $A' \bigcap B = \emptyset$.*

2. *Let $\underline{A}, \underline{B}$ be L-structures, and let $f : \underline{A} \to \underline{B}$ be an embedding. Then $f(A)$ is the universe of a substructure of $\underline{B}$, isomorphic to $\underline{A}$.*

(Hint for (1): You may assume here, as a matter of set theory, that there exists some $*$ such that the ordered pair $(x, *)$ is never an element of B. In the usual well-founded axiomatizations of set theory and the usual definition of ordered sets, this holds with $* = \{B\}$. For $a \in A$, let $a^* := (a, *)$. For $a = (a_1, \ldots, a_n) \in A^n$, let $a^* = (a_1^*, \ldots, a_n^*)$. Let $A' = \{(a, *) : a \in A\}$. For any relation symbol R, let $R^{\underline{A}'} = \{(a^* : a \in R^{\underline{A}})\}$. Define $F^{\underline{A}'}$ similarly for function symbols, and prove that $a \mapsto a^*$ is an isomorphism.)

The following lemma can be read to say that "any L-embedding is isomorphic to an inclusion"; it has the consequence that in many proofs regarding embeddings we may take them to be inclusions. It has no model-theoretic content and is purely about the ambient set theoretic presentation of the structures.

Lemma 3.4 (Renaming lemma). *Let $\underline{A}, \underline{B}$ be L-structures, and let $f : A \to B$ be an embedding. Then there exists an L-structure $\underline{B}'$ such that $\underline{A} \leq \underline{B}'$, and an isomorphism $g : \underline{B}' \to \underline{B}$, such that $f = g|_A$.*

Proof. We seek $\underline{B}'$ isomorphic to $\underline{B}$, but whose universe contains A. We will simply 'rename' or replace the elements of $f(A)$, by their pre-images in A; but in order not to create clases with elements of $B \setminus f(A)$, we also replace them by some elements not in A. For this purpose, as in Exercise 3.3, let $b \mapsto b^*$ be any injective function on B, whose image is disjoint from A. For $b \in f(A)$, let b' be the unique element of A with $f(b') = b$. For $b \in B \setminus f(A)$, let $b' = b^*$. Let $B' = \{b' : b' \in B'\}$. For a relation symbol R, let $R = \{(b'_1, \ldots, b'_n) : (b_1, \ldots, b_n) \in R^{\underline{B}}\}$. Similarly for function symbols. Prove as an exercise that the map $b \mapsto b'$ is an isomorphism $\underline{B} \to \underline{B}'$, whose inverse g satisfies $f = g|_A$. □

Theorem 3.5 (Upward Lowenheim-Skolem Theorem, Tarski). *For any infinite L-structure $\underline{A}$ and a cardinal $\kappa \geq \max\{|L|, ||\underline{A}||\}$ there is an L-structure $\underline{B}$ of cardinality κ such that $\underline{A} \preccurlyeq \underline{B}$.*

Proof Let M be a set of cardinality κ. Consider an extension $L_{A,M}$ of language L obtained by adding to L_A constant symbols c_i for each $i \in M$. Consider now the set of $L_{A,M}$-sentences

$$\Sigma = \text{CDiag}(\underline{A}) \cup \{\neg c_i \doteq c_j : \ i \neq j \in M\}.$$

We claim that Σ is finitely satisfiable Indeed, consider a finite subset $S \subseteq \Sigma$. Obviously

$$S \subseteq S_0 \cup \{\neg c_i \doteq c_j : \ i \neq j \in M_0\}$$

for some $S_0 \subseteq \text{CDiag}(\underline{A})$ and $M_0 \subset M$, both finite. By definition $\underline{A}^+ \vDash S_0$. Now, since A is infinite, we can expand $\underline{A}^+$ to the model of S by assigning to c_i $(i \in M_0)$ distinct elements of A. This proves the claim.
It follows from the compactness theorem that Σ has a model of cardinality $|L_{A,M}|$, which is equal to κ. Let $\underline{B}^*$ be such a model. The L-reduct $\underline{B}$ of $\underline{B}^*$, by the method of diagrams, satisfies the requirement of the theorem. □

Lemma 3.6 (Tarski-Vaught criterion). *Suppose $\underline{A} \leq \underline{B}$ are L-structures with domains $A \subseteq B$. Then $\underline{A} \preccurlyeq \underline{B}$ iff the following condition holds:*
for all L-formulas $\varphi(v_1, \ldots, v_n)$ and all $a_1, \ldots, a_{n-1} \in A$, $b \in B$ such that $\underline{B} \vDash \varphi(a_1, \ldots, a_{n-1}, b)$ there is $a \in A$ with $\underline{B} \vDash \varphi(a_1, \ldots, a_{n-1}, a)$

Proof Obviously, given $\bar{a} = \langle a_1, \ldots, a_{n-1} \rangle$ the existence of $b \in B$ as above is equivalent to $\underline{B} \models \exists v \varphi(\bar{a}, v)$.

Suppose $\underline{A} \preccurlyeq \underline{B}$. Then $\underline{B} \models \exists v \varphi(\bar{a}, v)$ is equivalent to $\underline{A} \models \exists v \varphi(\bar{a}, v)$ which is equivalent to the existence of an $a \in A$ with $\underline{A} \models \varphi(\bar{a}, a)$. The latter by $\underline{A} \preccurlyeq \underline{B}$ implies $\underline{B} \models \varphi(\bar{a}, a)$.
For the converse, we assume that for all φ

$$(*) \quad \underline{B} \models \exists v \varphi(\bar{a}, v) \text{ implies that for some } a \in A \quad \underline{B} \models \varphi(\bar{a}, a)$$

and want to prove that

$$(**) \quad \underline{A} \models \psi(\bar{a}) \text{ iff } \underline{B} \models \psi(\bar{a})$$

for all L-formulas $\psi(\bar{v})$.
Induction on the complexity of ψ. For ψ atomic (**), this follows from Exercise 1.9. The cases of $\psi = \psi_1 \wedge \psi_2$ and $\psi = \neg\psi_1$ are easy. In the case $\psi = \exists v \varphi$ the $\Rightarrow$ side of (**) follows immediately from the induction hypothesis and the meaning of $\exists$.
Proof of $\Leftarrow$:
$\underline{B} \models \exists v \varphi(\bar{a}, v)$ implies $\underline{B} \models \varphi(\bar{a}, b)$, some $b \in B$, implies $\underline{B} \models \varphi(\bar{a}, a)$, some $a \in A$, implies, by the induction hypothesis, $\underline{A} \models \varphi(\bar{a}, a)$, implies $\underline{A} \models \exists v \varphi(\bar{a}, v)$.
□

Theorem 3.7 (Downward Lowenheim-Skolem Theorem)**.** *Let $\underline{B}$ be an L-structure, S a subset of $B = \text{dom}\ (\underline{B})$. Then there exists $\underline{A} \preccurlyeq \underline{B}$ such that $S \subseteq A = \text{dom}\ (\underline{A})$ and $||\underline{A}|| \leq \max\{\text{card}\ (S), |L|\}$. In particular, given $\underline{B}$ and a cardinal $||\underline{B}|| \geq \kappa \geq |L|$ we can have $\underline{A} \preccurlyeq \underline{B}$ of cardinality κ.*

Proof Fix some $b_0 \in B$. For each L-formula $\phi(v_1, \ldots, v_n)$ define a function $g_\phi : B^{n-1} \to B$ by

$$g_\phi(b_1, \ldots, b_{n-1}) = \begin{cases} \text{an element } b \in B : \underline{B} \models \phi(b_1, \ldots, b_{n-1}, b) \\ \qquad\qquad\qquad\qquad \text{if such one exists} \\ b_0 \quad \text{if not} \end{cases}$$

(g_ϕ are called *Skolem functions*).
Notice that for ϕ of the form $\tau(v_1, \ldots, v_{n-1}) \doteq v_n$, where τ is an L-term, g_ϕ coincides with the function $\tau^{\underline{B}}$.

Let A be the closure of S under all the g_ϕ, i.e.

$$A = \bigcup_{i \in \mathbb{N}} S_i : \quad S_0 = S \text{ and}$$

$$S_{i+1} = \{g_\phi(b_1, \ldots, b_{n-1}) : b_1, \ldots, b_{n-1} \in S_i,\ \phi(v_1, \ldots, v_n)\ L-\text{formulas}\}.$$

Notice that card $A \leq$ card $S + |L|$.
Define an L-structure $\underline{A}$ on the domain A interpreting the relation, function and constant symbols of L on A as induced from $\underline{B}$:
(i) for an n-ary relation symbol P or the equality symbol,
$P^{\underline{A}} = P^{\underline{B}} \cap A^n$;
(ii) for an m-ary function symbol f and $\bar{a} \in A^m$, $a \in A$,
$f^{\underline{A}}(\bar{a}) = a$ iff $f^{\underline{B}}(\bar{a}) = a$;
(iii) for a constant symbol c, $\quad c^{\underline{A}} = c^{\underline{B}}$.
(ii) and (iii) are possible since A is closed under L-terms.

Clearly then $\underline{A} \leq \underline{B}$ and the condition of Tarski-Vaught criterion is satisfied, for if $\underline{B} \vDash \exists v \phi(\bar{a}, v)$ then $\underline{B} \vDash \phi(\bar{a}, g_\phi(\bar{a}))$. Thus the lemma finishes the proof. □

Corollary 3.8. *Let Σ be a set of L-sentences which has an infinite model. Then for any cardinal $\kappa \geq |L|$ there is a model of Σ of cardinality κ.*

Example Let $\mathcal{M}$ be a model of ZF (or any axiomatization of set theory) in the language with one binary predicate symbol $\in$. Then there is a countable elementary submodel

$$\mathcal{M}_0 \preccurlyeq \mathcal{M}.$$

Definition 3.9. *Let $\underline{A}$ be a structure, and $f : A^n \to A$ a function. We say f is* definable *if the graph of f is definable, i.e. for some formula $\phi(x_1, \ldots, x_n, y)$, $\underline{A} \models \phi(a_1, \ldots, a_n, b)$ iff $b = f(a_1, \ldots, a_n)$.*

Example 3.10. $\underline{\mathbb{N}} := (\mathbb{N}, +, \cdot, <, 0, 1)$ *has definable Skolem functions, i.e. if $\phi(x, y)$ is any formula, there exists a definable function $f(x)$ such that $\underline{\mathbb{N}} \models \phi(a, f(a))$ whenever $\underline{\mathbb{N}} \models (\exists x)\phi(a, x)$.*
Indeed, let $f(a) = 0$ if $\underline{\mathbb{N}} \models \neg(\exists x)\phi(a, x)$. Let $f(a) = b$ if $\underline{\mathbb{N}} \models \neg(\exists x)\phi(a, x)$, and b is the smallest natural number such that $\underline{\mathbb{N}} \models \phi(a, b)$. Then f is definable; indeed $f(t) = x$ iff

$$(x = 0 \& \neg(\exists x)\phi(t, x)) \bigvee (\phi(t, x) \& (\forall x' < x)\neg\phi(t, x'))$$

Definition A theory T *admits Skolem functions* if for any n-tuple of variables $x = (x_1, \ldots, x_n)$ and any formula $\phi(x, y)$, for some term t, $T \models (\forall x)((\exists y)\phi(x, y) \to \phi(x, t(x)))$. The term t is called a Skolem term for ϕ.

Exercise 3.11. Assume T admits Skolem functions. Let $\underline{A} \leq \underline{B}$. If $\underline{B} \models T$, show that $\underline{A} \prec \underline{B}$.

Solution. We will use the Tarski-Vaught criterion. Let $\varphi(v_1, \ldots, v_n)$ be a formula, $a = (a_1, \ldots, a_{n-1}) \in A^{n-1}$, $b \in B$, such that $\underline{B} \vDash \varphi(a, b)$. We have to show that there exists $c \in A$ with $\underline{B} \vDash \varphi(a, c)$. Indeed let t be a Skolem term for $\phi(x, y)$ (where $x = (v_1, \ldots, v_{n-1})$ and $y = v_n$), and let $c = t^{\underline{B}}(a) = t^{\underline{A}}(a) \in A$. Since $\underline{B} \models T$, we have

$$\underline{B} \models (\forall x)((\exists y)\phi(x, y) \to \phi(x, t(x)))$$

So

$$\underline{B} \models (\exists y)\varphi(a, y) \to \varphi(a, t(a))$$

Now $\underline{B} \models \varphi(a, b)$ so $\underline{B} \models (\exists y)\varphi(a, y)$. Thus $\underline{B} \models \varphi(a, c)$, as was to be proved.

Remark The Löwenheim-Skolem theorem is due to Löwenheim 1915, Skolem 1920. Following further work of Skolem in 1923, a clear statement of the completeness theorem appeared in Gödel's 1929 thesis. Gödel proof used something like Skolem functions. Skolem in 1934 showed the existence of a proper elementary extension of $(\mathbb{N}, +, \cdot)$, by an ultrapower construction that can also be used to prove compactness. Ultraproducts in general were defined by Łoś in 1955. The use of constants as witnesses comes from Henkin's 1949 thesis.

Exercise 3.12. *Let $L = (<, +, \cdot, 0, 1, F)$ be the language of rings, with an additional unary function symbol F. Let $\underline{R} = (\mathbb{R}; <, +, \cdot, 0, 1, f)$ be an L-structure, with $(\mathbb{R}; <, +, \cdot, 0, 1)$ the ordered field of real numbers, and $f = F^{\underline{R}}$ a unary function, with $f(0) = 0$.*

1. *Show that there exists a model $\underline{A} \succ \underline{R}$ containing a nonzero infinitesimal element, i.e. an element $\epsilon \neq 0$ such that for any $n \in \mathbb{N}$, $\underline{A} \models |n\epsilon| < 1$.*

2. *Assume f is continuous at 0. Show that for any infinitesimal ϵ of any such $\underline{A}$, $F^{\underline{A}}(\epsilon)$ is also infinitesimal.*

3. *Prove the converse to (2): f is continuous at 0 iff for any infinitesimal ϵ of any $\underline{A} \succ \underline{R}$, $F^{\underline{A}}(\epsilon)$ is also infinitesimal.*

Solution.

1. Let $L' = L_{\underline{R}} \bigcup \{e\}$ with e a new constant symbol. Let $T = CDiag(\underline{R})$, and let $T' = T \bigcup \{|ne| < 1 : n = 1, 2, \ldots\}$. Let $\underline{R}'_k$ be the expansion of $\underline{R}$ to L' obtained by interpreting $\underline{a}$ as a ($a \in \mathbb{R}$) and e as $1/k$. Then $\underline{R}'_k \models Th(\underline{R}_{\underline{R}}) \bigcup \{|ne| < 1 : n = 1, 2, \ldots, k-1\}$. Thus $CDiag(\underline{R}) \bigcup \{|n\, e| < 1 : n = 1, 2, \ldots\}$ is finitely satisfiable. By the compactness theorem, it has a model $\underline{A}''$. Let $\underline{A}'$ be the reduct to $L_{\underline{R}}$ of $\underline{A}''$. Note that $\underline{A}'$ has an infinitesimal element, namely $e^{\underline{A}''}$. Now $\underline{A}' \models CDiag(\underline{R})$, so by Theorem 3.1, it corresponds to a structure $\underline{A}$ with an infinitesimal element, along with an elementary embedding $\underline{R} \to \underline{A}'$. By Lemma 3.4, we may take $\underline{A}$ to be an elementary extension of $\underline{R}$.

2. Assume f is continuous at 0. Let f^* be the function $F^{\underline{A}}$. Let $\underline{R} \prec \underline{A}$ and let $\epsilon \in \underline{A}$ be infinitesimal. We have to show that $f^*(\epsilon)$ is also infinitesimal, i.e. for each $n \in \mathbb{N}$ we must show that $|nf^*(e)| < 1$. Now continuity of f at 0 implies existence of some $m \in \mathbb{N}$ such that in $\underline{R}$, if $|m\epsilon| < 1$ then $|nf(\epsilon)| < 1$. So

$$\underline{R} \models (\forall x)(|mx| < 1 \to |nF(x)| < 1)$$

Thus

$$\underline{A} \models (\forall x)(|mx| < 1 \to |nF(x)| < 1)$$

But as ϵ is infinitesimal, $\underline{A} \models |m\epsilon| < 1$. So $\underline{A} \models |nf^*(\epsilon)| < 1$, as required.

3. Now assume f is discontinuous at 0. So for some m there exist elements $r \in \mathbb{R}$ arbitrarily close to 0, with $|f(r)| > 1/m$. In particular for any $n \in \mathbb{N}$, we can find such an r with $n|r| < 1$. Thus the theory T' of (1) remains consistent even when one adjoins to it the sentence: $m|F(e)| > 1$. By the proof of (1), there exists $\underline{A} \succ \mathbb{R}$ and an infinitesimal element $\epsilon \in A$ with $|F(\epsilon)| > 1/m$. So $F(\epsilon)$ is not infinitesimal.

4 Axiomatizable classes and preservation theorems

Exercise 4.1. Let $\phi_1, \dots, \phi_n$ be existential formulas. Prove that
(i) $(\phi_1 \vee \dots \vee \phi_n)$ and $(\phi_1 \wedge \dots \wedge \phi_n)$ are logically equivalent to existential formulas;
(ii) $(\neg\phi_1 \wedge \dots \wedge \neg\phi_n)$ and $(\neg\phi_1 \vee \dots \vee \neg\phi_n)$ are logically equivalent to universal formulas.

Given a set of sentences Σ denote $\Sigma_\exists$ its subset consisting of all existential formulas in Σ. Correspondingly, $\Sigma_\forall$ are the universal formulas of Σ.
Thus $\mathrm{Th}_\exists(\underline{A})$ is the set of all existential L-sentences which hold in $\underline{A}$.

Lemma 4.2. *Suppose $\underline{A} \leq \underline{B}$ and $a_1, \dots, a_n \in A$.*
(i) If $\underline{A} \models \varphi(a_1, \dots, a_n)$, for an existential formula $\varphi(v_1, \dots, v_n)$, then $\underline{B} \models \varphi(a_1, \dots, a_n)$.
(ii) If $\underline{B} \models \psi(a_1, \dots, a_n)$, for a universal formula $\psi(v_1, \dots, v_n)$, then $\underline{A} \models \psi(a_1, \dots, a_n)$.

Proof (i) Let $\varphi(v_1, \dots, v_n)$ be $\exists v_{n+1}, \dots, v_m \theta(v_1, \dots, v_n, v_{n+1}, \dots, v_n)$ and θ quantifier-free. Under this notation $\underline{A} \models \varphi(a_1, \dots, a_n)$ means that there are $a_{n+1}, \dots, a_m \in A$ such that $\underline{A} \models \theta(a_1, \dots, a_m)$. To prove the statement of the lemma it is enough to show that for quantifier-free θ

$$\underline{A} \models \theta(a_1, \dots, a_m) \Leftrightarrow \underline{B} \models \theta(a_1, \dots, a_m).$$

For θ atomic it is proved in Lemma 1.9. If the equivalence holds for θ_1 and θ_2, it holds by definitions for $\neg\theta_1$ and $(\theta_1 \wedge \theta_2)$. The statement (i) follows by induction.
(ii) Follows immediately from (i). □

Lemma 4.3. $\Sigma \cup \mathrm{Diag}(\underline{A})$ *is satisfiable iff* $\Sigma \cup \mathrm{Th}_\exists(\underline{A})$ *is satisfiable. (Here $\underline{A}$ is an L-structure; Σ is any set of L-sentences or even of a bigger language L', provided that the new constants used in* $\mathrm{Diag}(\underline{A})$ *do note appear in L'.)*

Proof In one direction we have in fact:

$$\Sigma \cup \mathrm{Diag}(\underline{A}) \vdash \mathrm{Th}_\exists(\underline{A})$$

To see this, let $\exists v_1, \dots v_n \theta(v_1, \dots, v_n) \in \mathrm{Th}_\exists(\underline{A})$, with θ quantifier-free. Since $\underline{A} \models \exists v_1, \dots v_n \theta(v_1, \dots, v_n)$, we have $\underline{A} \models \theta(a_1, \dots, a_n)$ for some $a_1, \dots a_n \in A$. Thus $\mathrm{Diag}(\underline{A}) \vDash \theta(c_{a_1}, \dots c_{a_n})$; and $\theta(c_{a_1}, \dots c_{a_n}) \vdash \exists v_1, \dots v_n \theta(v_1, \dots, v_n)$.

Conversely, consider a finite part of $\Sigma \cup \text{Diag}(\underline{A})$; by taking the conjunction it suffices to consider a single sentence $\theta(c_{a_1}, \ldots, c_{a_n})$ of $\text{Diag}(\underline{A})$, with θ quantifier-free. Let $\underline{B} \models \Sigma \cup \text{Th}_\exists(\underline{A})$. Since $\exists v_1, \ldots v_n \theta(v_1, \ldots, v_n) \in \text{Th}_\exists(\underline{A})$, there are $b_1, \ldots, b_n \in B$ with $\underline{B} \models \theta(b_1, \ldots, b_n)$. Let $\underline{B}'$ be $\underline{B}$ enriched to an L_A-structure, with c_{a_i} interpreted as b_i. Then $\underline{B}' \models \Sigma \bigcup \theta(c_{a_1}, \ldots, c_{a_n})$, as required.

A class C of L-structures is called **axiomatizable** if there is a set Σ of L-sentences such that

$$\underline{A} \in C \text{ iff } \underline{A} \vDash \Sigma.$$

We also write equivalently

$$C = \text{Mod }(\Sigma).$$

Σ is then called **a set of axioms for** C.
C is called **finitely axiomatizable** iff there is a finite set Σ of axioms for C.
An axiomatizable class C is said to be **$\exists$-axiomatizable** ($\forall$-axiomatizable) if Σ can be chosen to consists of existential (universal) sentences only.

Definition 4.4. *A* theory *in a language L is satisfiable, deductively closed set of L-sentences.*

Any satisfiable set of sentences Σ determines a theory, namely the set of logical consequences of Σ. The point of the definition is that we wish to view Σ, Σ' as equivalent if each is among the logical consequences of the other.
Given a nonempty class of L-structures C, **the theory of** C is

$$\text{Th}(C) = \{\sigma : L\text{-sentence}, \underline{A} \vDash \sigma \text{ for all } \underline{A} \in C\}.$$

If C consists of a one structure $\underline{A}$ then we denote $\text{Th}(\underline{A})$ the theory of this class and call it **the theory of** $\underline{A}$.

Exercise 4.5. Show that $\text{Th}(C)$ is deductively closed, for every nonempty class C of L-structures; $\text{Th}(\underline{A})$ is complete, for every structure $\underline{A}$. But as soon as C contains two non-isomorphic structures, $Th(C)$ may not be complete.

A set of axioms for a theory T is a set of sentences E that have the same logical consequences as T. Equivalently, T is the theory of the class of models of E.
Example The theory of rings is the set of logical consequences, in the language $+, -, \cdot, 0, 1$, of the group law for $+, -, 0$, the semigroup law for $\cdot, 1$ (associativity and the unit property), and the two distributive laws.
Informally we often say that these axioms *are* the theory of rings. In practice we always work with sets of axioms and not with the full theory, simply identifying two sets of axioms if they have the same logical consequences.

Exercise 4.6. 1. The class of groups in the language with one binary function symbol $\cdot$, one unary function symbol $^{-1}$ (taking the inverse) and one constant symbol e is $\forall$-axiomatizable.
2. The class of finite groups is not axiomatizable.
3. The class of fields of characteristic zero is axiomatizable but not finitely axiomatizable.

Example. Let $L = \{+, -, \cdot, 0, 1\}$ be the language of rings. Let TF be the theory of fields.
Let T_{dom} be the theory of integral domains; this is the theory of commutative rings, with the additional axiom that there are no zero-divisors. Thus T_{dom} is given by some universal axioms.
In $TF_\forall = T_{dom}$.
To see this, let $M \models T_{dom}$. The *field of fractions* construction shows that M embeds in some field K. Since $K \models TF$, we have $M \models T_\forall$. Thus any model of T_{dom} is a model of $TF_\forall$. Conversely, any field is an integral domain.

Theorem 4.7. *Let C be an axiomatizable class. Then the following conditions are equivalent:*
(i) C is $\forall$-axiomatizable;
(ii) If $\underline{B} \in C$ and $\underline{A} \leq \underline{B}$ then $\underline{A} \in C$.

Proof (i) implies (ii) by Lemma 4.2(ii).
To prove the converse consider $\mathrm{Th}(C)$, the theory of class C, and $\mathrm{Th}_\forall(C)$, its universal part. Let $\underline{A} \models \mathrm{Th}_\forall(C)$. We need to show that $\underline{A} \in C$ which would yield Mod $(\mathrm{Th}_\forall(C)) =$ Mod $(\mathrm{Th}(C)) = C$, as required.

Claim. $\mathrm{Th}(C) \cup \mathrm{Th}_\exists(\underline{A})$ is finitely satisfiable.
Indeed, otherwise, $\mathrm{Th}(C) \models \neg\sigma_1 \vee \ldots \vee \neg\sigma_n$, for some $\sigma_1, \ldots, \sigma_n \in \mathrm{Th}_\exists(\underline{A})$.
Also $\neg\sigma_1 \vee \ldots \vee \neg\sigma_n \equiv \neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ and $\underline{A} \models \sigma_1 \wedge \ldots \wedge \sigma_n$. On the

other hand $\neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$ is equivalent to an $\forall$-formula, and is a logical consequence of $\mathrm{Th}(C)$. So $\underline{A} \models \neg(\sigma_1 \wedge \ldots \wedge \sigma_n)$, the contradiction. Claim proved.

It follows from the claim and Lemma 4.3 that $\mathrm{Th}(C) \cup \mathrm{Diag}(\underline{A})$ is satisfiable. Let $\underline{B}^+$ be a model of $\mathrm{Th}(C) \cup \mathrm{Diag}(\underline{A})$ and $\underline{B}$ its reduct to the initial language. In particular, $\underline{B} \in C$ and, by Theorem 3.1, $\underline{A} \leq \underline{B}$. It follows by assumptions that $\underline{A} \in C$. □

Note that we have shown, within the proof, that for any theory T, if $\underline{A} \models T_\forall$ then $\underline{A}$ embeds into a model of T.

Exercise 4.8. Let T be a theory, $\phi(x)$ a formula. Show that the following conditions are equivalent:

- ϕ is T-equivalent to a universal formula, i.e. there exists a universal formula ϕ' such that $T \models (\forall x)(\phi \leftrightarrow \phi')$.
- σ is preserved under passing to submodels of models of T, i.e. whenever $\underline{A}, \underline{B}$ are models of T with $\underline{A} \leq \underline{B}$, we have
$$\phi(A) \subseteq \phi(B)$$

(You may want to do this for sentences ϕ first; in this case, the preservation condition is that if $\underline{B} \models \phi$ then $\underline{A} \models \phi$.)

Exercise 4.9. Let C be an axiomatizable class. Then the following conditions are equivalent:
(i) C is $\exists$-axiomatizable;
(ii) If $\underline{A} \in C$ and $\underline{A} \leq \underline{B}$ then $\underline{B} \in C$.

Exercise 4.10. Let $\underline{A}$ be a finite structure.

1. Find $\sigma_1 \in Th(\underline{A})$ such that any model of σ_1 has universe of the same cardinality as $\underline{A}$.
2. Let $\underline{B} \models Th_\exists(\underline{A}) \bigcup \{\sigma_1\}$. If also $\underline{B} \models \sigma_1$, show that $\underline{B} \cong \underline{A}$. In particular, any model of $Th(\underline{A})$ is isomorphic to $\underline{A}$.
3. Show that any model of $Th_\forall(\underline{A}) + \sigma_1$ is isomorphic to $\underline{A}$.
4. Assume L has finitely many symbols. Find a single existential sentence σ_2 such that any model of $\{\sigma_1, \sigma_2\}$ is isomorphic to $\underline{A}$.

Hints: In (4), Suppose $|A| = 1$ and $L = \{R\}$ with R a unary relation symbol. Write σ_2 explicitly- there will be two possibilities. Continue with some special cases with $|A| = 2$ till you see the general case. (2,3) can be done directly, but you can also make use of Lemma 4.3: form Diag($\underline{A}$) in L_A and Diag($\underline{B}$) in L_B as usual, ensuring that L_A, L_B use disjoint sets of new constant symbols. Prove using the method of diagrams that $\underline{A} \cong \underline{B}$ if $\mathrm{Diag}(\underline{A}) \bigcup \mathrm{Diag}(\underline{B}) \bigcup \{\sigma_1\}$ is consistent. Then apply Lemma 4.3, and connect to the hypotheses of (2,3).

Definition Let

$$\underline{A}_0 \leq \underline{A}_1 \leq \ldots \leq \underline{A}_i \leq \ldots \tag{1}$$

be a sequence of L-structures, $i \in \mathbb{N}$, forming a chain with respect to embeddings.
Denote $\underline{A}^* = \bigcup_n \underline{A}_n$ the L-structure with:
the domain $A^* = \bigcup_n A_n$,
predicates $P^{\underline{A}^*} = \bigcup_n P^{\underline{A}_n}$, for each predicate symbol P of L,
operations $f^{\underline{A}^*} : (A^*)^m \to A^*$ sending $\bar{a}$ to b iff $\bar{a}$ is in A_n for some n and $f^{\underline{A}_n}(\bar{a}) = b$, for each function symbol f of L,
and $c^{\underline{A}^*} = c^{\underline{A}_0}$, for each constant symbol from L.

By definition $\underline{A}_n \leq \underline{A}^*$, for each n. The structure $\underline{A}^*$ will be called the *limit* of the chain. [8]
A formula equivalent to one of the form $\forall v_1 \ldots \forall v_m \exists v_{m+1} \ldots \exists v_{k+m} \theta$, where θ is a quantifier-free formula, is called an AE-formula.
The negation of an AE-formula is called an EA-formula.

Exercise 4.11. Given a chain of the form (1) and an AE-sentence σ assume that $\underline{A}_n \models \sigma$ for every $n \in \mathbb{N}$. Prove that

$$\underline{A}^* \models \sigma.$$

The sequence $\underline{A}_1 \preccurlyeq \underline{A}_2 \preccurlyeq \underline{A}_3 \preccurlyeq \cdots$ in the following exercise is referred to as an *elementary chain*.

Exercise 4.12. If, for each n, $\underline{A}_n \preccurlyeq \underline{A}_{n+1}$ then $\underline{A}_n \preccurlyeq \underline{A}^*$, for each n. (Hint: show first that $\underline{A}_n \preccurlyeq \underline{A}_{n'}$ when $n \leq n'$. Then show by induction on the complexity of a formula $\phi(x_1, \ldots, x_k)$ that for any n and any $a \in \underline{A}_n^k$, we have $\underline{A}^* \models \phi(a)$ iff $\underline{A}_n \models \phi(a)$. You may concentrate on the case where $\phi = (\exists y)\psi$.)

[8] In category theory this is referred to as a colimit.

Exercise 4.13. Assume $\underline{A}_1 \leq \underline{B} \leq \underline{A}_2$ and $\underline{A}_1 \prec \underline{A}_2$. Let σ be an EA-sentence. Show that if $\underline{A}_1 \models \sigma$ then $\underline{B} \models \sigma$.

Exercise 4.14. Assume $\underline{A}_1 \leq \underline{B}_1 \leq \underline{A}_2 \leq \underline{B}_2$ and $\underline{A}_1 \prec \underline{A}_2$, $\underline{B}_1 \prec \underline{B}_2$. Let σ be an EAE-sentence. If $\underline{A}_1 \models \sigma$, show that $\underline{B}_1 \models \sigma$.

Exercise 4.15. Formulas formed using $\forall, \exists, \wedge, \vee$ are called *positive*. Show that if $f : \underline{A} \to \underline{B}$ is a surjective homomorphism and ψ is a positive sentence,if $\underline{A} \models \psi$ then $\underline{B} \models \psi$.

We state without proof

Theorem 4.16. *Let C be an axiomatizable class. Then the following conditions are equivalent:*
(i) C is AE-axiomatizable;
(ii) For any chain of the form (1) with $\underline{A}_n \in C$ for all $n \in \mathbb{N}$, the union $\underline{A}^*$ is in C.
(The proof in Chang and Keisler of the above two theorems uses saturated models, which will be available to us later on.)

Similarly, C is axiomatizable by positive sentences if and only if it is preserved under homomorphic images.

4.17 Quantifier elimination

We say T *admits quantifier-elimination (QE)* if every formula ϕ is T-equivalent to a quantifier-free one. I.e. for any $x = (x_1, \ldots, x_n)$ and any formula $\phi(x)$ of L, there exists a quantifier-free formula $\phi'(x)$ such that $T \models (\forall x)(\phi \leftrightarrow \phi')$.

Example 4.18.
1. *$(\mathbb{R}, +, -, \cdot, 0, 1)$ does not admit QE: the set of non-negative numbers can be defined via $P(x) \equiv (\exists y)(y^2 = x)$, but it cannot be defined in a quantifier-free way.*
2. *$(\mathbb{R}, +, -, \cdot, \leq, 0, 1)$ does admit QE (Tarski).*
3. *For any finite set of relations on $\mathbb{N}$, the structure $(\mathbb{N}, +, \cdot, R_1, \ldots, R_k)$ does not admit QE (Gödel.)*

Exercise 4.19. *T admits QE if and only if for any quantifier-free formula $\phi(x, y_1, \ldots, y_n)$ of L, there exists a quantifier-free formula $\psi(y_1, \ldots, y_n)$ such that*

$$T \models (\forall y_1, \ldots, y_n)(\psi \iff (\exists x)\phi)$$

Let $\underline{M} \models T$, and $A \subset M$. Recall $L_A, \underline{M}_A$. We write $Diag_{\underline{M}}(A)$ for the set of quantifier-free L_A-sentences true in $\underline{M}_A$.
We say that a set of sentences Σ is complete when it is consistent and implies a complete theory, i.e. for any sentence ϕ in the given language, $\Sigma \models \phi$ or $\Sigma \models \neg\phi$.

Theorem 4.20. *Assume $T \bigcup Diag_M(B)$ is complete for any $M \models T$ and finite subset B of the universe of M. Then T admits QE.*

Before beginning the proof, let us dispose of a technical point.The assumption asserts the completeness of the quantifier-free theory of M_B, when a finite number of constants has been added to the language in order to name the elements of B. In the definition of L_B, we do not specify the identity of the new constants; it does not matter what constants we use, as long as they are not in the original language L, and distinct. But what if we use more constants than necessary, say using two distinct constants c_1, c_2 to name the same element b? Write M_1 for the expansion of M to $L_1 = L \bigcup \{c_1\}$ with c_1 interpreted as b, and M_{12} for the expansion of M to $L_{12} = L \bigcup \{c_1, c_2\}$ with c_1, c_2 both interpreted as b. If $\psi(c_1, c_2)$ holds in M_{12}, then $\psi(c_1, c_1)$ holds in M_1; and $\psi(c_1, c_2)$ follows logically from $\psi(c_1, c_1) \& (c_1 = c_2)$. In particular, adding a redundant name does not compromise the completeness. This shows:
Claim. The hypothesis of Theorem 4.20 can be restated thus: let L' be a language containing the symbols of L and in addition, constant symbols $c_1, \ldots, c_n$; let $\underline{A} \models T$; and let $\underline{A}'$ be an expansion of $\underline{A}$ to L'. Then $T \bigcup Th_{qf}(\underline{A}')$ is complete.

Proof. of Theorem 4.20:
Let $\phi(x)$ be a formula, and let $c = (c_1, \ldots, c_n)$ be new constant symbols corresponding to $x = (x_1, \ldots, x_n)$. we have to show that for some quantifier-free formula $\phi'(x)$ of L

$$T \models (\forall x)(\phi(x) \iff \phi'(x))$$

Equivalently,

$$T \models \phi(c) \leftrightarrow \phi'(c)$$

Let Q be the set of quantifier-free sentences $\phi'(c)$ of L' such that $T \models \phi(c) \rightarrow \phi'(c)$.
Claim. $T \bigcup Q \models \phi(c)$.

Proof. Otherwise, there exists $\underline{A}' \models T \bigcup Q \bigcup \{\neg\phi(c)\}$; let $\underline{A}$ be the reduct of $\underline{A}'$ to L, and let $a = c^{\underline{A}'} := (c_1^{\underline{A}'}, \cdots, c_n^{\underline{A}'})$. Let D_a be the set of qf L'-sentences $\psi(c)$ true in $(\underline{A}, a)$. By assumption, $T \bigcup D_a$ is complete; since $T \bigcup D_a \bigcup \{\neg\phi(c)\}$ is consistent (being part of $Th(\underline{A}, a)$), it must be that $T \bigcup D_a \vdash \neg\phi(c)$. By compactness it follows that $T \bigcup \{\sigma_1, \ldots, \sigma_m\} \vdash \neg\phi(c)$ for some finite subset $\sigma_1, \ldots, \sigma_m$ of D_a. Let $\sigma = \bigwedge_{i=1}^m \sigma_i$; then $\sigma \in D_a$, and $T \cup \{\sigma\} \vdash \neg\phi(c)$. Taking the contrapositive, we have $T \cup \{\phi(c)\} \vdash \neg\sigma$; so $\neg\sigma \in Q$. But $\underline{A}' \models Q$, so $\underline{A}' \models \neg\sigma$, contradicting the definition of D_a and the fact that $\sigma \in D_a$. This contradiction shows that $T \bigcup Q \models \phi(c)$. $\square$

By the compactness theorem, $T \bigcup Q_0 \models \phi(c)$ for some finite $Q_0 \subset Q$; consider the conjunction of all sentences in Q_0; it is a sentence of L', that can be written as $\phi'(c)$ for some formula $\phi'(x)$ of L. We have $T \vdash \phi'(c) \to \phi(c)$; by definition of Q we have also $T \vdash \phi(c) \to \phi'(c)$; so $T \vdash \phi(c) \leftrightarrow \phi'(c)$. $\square$

Lemma 4.21. *Let $M \models T$, and let M_0 be the minimal substructure of M; see Definition 1.18. If $T \bigcup Diag(M_0)$ is complete, then so is $T \bigcup Diag_M(\emptyset)$.*

Proof. Let $N \models T \bigcup Diag(\emptyset)$. We have to show that $Th(M) = Th(N)$. Let N_0 be the minimal substructure of N. By Exercise 1.25 (applied to $\Sigma = Diag(\emptyset)$), M_0 and N_0 are isomorphic; say $f : M_0 \to N_0$ is an isomorphism. For each $a \in M_0$, let c_a be a new constant symbol; let $L' = L \bigcup \{c_a : a \in M_0\}$; let M' be the result of interpreting c_a as a, and N' the result of interpreting c_a as $f(a)$. Then $M' \models T \bigcup Diag(M_0)$, and $N' \models T \bigcup Diag(M_0)$. By assumption, $Th(M') = Th(N')$. A fortiori, $Th(M) = Th(N)$. $\square$

Let $M \models T$, and $A \subset M$, as in Theorem 4.20. The *substructure generated by* A, denoted $< A >$, is by definition the substructure of M, whose universe is

$$\{t(a_1, \ldots, a_n) : n \in \mathbb{N}, a_1, \ldots, a_n \in A, t \in \mathrm{Term}_n\}$$

where Term_n denotes the set of terms $t = t(x_1, \ldots, x_n)$ of the language. This is the smallest substructure of M containing A. It can also be described as the L-reduct of the minimal L_A-structure of M_A, see Definition 1.18.
A substructure of M is called *finitely generated* if it is generated by a finite (or empty) subset of M.

Exercise 4.22. *Let T be a theory, $M \models T$, and $A \subset M$. Assume*

$$T \bigcup Diag(< A >)$$

is complete. Then $T \bigcup Diag_M(A)$ is complete. (Hint: the case $A = \emptyset$ is Lemma 4.21. You can either reduce to this case by moving to the language L_A, or follow the proof of the lemma while modifying as needed; or give a direct proof.) Deduce Corollary 4.23.

Corollary 4.23. *Assume $T \bigcup Diag(\underline{A})$ is complete, for any $M \models T$ and any finitely generated substructure $\underline{A} \leq M$. Then T admits QE.*

Remark Note that $Diag(\underline{A})$ (and even $Diag(\emptyset)$) includes the quantifier-free sentences true in $\underline{A}$. For instance if T is a theory of fields, then $T_{\underline{A}}$ will include the sentence $1 + 1 = 0$ or its negation; so $T_{\underline{A}}$ will determine whether the characteristic is 2 or otherwise (and similarly for every other prime.) Quantifier-elimination implies that every sentence is equivalent to a quantifier-free sentence, but this does not imply that the theory is complete.
Definition Let T_0 be a universal theory in a language L. We say that a theory T in L is a *model completion* of T_0 if T admits quantifier elimination, and has universal part T_0 (i.e. $T_0 \models T_\forall, T\forall \models T_0$).
The next theorem shows that a universal theory T_0 admits at most one model completion.

Theorem 4.24. *Assume T, T' are theories of L that admit quantifier elimination, and with $T_\forall = T'_\forall$. Then $T = T'$.*

Proof. We have to prove that $Mod(T) = Mod(T')$; by the symmetry between T and T', it suffices to prove that $Mod(T) \subseteq Mod(T')$. Let $\underline{A}_1 \in Mod(T)$; we will show that $\underline{A}_1 \in Mod(T')$. Define inductively $\underline{A}_k \in Mod(T)$ and $\underline{B}_k \in Mod(T)$, as follows. Assume $\underline{A}_k$ has been defined. Since $\underline{A}_k \models T_\forall = T'_\forall$, there exists $\underline{B} \in Mod(T')$ with $A_k \leq \underline{B}$. (See remark after the proof of Theorem 4.7.) Let $\underline{B}_k$ be such a $\underline{B}$. Now since $\underline{B}_k \models T'_\forall = T_\forall$, there exists $\underline{A} \models T$ with $\underline{B}_k \leq \underline{A}$; let $\underline{A}_{k+1} = \underline{B}_k$. In this way we defined inductively $\underline{A}_k, \underline{B}_k$ with

$$\underline{A}_1 \leq \underline{B}_1 \leq \underline{A}_2 \leq \cdots$$

Now since all formulas are T-equivalent to quantifier-free ones, for models of T there is no difference between *embeddings* and *elementary embeddings*; so the $\underline{A}_i$ form an elementary chain.

$$\underline{A}_1 \prec \underline{A}_2 \prec \cdots$$

Similarly

$$\underline{B}_1 \prec \underline{B}_2 \prec \cdots$$

Let $\underline{A}$ be the limit structure of the $\underline{A}_i$-chain, and $\underline{B}$ the limit structure of the $\underline{B}_i$-chain (as in 1.) Then by Exercise 4.12 we have $\underline{A}_1 \prec \underline{A}$ and $\underline{B}_1 \prec \underline{B}$. But $\underline{A} = \underline{B}$. So $Th(\underline{A}_1) = Th(\underline{A}) = Th(\underline{B}) = Th(\underline{B}_1)$ and in particular, $\underline{A}_1 \models T'$.

$\square$

5 Categoricity

A theory T is said to be **categorical in power** κ (κ-categorical) if there is a model $\underline{A}$ of T of cardinality κ and any model of T of this cardinality is isomorphic to $\underline{A}$.

At this stage, categoricity will serve us to prove completeness of certain significant theories, and thus develop a small repertoire of comprehensible theories. Though categoricity is much stronger than completeness, it is purely semantical and sometimes easier to verify. Later on, with saturated models, a similar completeness test can be described which is general enough to apply to any theory.

Theorem 5.1 (Łoś-Vaught Test)**.** *Let T be a theory with no finite models. Let $\kappa \geq |L|$ be a cardinal. If T is κ-categorical, then T is complete.*

Proof Let σ be an L-sentence and $\underline{A}$ the unique, up to isomorphism, model of T of cardinality κ. The either σ or $\neg\sigma$ holds in $\underline{A}$, let it be σ. Then $T \cup \{\neg\sigma\}$ does not have a model of cardinality κ, which by the Lowenheim-Skolem theorems means $T \cup \{\neg\sigma\}$ does not have an infinite model, which by our assumption means it is not satisfiable. It follows that $T \vDash \sigma$. □

Example 0 The *language of pure equality* $L_=$ has no non-logical symbols (we view the equality symbol as a logical symbol.) The *theory of pure equality* is axiomatised by $\emptyset$. This theory is categorical in every power. Indeed, any set A determines a model $\underline{A} = \langle A \rangle$; any bijection is an L-isomorphism; hence by definition of cardinality, any model of the same cardinality is isomorphic to $\underline{A}$.
Note that $T_=$ is not complete; however the theory T_∞ asserting that the model is infinite, is complete by the Łoś–Vaught test.

5.2 Vector spaces

Example 1 Let K be a field (or division ring) and L_K be the language with alphabet $\{+, \lambda_k, 0\}_{k \in K}$ where $+$ is a symbol of a binary function and λ_k symbols of unary functions, 0 constant symbol. Define $Vect_K$ to be the theory of vector spaces over K, i.e. $Vect_K$ is axiomatised by:
$\forall v_1 \forall v_2 \forall v_3 \;\; (v_1 + v_2) + v_3 \doteq v_1 + (v_2 + v_3);$
$\forall v_1 \forall v_2 \;\; v_1 + v_2 \doteq v_2 + v_1;$

$\forall v \;\; v + 0 \simeq v;$
$\forall v_1 \exists v_2 \;\; v_1 + v_2 \simeq 0;$
$\forall v_1 \forall v_2 \;\; \lambda_k(v_1 + v_2) \simeq \lambda_k(v_1) + \lambda_k(v_2)$ an axiom for each $k \in K$;
$\forall v \;\; \lambda_1(v) \simeq v;$
$\forall v \;\; \lambda_0(v) \simeq 0;$
$\forall v \;\; \lambda_{k_1}(\lambda_{k_2}(v)) \simeq \lambda_{k_1 \cdot k_2}(v)$ an axiom for each $k_1, k_2 \in K$;
$\forall v \;\; \lambda_{k_1}(v) + \lambda_{k_2}(v) \simeq \lambda_{k_1 + k_2}(v)$ an axiom for each $k_1, k_2 \in K$.

Mod $Vect_K$ is exactly the class of vector spaces over K.
To discuss the theory further let us recall the basic facts and definitions of the theory of vector spaces.
A **basis of a vector space** $\underline{A}$ is a maximal linearly independent subset of $\underline{A}$. By Zorn's Lemma any independent subset can be extended to a basis, so a basis exists in any vector space (and in general can be infinite).
If B_1 and B_2 are bases of the same vector space, then card B_1 = card B_2.
This allows to define **the dimension of a vector space** to be the cardinality of a basis of the vector space.
If B_1 is a basis of $\underline{A}_1$ and B_2 a basis of $\underline{A}_2$, vector spaces over K, and $\pi : B_1 \to B_2$ a bijection, then π can be extended in a unique way (linearly) to an isomorphism between the vector spaces. In other words the isomorphism type of a vector space over a given field is determined by its dimension.
Let $\underline{A}$ be a model of $Vect_K$ of cardinality $\kappa > |L_K| = \max\{\aleph_0, \text{card } K\}$. Then $||\underline{A}|| = \dim \underline{A}$, the dimension of the vector space (check it). It follows that, if $\underline{B}$ is another model of $Vect_K$ of the same cardinality, $\underline{A} \cong \underline{B}$. Thus we have checked the validity of the following statement.

Theorem 5.3. *$Vect_K$ is categorical in any infinite power $\kappa >$ card K.*

Recall the set of sentences T_∞ of $L_=$, asserting that there exist infinitely many distinct elements.
Using the Los-Vaught text we obtain:

Corollary 5.4. *$Vect_K \bigcup T_\infty$ is complete.*

Exercise 5.5. Show that if U, V are models of $Vect_K$ of equal cardinality $> \max(\aleph_0, |K|)$, and $\underline{A}$ is a common subspace of U, V with $\underline{A}$ finite-dimensional, then there exists an isomorphism $f : U \to V$ which fixes the points of A. (Hint: choose a basis I_0 of $\underline{A}$, and extend it to bases I of U and J of V. Find a bijection $f : I \to J$ with $f(x) = x$ for $x \in I_0$; extend f to an isomorphism

$U \to V$.) Conclude using Corollary 4.23 that $Vect_K \bigcup T_\infty$ admits quantifier-elimination.

5.6 Dense linear order

Example 2 Let L be the language with one binary symbol $<$ and DLO be the theory of dense linear order with no end elements:
$\forall v_1 \forall v_2 \ \ (v_1 < v_2 \to \neg\ v_2 < v_1)$;
$\forall v_1 \forall v_2 \ \ (v_1 < v_2 \vee v_1 \doteq v_2 \vee v_2 < v_1)$
$\forall v_1 \forall v_2 \forall v_3 \ (v_1 < v_2 \wedge v_2 < v_3) \to v_1 < v_3$;
$\forall v_1 \forall v_2 \ \ (v_1 < v_2 \to \exists v_3 \ (v_1 < v_3 \wedge v_3 < v_2))$;
$\forall v_1 \exists v_2 \exists v_3 \ \ v_1 < v_2 \wedge v_3 < v_1$.

Cantor's Theorem *Any two countable models of* DLO *are isomorphic. In other words* DLO *is* $\aleph_0$*-categorical.*

To prove that any two countable models of DLO are isomorphic we enumerate the two ordered sets and then apply the *back-and-forth construction* of a bijection preserving the orders. Compare the proof of Proposition 7.2.
Proof Let $\underline{A}, \underline{B}$ be countable models of DLO. Enumerate

$$A = \{a_1, a_2, \ldots\}, \quad B = \{b_1, b_2, \ldots\}.$$

We will construct inductively new enumerations $\{a'_1, a'_2, \ldots\}$ and $\{b'_1, b'_2, \ldots\}$ of the sets so that the correspondence $a'_i \mapsto b'_i$ is bijective, and indeed an isomorphism.
Suppose $a'_1, \ldots, a'_{n-1} \in A$ and $b'_1, \ldots, b'_{n-1} \in B$ have been defined, with a'_i distinct elements of A and b'_j distinct elements of B, and the correspondence $a'_i \mapsto b'_i$ is order-preserving; in other words, for $i, j < n$,
(*) $a'_i < a'_j$ iff $b'_i < b'_j$
We define a'_n and b'_n. Assume first that n is odd and a'_n be the first element $A = \{a_1, a_2, \ldots\}$ not occurring among $a'_1, \ldots a'_{n-1}$.; i.e. $a'_n = a_m$ with $a_m \notin \{a'_1, \ldots, a'_{n-1}\}$ and m least such.
We finally take into account the ordering we really care about, namely the order $<^{\underline{A}}$; in this ordering, either $a'_n < a'_i$ for all $i < n$, or $a'_n > a'_i$ for all $i < n$, or there exist $l, r < n$ such that $a'_i < a'_n$ iff $a'_i \leq a'_l$ and $a'_n < a'_i$ iff $a'_r \leq a'_i$. Choose $b'_n \in B$ such that $b'_l < b'_n < b'_r$ (and similarly in the first two cases, choose b'_n below or above all elements $b'_i, i < n$, respectively.) Note now that (*) continues to hold for $i \leq n$.

Similarly, when n is even, let b'_n be the first element in $B = \{b_1, b_2, \ldots\}$ not occurring among $b'_1, \ldots b'_n$. Then find $a'_{n+1} \in A$ such that (*) continues to hold.
Hence we may inductively construct in this way $A = \{a'_1, a'_2, \ldots a'_n \ldots\}$, $B = \{b'_1, b'_2, \ldots b'_n \ldots\}$ satisfying (2) for all n.
Claim: Every element of A occurs as some a'_i.
Proof: We show by induction on k that a_k occurs in this way. We may thus assume that each $a_l, l < k$ does occur as some $a'_{i(l)}$; let n be an odd integer greater than each $i(l)$, $l < k$. At stage n of the construction, if $a_k = a'_i$ for some $i < n$, we are done. If not, then by construction we have $a'_n = a_k$, and again we are done.
Similarly, every element of B is some b'_j. Hence $a'_i \mapsto b'_j$ is bijective. By (*), it is an isomorphism.

□

Proposition For any finite linear ordering $\mathcal{C}$, DLO $\bigcup$ $Diag(\mathcal{C})$ is $\aleph_0$-categorical.
We offer two proofs.
Proof 1: Repeat the proof of Cantor's theorem, but define $a'_i = c_i^A$ and $b'_i = c_i^B$ where $c_1, \ldots, c_n$ are the new constant symbols of $L_{\mathbb{C}}$, and continue the recursive definition at stage $n+1$.
Proof 2: The universe of $\mathcal{C}$ has a finite number n of elements, $a_1, \ldots, a_n$; we can number them in such a way that $a_1 < \cdots < a_n$. The language of DLO $\bigcup$ $Diag(\mathcal{C})$ has n constant symbols, $c_1, \ldots, c_n$; and $Diag(\mathcal{C})$ includes the sentences $c_1 < c_2, \cdots, c_{n-1} < c_n$. We will show that any model is isomorphic to $(\mathbb{Q}, 1, 2, \cdots, n)$ where c_i is interpreted by i.
Let $\underline{A} = (A, <, b_1, \ldots, b_n) \models$ DLO $\bigcup$ $Diag(\mathcal{C})$ be an arbitrary countable model. Then $(A, <) \models DLO$. By Cantor's theorem there exists an isomorphism $f : (A, <) \to (\mathbb{Q}, <)$. Define $\underline{B} = (\mathbb{Q}, <, f(b_1), \ldots, f(b_n))$. Then $\underline{A} \cong \underline{B}$. Thus it suffices to prove that $\underline{B} \cong (\mathbb{Q}, 1, 2, \cdots, n)$. This can be seen by an explicit isomorphism, defined so as to take the segment $[i, i+1]$ to the segment $[b_i, b_{i+1}]$ in an order preserving way. For instance, define $g : \mathbb{Q} \to \mathbb{Q}$ by: $g(x) = x + b_1$ for $x \leq 0$; $g(x) = (b_2 - b_1)x + b_1$ for $0 < x \leq 1$; $g(x) = (b_3 - b_2)(x-1) + b_2$; etc.

Theorem 5.7. *DLO is complete and admits QE.*

Proof. By Cantor's theorem, the above Proposition, and Corollary 4.23. □

Exercise 5.8. Show that DLO is not κ-categorical, where $\kappa = 2^{\aleph_0}$. (Hint: let $\underline{A} = (\mathbb{R}, <)$, let $\underline{B}$ be a countable model of DLO with universe disjoint from A, and define a linear ordering on $C = A \bigcup B$ so that $\underline{A} \leq \underline{C}, \underline{B} \leq \underline{C}$, and any element of A is $<$ any element of B. Show that for some $c \in C$, the interval (c, ∞) is countable. Now find a linear ordering of the same cardinality without this property.)

5.9 Algebraically closed fields

Example 3 ACF, the theory of *algebraically closed fields* is given by the following axioms in the language of fields L_{fields} with binary operations $+$, $\cdot$ and constant symbols 0 and 1 :

Axioms of fields:
$\forall v_1 \forall v_2 \forall v_3$

$$\begin{aligned}
&(v_1 + v_2) + v_3 \doteq v_1 + (v_2 + v_3) \\
&(v_1 \cdot v_2) \cdot v_3 \doteq v_1 \cdot (v_2 \cdot v_3) \\
&v_1 + v_2 \doteq v_2 + v_1 \\
&v_1 \cdot v_2 \doteq v_2 \cdot v_1 \\
&(v_1 + v_2) \cdot v_3 \doteq v_1 \cdot v_3 + v_2 \cdot v_3 \\
&v_1 + 0 \doteq v_1 \\
&v_1 \cdot 1 \doteq v_1.
\end{aligned}$$

$\forall v_1 \exists v_2 \;\; v_1 + v_2 \doteq 0$
$\forall v_1 (\neg v_1 \doteq 0 \rightarrow \exists v_2 \; v_1 \cdot v_2 \doteq 1)$.

Solvability of polynomial equations axioms, one for each positive integer n :

$$\forall v_1 \ldots \forall v_n \exists v \;\; v^n + v_1 \cdot v^{n-1} + \ldots + v_i \cdot v^i + \ldots + v_n \doteq 0.$$

Basic facts and definitions of dimension theory in algebraically closed fields We give below a rapid survey of it. The definition of *independence* differs from the case of vector spaces in that nonlinear polynomials are used; beyond this, the theories are very similar.

We consider a field F with a subring A. (When A is the prime field, i.e. $\mathbb{Q}$ or $\mathbb{F}_p$, it need not be mentioned.)
A finite subset $\{a_1, \ldots, a_n\}$ of F is said to be **algebraically independent** over A if, for any nonzero polynomial in n variables $P(v_1, \ldots, v_n) \in A[v_1, \ldots, v_n]$

$$P(a_1, \ldots, a_n) \neq 0.$$

When $n = 1$, we say that a_1 is *transcendental* over A.
A **transcendence basis of a field** F over A is a maximal algebraically independent subset of F.
By Zorn's Lemma any independent subset can be extended to a basis, so a basis exists in any field.
Proposition If B_1 and B_2 are bases of the same field, then card B_1 = card B_2..
This is proved in a way analogous to the case of vector spaces. In fact it is possible to abstract a notion of a *dependence relation* and prove that linear and algebraic dependence are examples, and that any dependence relation has bases of constant cardinality. (Steinitz.) The main property that needs to be proved, in this formuation, is *transitivity of dependence*. Let us say that $a \in F$ *depends on* a set B over A if for some $k \geq 0$ and some algebraiclly independent $b_1, \ldots, b_k \in B$, the set $a, b_1, \ldots, b_k$ is *not* independent. Transitivity is the assertion that if a depends on $B \bigcup \{c\}$ and c depends on B, then a depends on B. For instance if b, c are each transcendental over A, and $P(a, b) = 0, P'(b, c) = 0$ for some nonzero polynomials P, P', then $P''(a, c) = 0$ for some nonzero polynomial P''.
The above proposition allows to define **the transcendence degree of a field** F over A to be the cardinality of any transcendence basis of F over A; it denoted $\mathrm{tr.d.}_A F$. .

Let K_1, K_2 be algebraically closed fields, each containing the subring A. For $i = 1, 2$, let B_i be a transcendence basis for K_i over A. If $\pi : B_1 \to B_2$ is a bijection, then π can be extended to a field isomorphism $K_1 \to K_2$, fixing A.

In particular, taking A to be the prime field, the isomorphism type of an algebraically closed field of a given characteristic is determined by its transcendence degree.
As in the case of vector spaces, if $|F|$ is uncountable and $|F| > |A|$, then $|F| = tr.d._A(F)$. We thus have:

Proposition For any countable integral domain A, and any uncountable cardinal λ, ACF_A is categorical in power λ.

By Corollary 4.23,

Corollary 5.10. *ACF admits quantifier-elimination.*

We can complement ACF by axioms stating that the field is of characteristic zero, one for each positive integer n :; the result is called ACF_0.

$$\neg \ (\underbrace{1 + \ldots + 1}_{n} \doteq 0),$$

Similarly, for any prime p we can add an axiom $p \doteq 0$, where p is represented by a term of the form $1 + \ldots + 1$; we obtain a theory ACF$_p$.
It follows that, if F_1 and F_2 are two models of ACF$_0$ of an uncountable cardinality κ, then $F_1 \cong F_2$. Thus ACF$_0$ is categorical in any such power κ. Likewise for ACF_p.

Any field F of characteristic zero contains a copy of rational numbers $\mathbb{Q}$. Indeed,

$$\underbrace{1^F + \ldots + 1^F}_{n} \in F,$$

is an element representing integer n, denote it n^F. Then the additive inverse of n^F represents $-n$, and correspondingly we can represent n^{-1} and in general any rational number m/n by a unique element of F. So we may just assume $\mathbb{Q} \subseteq F$.

Similarly, a field of positive characteristic $p > 0$ contains the p-element field $\mathbb{F}_p$.
From the Łoŝ-Vaught test we obtain:

Corollary 5.11. *ACF_p is complete (for any $p = 2, 3, 5, \cdots$ or $p = 0$.)*

Using Theorem 4.20, or Corollary 4.23, we conclude

Corollary 5.12. *The theories $DLO, Vect_F, ACF$ admit QE.*

Example The theory of successor, T_S.
The language contains a unary function symbol s and a constant symbol 0. The axioms are:

(a) $\forall v_1 \forall v_2 \ (s(v_1) \doteq s(v_2) \to v_1 \doteq v_2)$;
(b) $\forall v_1 \exists v_2 \ (\neg v_1 \doteq 0 \to v_1 \doteq s(v_2))$;
(c)$_n$ $\forall v \ \neg \ s^n(v) \doteq v$ for any positive integer n, where $s^n(v) = s(...(s(v))...)$, n times;
(d) $\forall v \ \neg s(v) \doteq 0$.

Exercise 5.13. 1. Prove that the theory T_S is categorical in all uncountable cardinalities. (Hint: show that any model of T_S is the disjoint union of an isomorphic copy of $(\mathbb{N}, S)$, and a number of copies of $(\mathbb{Z}, S)$. It is rare that a structure can be written as the disjoint union of proper substructures, but this is the case here.)

2. Show that T_S has QE.

3. (extra credit.) Let L'_S be the language consisting of the S alone, without 0, and let T'_S have axioms (a), $(c)_n$ and an axiom (b') stating that the image of S consists of all but one element. Show that T'_S is a categorical theory, but does not admit QE. (Hint: show that the element 0 of $\mathbb{N}$ is definable using a universal formula, but cannot be defined using an existential formula: there is an extension $\underline{A}$ where 0 is the successor of something.)

Definition A subset X of M is called *cofinite* (in M) if $M \setminus X$ is finite.

Exercise 5.14. Let $L = \{+, \cdot, 0, 1, -\}$ be the language of rings, and let TF be the theory of fields.

1. Let $t(x_1, \ldots, x_n)$ be a term of L. Show that there exists a polynomial $F(x_1, \ldots, x_n)$ over $\mathbb{Z}$ in the same variables, such that $TF \models (\forall x)(t(x_1, \ldots, x_n) = F(x_1, \ldots, x_n))$.

2. $\phi(x)$ be a basic formula in one variable x; so $\phi(x)$ has the form $t_1(x) = t_2(x)$. Let $M \models T$. Show that ϕ^M is either finite, or equal to all of M.

3. Let $\phi(x)$ be a quantifier-free formula in one variable x. Let $M \models T$. Show that $\phi(M)$ is either finite, or cofinite. Conclude that any definable subset of the field $\mathbb{C}$ is finite or cofinite. (You may assume Gauss's theorem that $\mathbb{C} \models ACF$.)

4. Deduce from (2) that $\mathbb{R}$ does *not* have QE. (Optional, more difficult: if an infinite field K has QE, then every element of K is a square.)

5. Let $\phi(x)$ be a quantifier-free formula in the language $L = \{+, \cdot, <, 0, 1, -\}$. Let OF be the theory of *ordered* fields: add to the theory of fields the axioms asserting that $<$ is a linear ordering, and $x < y$, implies $x + u < y + u$, $-y < -x$, and if $u > 0$ also $ux < uy$. Follow the above steps to show that for any quantifier-free ϕ, $\phi^{\mathbb{R}}$ is a finite

union of intervals (open,closed and half-open, and including $(-\infty, \infty)$ $(-\infty, a)$, and (a, ∞).)

6. Assume Tarski's theorem, that the theory of the real field has QE. Show that any definable subset of the field $\mathbb{R}$ is a finite union of intervals.

Remark. The property in (6) is called *o-minimality*. A theory T such that (3) holds in every model of T is called *strongly minimal*. Both are very interesting, and intensively studied classes of theories.

Exercise 5.15. Show that $Th(\mathbb{N}, <)$ does not admit QE.

Exercise 5.16. Let $K \models ACF$. Let $f_1, \ldots, f_k, g \in K[X_1, \ldots, X_n]$ be polynomials in n variables with coefficients in M. Assume: in some field $L \geq K$, there exists $a = (a_1, \ldots, a_n)$ with $f_1(a) = \cdots = f_k(a) = 0$ and $g(a) \neq 0$. Show that such an n-tuple exists in K^n. (This is a form of Hilbert's (1893) Nullstellensatz. You may use the fact that any field extends to an algebraically closed field.)

Exercise 5.17. [Disjoint unary predicates]

1. For $n \geq 1$, let L_n be the language with n unary predicate symbols $P_1, \ldots, P_n$. Show there exists a theory T_n asserting that each P_i is infinite, the P_i are disjoint $(\neg(\exists x)(P_1(x) \& P_2(x))$, etc.) and there are infinitely many elements not in any P_i, $i \leq n$.

2. Show T_n is $\aleph_0$-categorical. Conclude that T_n is complete.

3. Let $T = \bigcup_n T_n$. Show that T is a complete theory.

4. How many models of cardinality $\aleph_1$ (up to isomorphism) does T_2 have?

5. Is T $\aleph_0$-categorical? If not how many countable models does T have?

Remark. One of the earliest QE results is that $Th((\mathbb{R}, +, <))$ admits quantifier eliminiation. See the wikipedia entry on Fourier-Motzkin elimination. This cannot be proved by the method used in this section, since the theory is not categorical. It can however be proved by a generalization of this method using *saturated models* that we will encounter later on. The particular case of $Th((\mathbb{R}, +, <))$ is also simple to prove directly 'by hand'.

Exercise 5.18. 1. Prove the Lefschetz principle: a sentence ϕ is true in the field $\mathbb{C}$ iff it is true in the field $\mathbb{F}_p^a$ for all but finitely many primes p. Here $\mathbb{F}_p^a$ denotes the algebraic closure of the p-element field; it is a model of ACF_p.

2. Show that any finite field has the *Ax property:* any injective polynomial map $K^n \to K^n$ is surjective.

3. Show that $\mathbb{F}_p^a$ has the Ax property. (You may assume $\mathbb{F}_p^a$ is the union of finite subfields.)

4. Show that $\mathbb{C}$ has the Ax property.

6 Types

We fix a language L, a theory T in L, and set of variables $x = (x_1, \ldots, x_n)$.
A set of formulas P with free variables among $\{x_1, \ldots, x_n\}$ is *satisfiable* if there exists a structure $\underline{A}$ and $a_1, \ldots, a_n \in \underline{A}$ such that $\underline{A} \models \phi(a_1, \ldots, a_n)$ for all $\phi \in P$. It is *satisfiable in a model of* T if $\underline{A}$ can be taken to be a model of T, equivalently if $P \bigcup T$ is satisfiable.
If $c = (c_1, \ldots, c_n)$ are new constant symbols, and $P' = \{\phi(c) : \phi \in P\}$, then P is satisfiable iff P' is satisfiable.
Definition A *partial type* of a theory T in variables x is a satisfiable set P of formulas in the variables x, containing T and closed under logical deduction.
By abuse of notation, we will sometimes say that a set A of formulas *is* a partial type when we mean that A *generates* a partial type, namely the deductive closure of $T \bigcup A$. The only requirement is thus that $T \bigcup A$ be satisfiable.
The main practical benefit of being deductively closed is being closed under conjunctions; i.e. if $\varphi, \psi \in p$ then $(\varphi \wedge \psi) \in p$.. This often allows us to use a single formula from P where otherwise we would need finitely many.

Exercise 6.1. Let T be a complete theory. Let P be a set of formulas, closed under conjunctions. Show that P is satisfiable in a model of T iff

$$(*) \text{ for all } \varphi \in p,\ T \models \exists \bar{v} \varphi(\bar{v}).$$

Is closure under conjunctions necessary?

Definition A *type* p in the set of variables x, for the theory T, is a partial type for T satisfying: for any formula $\varphi(x)$, either $\varphi \in p$ or $\neg\varphi \in p$.

For emphasis, we sometimes say *complete* type in place of just *type*.
A type p in $v_1, \ldots, v_n$ will also be called an n-type.
Suppose $\bar{a} \in A^n$. Then we define **the type of $\bar{a}$ in $\underline{A}$**.

$$\mathrm{tp}_{\underline{A}}(\bar{a}) = \{\varphi(x) : \ \underline{A} \models \varphi(\bar{a})\}.$$

Clearly, $\mathrm{tp}_{\underline{A}}(\bar{a})$ is a complete n-type.

When $\underline{A} \leq \underline{B}$ then $\mathrm{tp}_{\underline{A}}(a)$ and $\mathrm{tp}_{\underline{B}}(a)$ may be different. But it follows immediately from definitions that

$$\underline{A} \preccurlyeq \underline{B} \text{ implies } \mathrm{tp}_{\underline{A}}(a) = \mathrm{tp}_{\underline{B}}(a).$$

We say that a partial type P is **realised** in $\underline{A}$ if there is $\bar{a} \in A^n$ such that $P \subseteq \mathrm{tp}_{\underline{A}}(\bar{a})$. When $P = p$ is a complete type, this can also be stated as: $p = \mathrm{tp}_{\underline{A}}(\bar{a})$.
If there is no such $\bar{a}$ in $\underline{A}$ we say that p is **omitted** in $\underline{A}$.
Remark If $\pi : \underline{A} \to \underline{B}$ is an isomorphism, $\bar{a} \in A^n$, $\bar{b} \in B^n$, and $\pi : \bar{a} \to \bar{b}$ then $\mathrm{tp}_{\underline{A}}(\bar{a}) = \mathrm{tp}_{\underline{B}}(\bar{b})$.

In particular, if σ is an automorphism of $\underline{A}$, then for any $a \in A$, $a, \sigma(a)$ realise the same type.

Exercise 6.2. Let T be the theory of infinitely many disjoint infinite unary predicates, described in Exercise 5.17 . Determine all the 1-types of T. Show there is a unique 1-type p_n including $P_n(x)$; and one more other than these.

Solution. In any model $\underline{A} \models T$, any two elements of $P_n^{\underline{A}}$ realize the same type. To see this, let $a, b \in P_n^{\underline{A}}$. Let σ be the permuation of A with $\sigma(a) = b, \sigma(b) = a$, and $\sigma(c) = c$ for $c \neq a, b$. Then σ is a permuation of A, preserving all the P_k; so an automorphism of $\underline{A}$. Thus $tp_{\underline{A}}(a) = tp_{\underline{A}}(\sigma(a)) = tp_{\underline{A}}(b)$.
Now let p, q be two types including $P_n(x)$. Let $\underline{A} \models T$ and $a, b \in A$ with $p = tp_{\underline{A}}(a), q = tp_{\underline{A}}(b)$. Then by the above, $p = q$. Thus there exists a unique 1-type including $P_n(x)$; we denote it p_n.
Let r be a 1-type not equal to any p_n. Then $P_n(x) \notin r$, for each n. By the same argument as above, in any $\underline{A} \models T$, permuting two elements of $A \setminus \bigcup_n P_n^{\underline{A}}$ is an automorphism; and so any two elements of $A \setminus \bigcup_n P_n^{\underline{A}}$ realize the same type in $\underline{A}$. It follows as above that if r' is any 1-type not equal to p_n, then $r = r'$.

Proposition 6.3. *Let $\underline{A}$ be an L-structure, and $P = \{p^\alpha : \alpha < \kappa\}$ of partial types for $Th(\underline{A})$. For any cardinal $\kappa \geq \max\{|\underline{A}|, |L|\}$ there is $\underline{B} \succcurlyeq \underline{A}$ of cardinality κ such that all types from P are realised in $\underline{B}$. In particular, for countable L and a complete theory T of L, given a partial type p there is a countable model $\underline{B}$ of T which realises p.*

Proof Consider the expansion L^+ of $L_{\underline{A}}$ by new constants

$$\{c_1^\alpha, \ldots, c_n^\alpha : \alpha < \kappa\},$$

and the theory

$$T^+ = \mathrm{CDiag}(\underline{A}) \cup \{\varphi(c_1^\alpha, \dots, c_n^\alpha) : \ \varphi \in p^\alpha, \ \alpha < \kappa\}$$

We claim that T^+ is finitely satisfiable in $\underline{A}$. Indeed, any finite subset S of T^+ contains only finitely many formulas φ from the types. Since types are closed under conjunction, we may assume that there is at most one formula of the form $\varphi(c_1^\alpha, \dots, c_n^\alpha)$ in S for a type p^α. Since $\exists \bar{v} \varphi(\bar{v})$ holds in $\underline{A}$, we can find in $\underline{A}$ for $\varphi(c_1^\alpha, \dots, c_n^\alpha)$ an interpretation of $c_1^\alpha, \dots, c_n^\alpha$ which makes each such formula true in the corresponding expansion of $\underline{A}$.
By the compactness theorem there is a model $\underline{B}^+ \models T^+$ of cardinality κ. Since $\underline{B}^+ \models \mathrm{CDiag}(\underline{A})$ the L-reduct $\underline{B}$ of $\underline{B}^+$ is an elementary extension of $\underline{A}$. Let, for each α, $a_1^\alpha, \dots, a_n^\alpha$ be the elements assigned to $c_1^\alpha, \dots, c_n^\alpha$ in $\underline{B}^+$. By the construction $\langle a_1^\alpha, \dots, a_n^\alpha \rangle$ realize p^α in $\underline{B}$.
If we start with a countable model $\underline{A}$ of T and $\kappa \leq \aleph_0$, then $\underline{B}$ can be chosen countable. □

Corollary 6.4. *For any partial type there is $p' \supseteq p$ which is a complete type in the same variables.*

Indeed, put $p' = \mathrm{tp}_{\underline{B}}(\bar{a})$ for $\bar{a}$ in $\underline{B}$ realising p.
(Alternatively, one can use Zorn's lemma to take a maximal partial type containing and check that it is complete.)

Example There is a countable elementary extension of the group of integers $\mathbb{Z} = (\mathbb{Z}; +; 0)$ which is not isomorphic to $\mathbb{Z}$.
Given $n > 0$ denote by $n|v$ the formula $\exists w(v = w + \ldots + w)$ (n summands).
Let

$$p = \{v \neq 0\} \bigcup \{n|v : \ n = 1, 2, \dots \in \mathbb{N}\}.$$

p is clearly is finitely satisfiable in $\mathbb{Z}$ (the first k sentences are realized by $k!$.) Thus it is realised in some countable elementary extension. But p is obviously omitted in $\mathbb{Z}$.

We denote $\mathrm{S}_n(T)$ the set of all complete n-types of T.
Remark $S_n(T)$ can profitably be viewed as a topological space, the Stone space of the n'th Lindenbaum algebra of T.; a basic open set has the form

$$X_\phi := \{p : \phi \in p\}$$

but here we consider it simply as a set. (Optional problem: show that $S_n(T)$ is a compact Hausdorff space. The (topological) compactness of $S_n(T)$ follows from the (logical) compactness theorem.)
Definition A partial type p is *principal* if there is $\varphi \in F_n$ such that $T \models \exists \bar{v}\varphi(\bar{v})$ and for any $\psi \in p \quad T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v}))$.

Exercise 6.5. *When p is a principal* type*, the formula φ above must be in p.*

Solution. Since p is a type, if φ is not in p then $\neg\varphi$ lies in p. Let c be a new constant symbol. As p is principal, $T \bigcup \varphi(c) \vdash p(c)$; but $\neg\varphi \in p$; so $T \bigcup \{\varphi(c)\} \models \neg\varphi(c)$. Thus $T \models \neg\varphi(c)$ and by generalisation, $T \models \neg(\exists x)\varphi(x)$. This contradicts the assumed consistency of φ with T.

Exercise 6.6. A principal partial type is realised in any model $\underline{A}$ of T.

Theorem 6.7 (Omitting a type)**.** *Let p be a non-principal partial type in a complete theory T of a countable language L. Then there is a countable model of T which omits p.*

Proof Let $L' = L \cup C$, C a set of countably many new constant symbols. Let $\bar{c}_1, \ldots, \bar{c}_k, \ldots$ be an enumeration of all n-tuples of constant symbols of L' and $\phi_1, \ldots, \phi_l, \ldots$ an enumeration of all sentences in L'.
We construct a chain of finite sets of L'-sentences

$$S_0 \subseteq \ldots S_m \subseteq \ldots$$

by induction on $m \geq 1$ so that
(i) $T \cup S_m$ are satisfiable,
(ii) for $m \geq 1$ either ϕ_m or $\neg\phi_m$ is in S_m, ,
(iii) if ϕ_m is in S_m and has the form $\exists v\ \varphi(v)$, for some 1-variable L'-formula $\varphi(v)$, then $\varphi(c) \in S_m$ for some $c \in C$
(iv) for $m \geq 1$ there is a formula $\psi \in p$ such that $\neg\psi(\bar{c}_m) \in S_m$.

Start with $S_0 = \emptyset$.
Suppose $S_0 \subseteq \ldots S_{m-1}$ are constructed.
If $T \cup S_{m-1} \cup \{\phi_m\}$ is satisfiable then put $S'_m = S_{m-1} \cup \{\phi_m\}$. Otherwise $S'_m = S_{m-1} \cup \{\neg\phi_m\}$. It is easy to see that $T \cup S'_m$ is satisfiable.

Claim. There exists $\psi \in p$ such that $T \cup S'_m \cup \{\neg\psi(\bar{c}_m)\}$ is satisfiable.

Proof of Claim. Suppose for all $\psi \in p$ the converse holds. Let $\Phi = \bigwedge S'_m$. We can represent Φ as $\varphi(c_{m,1}, \ldots, c_{m,n}, d_1, \ldots, d_p)$, where $\varphi(v_1, \ldots v_n, u_1, \ldots, u_p)$ is an L-formula with free variables $v_1, \ldots v_n, u_1, \ldots, u_p$ and $\langle c_{m,1} \ldots, c_{m,n}\rangle = \bar{c}_m$, $d_1, \ldots, d_p$ constant symbols not in L and different from $c_{m,i}$'s. We write corresponding formulas in the short form $\varphi(\bar{c}_m, \bar{d})$ and $\varphi(\bar{v}, \bar{u})$.
Then, by our assumption, for any $\psi \in p$

$$T \models (\varphi(\bar{c}_m, \bar{d}) \to \psi(\bar{c}_m)).$$

Since no component of $\bar{c}_m$ and $\bar{d}$ do occur in T, it follows

$$T \models \forall \bar{v} \forall \bar{u}(\varphi(\bar{v}, \bar{u}) \to \psi(\bar{v})).$$

The formula can be equivalently rewritten as $\forall \bar{v}(\exists \bar{u} \varphi(\bar{v}, \bar{u}) \to \psi(\bar{v}))$, so

$$T \models \forall \bar{v}(\exists \bar{u} \varphi(\bar{v}, \bar{u}) \to \psi(\bar{v}))$$

for every $\psi \in p$. This means that $\exists \bar{u} \varphi(\bar{v}, \bar{u})$ is a principal formula for p. The contradiction, which proves the claim.

Now take $S''_m = S'_m \cup \{\neg \psi(\bar{c}_m)\}$.
Suppose ϕ_m is in S''_m and has the form $\exists v\ \varphi(v)$. Choose $c \in C$ which does not occur in S''_m. Then $T \cup S''_m \cup \{\varphi(c)\}$ has a model: any model $\underline{A}$ of $T \cup S''_m$ in the language $L \cup \{$ constants of $S''_m\}$ can be expanded by assigning to c the values of v for $\exists v \varphi(v)$.
Denote $S_m = S''_m \cup \{\varphi(c)\}$. If ϕ_m does not have this form then put $S_m = S''_m$. This S_m satisfies (i)-(iv) by the construction.
To finish the proof of the theorem consider now

$$T^* = T \cup \bigcup_{m \in \mathbb{N}} S_m.$$

By the properties (i)-(iii) T^* is satisfiable, complete and witnessing set of sentences. By Theorem 1.19 T^* has a L-minimal model $\underline{A}$. Notice that by (iii) for any closed term λ T^* says $\lambda = c$ for some $c \in C$. Thus all elements of the L-minimal model $\underline{A}$ are named by symbols from C. Consequently, (iv) says that no n-tuple in $\underline{A}$ realises the partial type p. □

A similar proof, with a little more book-keeping, shows that countably many types may simultaneously be omitted.

Theorem 6.8 (Omitting types). *Let P be a countable set of partial types in a complete theory T of a countable language L. Assume no element of P is principal. Then there is a countable model of T which omits every type in P.*

Proof. In the proof of the omitting types theorem, the goal of omitting p is subdivided into $\aleph_0$ smaller tasks, to be taken care of in turn: at stage m, we took care that $\bar{c}_m$ will not realize p. If we wish to omit countably many partial types $p_1, p_2, \cdots$, with p_j a partial type in variables $x_1, \ldots, x_{\alpha(j)}$, it suffices to use an enumeration $(\bar{c}_1, j_1), (\bar{c}_2, j_2), \cdots$ of all pairs $(\bar{c}, j)$, with $j \in \mathbb{N}$ and $\bar{c}$ an $\alpha(j)$-tuple from the new constants $c_1, c_2, ...$ At the stage m, we ensure that $\bar{c}_m$ will not realize p_{j_m}. The proof is otherwise identical. $\square$

7 Atomic models and $\aleph_0$-categoricity

Fix a *countable* language L. Henceforce T denotes a complete L-theory having an infinite model. By the Lowenheim-Skolem downward Theorem, T has a countable model $\underline{A}$. As T is complete, we have $T = \mathrm{Th}(\underline{A})$.

Denote F_n the set of all L-formulas with free variables $v_1 \dots v_n$ (abbreviated $\bar{v}$). Denote $\equiv_T$ the binary relation on F_n defined by

$$\varphi(\bar{v}) \equiv_T \psi(\bar{v}) \quad \text{iff} \quad T \models \forall \bar{v}(\varphi(\bar{v}) \leftrightarrow \psi(\varphi)).$$

Equivalently, since T is complete, $\varphi(\underline{A}) = \psi(\underline{A})$.

Thus, $\equiv_T$ is an equivalence relation respecting the Boolean operations $\wedge$, $\vee$ and $\neg$.

Given a theory T and a number n, $F_n/{\equiv_T}$ is called the **nth Lindenbaum algebra of T**. As was mentioned above, its elements are in a one-to-one correspondence with definable subsets of $\underline{A}$ and $\wedge$, $\vee$ and $\neg$ correspond to the usual Boolean operations $\cap$, $\cup$ and the complement, on the sets.
T
Let S_n be the set of types of T in variables $v_1, \dots, v_n$.
Assume $\varphi \in F_n$, $p \in S_n$, $\varphi \in p$, and p is the *unique* element of S_n including φ.
Equivalently, $T \models \exists \bar{v} \varphi(\bar{v})$ and for any $\psi \in p \quad T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v}))$.
In this situation, we will say that p is a *principal type*, that ϕ is a *principal formula*, and also that ϕ *is a principal formula for* p.

Exercise 7.1. $\varphi \in L_n$ is a principal formula (for some type) iff $T \models \exists \bar{v} \varphi(\bar{v})$ and for any $\psi \in L_n$, either $T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v}))$, or $T \models \forall \bar{v}(\varphi(\bar{v}) \to \neg\psi(\bar{v}))$

A type which is not principal is called **non-principal**.

Call a model of T *atomic* if every n-tuple in $\underline{A}$ satisfies a principal formula. [9]

Proposition 7.2. *Let $\underline{A}, \underline{B}$ be two countable, atomic models of T. Then $\underline{A} \cong \underline{B}$.*

[9] The word 'atomic' here refers to the atoms of the Lindenbaum algebra, i.e. minimal nonzero elements, and has nothing to do with atomic formulas.

Proof Enumerate

$$A = \{a_1, a_2, \ldots\}, \quad B = \{b_1, b_2, \ldots\}.$$

We will construct new enumerations $\{a'_1, a'_2, \ldots\}$ and $\{b'_1, b'_2, \ldots\}$ of the sets so that the enumerations establish a correspondence between the sets preserving L-formulas, by the **back-and-forth method:**
Suppose $a'_1, \ldots, a'_{n-1} \in A$ and $b'_1, \ldots, b'_{n-1} \in B$ satisfy for all $\psi \in F_{n-1}$

$$(*) \quad \underline{A} \models \psi(a'_1, \ldots, a'_{n-1}) \text{ iff } \underline{B} \models \psi(b'_1, \ldots, b'_{n-1}).$$

Notice that (*) is true for $n = 1$ since $\underline{A} \equiv \underline{B}$. Let n be odd and a'_n be the first member in $A = \{a_1, a_2, \ldots\}$ not occurring among $a'_1, \ldots a'_{n-1}$. Let φ be a principal formula for $a'_1, \ldots, a'_n$. Then $\underline{A} \models \varphi(a'_1, \ldots, a'_n)$ and so, $\underline{A} \models \exists v \varphi(a'_1, \ldots, a'_{n-1}, v)$. By (*) $\underline{B} \models \exists v \varphi(b'_1, \ldots, b'_{n-1}, v)$. Hence we may choose $b'_n \in B$ such that $\underline{B} \models \varphi(b'_1, \ldots, b'_n)$.
Now suppose $\psi \in F_n$ and $\underline{A} \models \psi(a'_1, \ldots, a'_n)$. Since φ is principal

$$T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v})).$$

Hence $\underline{B} \models \psi(b'_1, \ldots, b'_n)$.
Thus (*) is satisfied for $a'_1, \ldots a'_n$ and $b'_1, \ldots b'_n$, too.
Similarly, when n is even, let b'_n be the first element in $B = \{b_1, b_2, \ldots\}$ not occurring among $b'_1, \ldots b'_n$. Then we can find $a'_n \in A$ such that (*) is satisfied for $a'_1, \ldots a'_n$ and $b'_1, \ldots b'_n$.
Hence we may inductively construct in this way $A = \{a'_1, a'_2, \ldots a'_n \ldots\}$, $B = \{b'_1, b'_2, \ldots b'_n \ldots\}$ satisfying (*) for all n. Our construction guarantees (as in the proof of Cantor's theorem) that we get all of A and all of B. Now it follows from (*) that $a'_i \to b'_i$ is an isomorphism (in particular, a well-defined injective map; to see this take ψ in (*) to be $x_i \doteq x_j$). □

Definition A model $\underline{A}$ of T is called **prime** if for any model $\underline{B}$ of T there exists an elementary embedding $\pi : \underline{A} \to \underline{B}$.

Proposition 7.3. *Any countable atomic model of a complete theory T is prime.*

Proof. The proof resembles that of Proposition 7.2, but with the "forth" part of the construction alone.
Let $\underline{A}$ be a countable atomic model of T. Let $\underline{B}$ be any model of T; we have to find an elementary embedding $\underline{A} \to \underline{B}$.
Enumerate

$$A = \{a_1, a_2, \ldots\}$$

We will recursively define elements $b_1, b_2, \ldots$ of B so that $a_i \mapsto b_i$ is an elementary embedding.
Assume $b_1, \ldots, b_{n-1}$ have been defined, and satisfy for all $\psi \in F_{n-1}$

$$(*) \quad \underline{A} \models \psi(a_1, \ldots, a_{n-1}) \text{ iff } \underline{B} \models \psi(b_1, \ldots, b_{n-1}).$$

Notice that (*) is true for $n = 1$ since $\underline{A} \equiv \underline{B}$. Assume inductively it holds at $n - 1$.
As $\underline{A}$ is atomic, there exists a principal formula φ realized by $a_1, \ldots, a_n$. Then $\underline{A} \models \varphi(a_1, \ldots, a_n)$ and so, $\underline{A} \models \exists v \varphi(a_1, \ldots, a_{n-1}, v)$. By (*) $\underline{B} \models \exists v \varphi(b_1, \ldots, b_{n-1}, v)$. Hence we may choose $b_n \in B$ such that $\underline{B} \models \varphi(b_1, \ldots, b_n)$. Picking such a b_n finishes the inductive step of the construction, once we prove (*) for n.
Thus suppose $\psi \in F_n$ and $\underline{A} \models \psi(a_1, \ldots, a_n)$. Since φ is principal

$$T \models \forall \bar{v}(\varphi(\bar{v}) \to \psi(\bar{v})).$$

Hence $\underline{B} \models \psi(b_1, \ldots, b_n)$.
Thus (*) is satisfied for $a_1, \ldots a_n$ and $b_1, \ldots b_n$, too.
It follows from (*) for the special case that ψ is the formula $x_i \doteq x_j$, that $a_i \mapsto b_i$ is a well-defined map, and is injective. By (*) again, it is an elementary embedding of $\underline{A}$ into $\underline{B}$. □

Proposition 7.4. *Let $\underline{A}, \underline{B}$ be two countable, atomic models of T. Let $c_1, \ldots, c_n \in A$ and $d_1, \ldots, d_n \in B$, and assume, for all formulas $\psi(u_1, \ldots, u_n)$ we have:*

$$\underline{A} \models \psi(c_1, \ldots, c_n) \textit{ iff } \underline{B} \models \psi(d_1, \ldots, d_n). \tag{2}$$

Then there exists an isomorphism $F : \underline{A} \to \underline{B}$ such that $F(c_i) = d_i$.

Proof. Follow the proof of Proposition 7.2, only start at stage n with $a'_i = c_i, b'_i = d_i$ for $i \leq n$. □

Corollary 7.5. *Let $\underline{A}$ be a countable atomic model. Then $\underline{A}$ is* homogeneous: *For any $c_1, \ldots, c_n, d_1, \ldots, d_n \in A$, with $tp(c_1, \ldots, c_n) = tp(d_1, \ldots, d_n)$, there exists an automorphism g of $\underline{A}$ with $g(c_i) = d_i$.*

Let us draw some corollaries.

Theorem 7.6. *A countable model of T is prime if and only if it is atomic.*

Proof. We already showed that countable atomic models are prime. Conversely, if M_0 is a prime model of T and a type p is realized in M_0, then p is realized in every $M \succ M_0$ and hence as M_0 is prime, in every model of T. By the omitting types theorem, p is principal. □

Definition T is a *small* theory if for each n, it has only countably many n-types; i.e. card $\mathrm{S}_n(T) \leq \aleph_0$ for all $n \in \mathbb{N}$.

Proposition 7.7. *Assume T is small . Then T has a countable atomic model.*

Proof. Since T is small , there are only countably many types and in particular countably many non-principal types in $\bigcup_n \mathrm{S}_n(T)$. By the omitting types theorem, there is a countable model $\underline{A}$ of T which omits all the non-principal types. This $\underline{A}$ is atomic by definition. □

Lemma 7.8. *$F_n/\equiv_T$ is finite if and only if every n-type of T is principal.*

Proof. Assume $F_n/\equiv_T$ is finite, and let $p(x)$ be any n-type, $x = x_1, \ldots, x_n$. Let $\psi_1, \ldots, \psi_k$ be a maximal set of pairwise inequivalent (under $\equiv_T$) formulas of F_n, with $\psi_i \in p$. Let $\psi = \psi_1 \wedge \cdots \wedge \psi_k$. Then $\psi \in p$, and as any formula ϕ in p is T-equivalent to some ψ_i, we have $T \vdash \psi \rightarrow \phi$. This proves that p is principal (via ψ.)

Assume conversely that $F_n/\equiv_T$ is infinite. Let us prove that T has a non-principal n-type.

Let $P = \{\neg\varphi_1 \wedge \ldots \wedge \neg\varphi_k \in F_n : \varphi_i$ principal formulae $\}$. We claim that P generates a partial type for T.

Suppose not. Then

$$T \models \forall \bar{v}(\varphi_1(\bar{v}) \vee \ldots \vee \varphi_k(\bar{v}))$$

for some principal formulas $\varphi_1, \ldots, \varphi_k \in F_n$.

Define for $\psi \in F_n$

$$W_\psi = \{i \in \{1, \ldots, k\} : \ T \models (\exists \bar{v})(\varphi_i(\bar{v}) \wedge \psi(\bar{v}))\}$$

Notice that since φ_i's are principal formulas
$T \models \exists \bar{v}(\varphi_i(\bar{v}) \wedge \psi(\bar{v})\}$ iff $T \models \forall \bar{v}(\varphi_i(\bar{v}) \rightarrow \psi(\bar{v}))\}$.
It follows that for any $\psi, \chi \in F_n$ $\psi \equiv_T \chi$ iff $W_\psi = W_\chi$. Thus card $F_n/\equiv_T = 2^k$. This contradicts the assumptions and proves the claim.
Take now a complete n-type extending P. It cannot be principal since the negation of every principal formula is already in P. □

Theorem 7.9 (Ryll-Nardzewski)**.** *T is $\aleph_0$-categorical iff $F_n/\equiv_T$ is finite for all $n \in$ N.*

Proof Assume $F_n/\equiv_T$ is finite for all $n \in \mathbb{N}$. By the above lemma, every type of T is principal. Hence by definition every model of T is atomic. But any two countable atomic models of T are isomorphic; so all countable models of T are isomorphic.
Conversely, assume $F_n/\equiv_T$ is infinite. Then there exists a non-principal n-type p of T. By the omitting types theorem there is a countable model $\underline{A}$ that omits p. On the other hand, by Lemma 6.3, there is a countable model $\underline{B}$ which realises p. It follows $\underline{A}$ is non-isomorphic to $\underline{B}$ and thus T is not $\aleph_0$ categorical. □

Remark The Ryll-Nardjewski theorem was in fact proved independently by him in Warsaw, by Engeler in Zurich and by Svenonius in Uppsala, in 1959, following precedents in the form of ω-logic. Vaught at Berkeley made the extension to omitting many types in 1961, in order to give the comprehensive theory of atomic and saturated countable models of countable complete theories.
This leads to a very fruitful connection to group theory.
Definition Let M be any set, and let G be a subgroup of the group $Sym(M)$ of permutations of M. For $a = (a_1, \ldots, a_n) \in M^n$, and $g \in G$, we define $ga := (ga_1, \ldots, ga_n)$. Two n-tuples $a, b \in M^n$ are said to be G-conjugate if there exists $g \in G$ with $ga = b$. This is an equivalence relation on M^n. The equivalence classes are called the G-orbits; the set of G-orbits is denoted M^n/G.
Definition A subgroup G of $Sym(M)$ is oligomorphic if for all $n = 1, 2, \cdots$, M^n/G is finite. The term is due to Peter Cameron.

Proposition 7.10. *Let M be a countable structure. Then $Th(M)$ is $\aleph_0$-categorical iff $G = Aut(M)$ is oligomorphic.*

Proof. Assume $Th(M)$ is $\aleph_0$-categorical, and let $n \in \mathbb{N}$, $x = (x_1, \ldots, x_n)$. Then $Th(M)$ has a finite number $\phi_1(x), \ldots, \phi_k(x)$ of formulas $\phi(x)$ up to equivalence. Since a type $p(x)$ is determined by the set of $\phi_i(x)$ it contains, there are at most 2^k types $p(x)$. [10]. Now if $a, b \in M^n$ and $tp(a) = tp(b)$, then there exists an automorphism $g \in G$ with $g(a) = b$. Thus the number of G-orbits is at most the number of types, so it is finite.
Conversely, if G is oligomorphic, say with m orbits on M^n, since every definable subset of M^n is G-invariant the number of definable subsets of M^n is at most 2^m. Thus $|F_n(T)/\equiv_T| \leq 2^m$. So T is $\aleph_0$-categorical. □

Thus an $\aleph_0$-categorical theory gives rise to an oligomorphic permutation group on a countable set.
Conversely, an oligomorphic permutation group on a countable set gives rise to an $\aleph_0$-categorical theory:

Exercise 7.11. *Let T be an $\aleph_0$-categorical theory, $M \models T$, $G = Aut(M)$. Let $S \subset M^n$. Show that S is definable if and only if it is G-invariant, i.e. $g(S) = S$ for every $g \in G$.*

Exercise 7.12. *(optional.) Let M be a countable set, and let G be an oligomorphic subgroup of $Sym(M)$. Let L be a language having one n-ary relation symbol R^n_c for each orbit c of G on M^n. Let $\underline{M}$ be the L-structure with universe M, and with R^n_c interpreted as c. Show that any qf definable subset of M^n is a (finite) disjunction $R^n_{c_1} \bigcup \cdots \bigcup R^n_{c_k}$. Show that the projection to M^{n-1} of any R^n_c is some $R^{n-1}_{c'}$. Thus $Th(\underline{M})$ admits quantifier elimination. Show that this theory is $\aleph_0$-categorical.*

Remark in the above exercise, $Aut(M)$ is not G but the *completion* of G in an appropriate sense.
Definition A model M is *minimal* if M has no proper elementary submodel (I.e. $N \prec M$ implies $N = M$.)
Note this differs from the earlier notion of an L-minimal model! Any finite structure is minimal, but not necessarily L-minimal.

Exercise 7.13. *1. Show that a prime model of T has size $\leq |L| + \aleph_0$.*

[10] In fact, there are only $\log_2(k)$ types $p(x)$

2. *Show that a minimal model of T has size $\leq |L| + \aleph_0$.*

3. *Show that any prime model of T is isomorphic to any minimal model of T. (Give a short direct argument, not requiring countability of L.) Conclude that if T has a prime model and a minimal model, then any two prime models are isomorphic, and any two minimal models are isomorphic.*

Solution. 1. By downward Loewenheim-Skolem, T has a model M of size $\leq |L| + \aleph_0$. Any prime model of T embeds into M, hence also has size $\leq |L| + \aleph_0$.
2. Let M be a minimal model of T. By Loewenheim-Skolem, M has an elementary submodel N of size $\leq |L| + \aleph_0$. By minimality of M we have $M = N$; so $|M| \leq |L| + \aleph_0$.
3. Let M be a prime model of T, and N a minimal model of T. As M is prime there exists an elementary embedding $f : M \to N$. Now $f(M) \prec N$, so by minimality of N we have $f(M) = N$. As f is a surjective L-embedding, it is an isomorphism. So $M \cong N$. Now if M' is another prime model, then by the above we have $M' \cong N$ and so $M' \cong M$; similarly if N' is another minimal model, then $N' \cong M$ and so $N' \cong N$.

Definition A countable model $\underline{A}$ of T is called $\aleph_0$-*universal* if, for any countable model $\underline{B}$ of T, there is an elementary embedding $\pi : \underline{B} \to \underline{A}$.

Exercise 7.14.

For each of the following theories, determine whether it has a minimal model, a prime model, a countable $\aleph_0$- universal model. How many isomorphism classes of countable models does each have?

1. *T^∞ (the theory of infinite sets in the language $\{\doteq\}$ with no nonlogical symbols.)*
   Solution. *Any two models of cardinality $\aleph_0$ are isomorphic; indeed any bijection between the universes is an isomorphism. Thus up to isomorphism there is a unique countable model. By the definitions of these terms, it is both prime and universal. However there is no minimal model: if A is an infinite set, any infinite subset A' of A is an elementary submodel.*

2. *ACF_0 (optional; if you do not feel comfortable with the algebra, do as much as you can.)*

   Solution. *An algebraically closed field of characteristic zero is entirely determined by its transcendence degree over $\mathbb{Q}$; for a countable field this can be any cardinal in $0, 1, 2, \cdots, \aleph_0$. Thus there are $\aleph_0$ isomorphism classes of countable models. Let K_n be an algebraically closed field of transcendence degree n over $\mathbb{Q}$, and K_∞ an algebraically closed field of transcendence degree $\aleph_0$ over $\mathbb{Q}$. Then K_0 is the prime model. It is also minimal, since any algebraically closed subfield of K_0 containing $\mathbb{Q}$ must contain all algebraic numbers in K_0; but all elements of K_0 are algebraic. K_∞ is a universal countable model.*

3. *The theory of nonzero $\mathbb{Q}$-vector spaces.*
   Solution. *A $\mathbb{Q}$-vector space is determined, up to isomorphism, by its dimension. Let $V_n = \mathbb{Q}^n$, and let V_∞ be any countable, infinite-dimensional $\mathbb{Q}$-vector space. Then any countable, nonzero $\mathbb{Q}$-vector space is isomorphic to some V_n with $n = 1, 2, \cdots$, or to V_∞. The V_n are pairwise non-isomorphic. Thus there are $\aleph_0$ isomorphism classes. We have embeddings of V_1 into V_2, of V_2 into V_3, etc. (map v to $(v, 0)$.) Because of quantifier elimination, any embedding is an elementary embedding. Hence, V_1 is prime, and V_∞ is a countable universal model. Since V_1 has no proper nonzero subspaces, it is minimal.*

4. *DLO*
   Solution. *There is one countable model up to isomorphism (Cantor's theorem.) It is thus prime and universal among countable models. However it is not minimal, and thus there is no minimal model. For instance, let $\underline{A}$ be the substructure of $(\mathbb{Q}, <)$ whose universe consists of rational numbers $a/2^n$ with $a \in \mathbb{Z}$. Then $\underline{A} \models DLO$, and $\underline{A} \prec \mathbb{Q}$ by quantifier-elimination; but $\underline{A} \neq \mathbb{Q}$.*

5. *$DLO_\mathbb{Q}$ (The theory of $(\mathbb{Q}, <)_\mathbb{Q}$, the rational order in a language including a constant symbol for each rational.)*

   Solution. *Let $\rho = I(DLO_\mathbb{Q}, \aleph_0)$ be the number of isomorphism types of countable models of $DLO_\mathbb{Q}$. We will show that $\rho = 2^{\aleph_0}$, by proving the two inequalities, beginning with: $\rho \leq 2^{\aleph_0}$. (This part of the argument can be generalized to any theory in a countable language.)*

Let $L = \{<, c_i : i \in \mathbb{Q}\}$ be the language. By the method of diagrams, there is a 1-1 correspondence between models of $DLO_{\mathbb{Q}}$ with universe $\mathbb{N}$, and models $\underline{A}$ of DLO with universe $\mathbb{N}$ along with embeddings of $\mathbb{Q}$ into $\underline{A}$. Now a model $\underline{A}$ of DLO with universe $\mathbb{N}$ is entirely determined by the choice of $<^{\underline{A}}$, a subset of $\mathbb{N}^2$; the number of possibilities is $2^{\aleph_0^2} = 2^{\aleph_0}$. The number of possible embeddings $\mathbb{Q} \to \underline{A}$ is at most the number of functions $\mathbb{Q} \to \mathbb{N}$, namely ${\aleph_0}^{\aleph_0} = 2^{\aleph_0}$. Thus there are at most $2^{\aleph_0}$ models of $DLO_{\mathbb{Q}}$ with universe $\mathbb{N}$. But any countable model of $DLO_{\mathbb{Q}}$ is isomorphic to one with universe $\mathbb{N}$. So $\rho \leq 2^{\aleph_0}$.
To show that $\rho \geq 2^{\aleph_0}$, we wil define a model A_α for any irrational $\alpha \in \mathbb{R}$, and show no two are isomorphic. Let $\underline{A}_\alpha$ be the substructure of $(\mathbb{R}, <)_{\mathbb{Q}}$ with universe $\mathbb{Q} \bigcup \{\alpha\}$. (Note the interpretation of the c_i lies in A_α.) Then $A_\alpha \models DLO_{\mathbb{Q}}$. If $\alpha \neq \beta$ there can be no L-isomorphism $f : A_\alpha \to A_\beta$, since such an L-isomorphism must fix every rational; thus it will have to take α to the unique irrational of A_β, namely β; but if γ is rational strictly between α and β, then $f(\gamma) = \gamma$, so f cannot be order-preserving. Thus $\underline{A}_\alpha \not\cong \underline{A}_\beta$. So $\rho \geq |\mathbb{R} \setminus \mathbb{Q}| = 2^{\aleph_0}$; and equality holds.
$\mathbb{Q}_{\mathbb{Q}}$ itself is a prime and minimal model. There can be no universal countable model $\underline{B}$, since such a model would embed every $\underline{A}_\alpha$, and the argument above shows that the distinct irrational numbers α would have distinct images in $\underline{B}$; as B is countable this is not possible.

Exercise 7.15. *Let $\phi(u_1, \ldots, u_n)$ be any formula, and let $U(\phi) = \{p \in S_n : \phi \in p\}$, where S_n is the set of n-types. Show that up to $\equiv_T$, ϕ is determined by $U(\phi)$. Conclude that if S_n is finite then there are finitely many formulas in $u_1, \ldots, u_n$, up to T-equivalence.*

Exercise 7.16. *Let $L = \{P_1, P_2, \ldots\}$; where the P_k are unary predicates.*

1. *Let $\underline{A}$ be the L-structure whose universe is $\mathbb{N}$, and such that P_k is the set of natural numbers divisible by the k'th prime p_k. For any finite subset S of $\mathbb{N}$, let P_S denote the conjunction $\bigwedge_{k \in S} P_k$, and let P'_S denote $\bigwedge_{k \in S} \neg P_k$. Note that for any two disjoint finite sets $S, S' \subset \mathbb{N}$, the intersection $P_S(\underline{A})$ with $P'_{S'}(\underline{A})$ is infinite.*

2. *Write explicitly a sentence $\alpha_{S,S',m}$ asserting that the intersection of P_S with $P'_{S'}$ has at least m points. Let T be the theory axiomatized by all these $\alpha_{S,S',m}$. Show that $T = Th(\underline{A})$ (you may first want to do the next two clauses.)*

3. *Let $L_n = \{P_1, \ldots, P_n\}$, and let T_n be the theory axiomatized by all $\alpha_{S,S',m}$ with S, S' disjoint subsets of $\{1, \ldots, n\}$, and $m \in \mathbb{N}$. Show that T_n is $\aleph_0$-categorical, and complete.*

4. *Conclude that T is complete.*

5. *Show that T has no principal 1-types, hence no atomic model.*

6. *Show that T has $2^{\aleph_0}$ 1-types.*

Exercise 7.17. *Let L be the language with a binary relation symbol E and a unary function symbol f. The axioms of T assert that E is an equivalence relation with infinitely many classes; that f is 1-1 and onto, and $f^n(x) \neq x$ for $n = 1, 2, \ldots$ (where $f^1 = f$ and $f^{n+1} = f \circ f^n$.); and that $E(x, f(x))$. You may assume that T is complete and admits QE. Describe the countable models, including the prime model, the saturated model and the universal models.*

Exercise 7.18. *Notation is as in problem 7.16.*

1. *How many models of cardinality $\aleph_n$ does T_1 have?*

2. *How many models of cardinality $\aleph_1$ does T_n have? Describe the $\aleph_1$-saturated one.*

3. *Show that T has $2^{\aleph_0}$ non-isomorphic countable models.*

Exercise 7.19. Let $L = \{<, c_1, c_2, \ldots\}$, and consider three L-structures M_1, M_2, M_3, all with universe $\mathbb{Q}$ and the usual interpretation of $<$, but different interpretations of the c_i; namely, $c_n^{M_1} = n$, $c_n^{M_2} = -1/n$, $c_n^{M_3} = (1+\frac{1}{n})^n$.

1. Show that these are three models of the same complete theory $T = DLO_{\mathbb{N}}$. (Hint: consider finite sublanguages.)
   Solution. Let $L_n = \{<, c_1, c_2, \ldots, c_n\}$, and let T_n be the theory axiomatized by DLO along with the sentences $c_i < c_{i+1}, i = 1, \cdots, n-1$. Then T_n is $\aleph_0$-categorical; see the Proposition following Cantor's theorem. By the Łoś-Vaught test, T_n is complete. Now let T be the set of logical consequences of $\bigcup_{n=1,2,\cdots} T_n$. Then $(\mathbb{Q}, 1, 2, \cdots) \models T$ so that T is consistent. Any sentence ψ of L uses finitely many symbols and so lies in some L_n; then $T_n \models \psi$ or $T_n \models \neg\psi$. As $T_n \subset T$, $T \models \psi$ or $T \models \neg\psi$. Thus T is complete. Each of the theories is a model of the axioms.

2. Are any two isomorphic? Which of them is prime? Which is universal? (Optional: Which of them is saturated?)
Solution. Let $p(x) = \{x > c_1, x > c_2, \ldots\}$. Then p is omitted in M_1, and realized in M_2 and M_3. Thus neither M_2 nor M_3 is isomorphic to M_1. Now in M_2, there is a *smallest* realisation of p, namely 0. In M_3 there is no smallest realization of p. So M_2, M_3 are not isomorphic.
Since p is realized is omitted in M_1 and realized in M_2, M_3 we see that M_1 does not embed M_2 as an elementary submodel, so it is not universal. And M_2, M_3 realize a nonprincipal type, so they are not atomic (hence not prime.)
We now show that M_1 is prime. It suffices to show that M_1 embeds elementarily into any countable model of T. Let $\underline{B}$ be such a model. For $n \geq 1$ let B_n be the interpretation in $\underline{B}$ of the interval $[c_{n-1}, c_n]$, i.e. $B_n = \phi_n(\underline{B})$ where $\phi_n(x) = (c_{n-1} < x \wedge x < c_n)$. Let $B_0 = \phi_0(B)$ where $\phi_n(x) = (x < c_0)$. Let $\underline{B}_i$ be the substructure of $(B, <)$ with universe B_i. Let C_i be the intervals defined similarly in $(\mathbb{Q}, <)$. Then $\underline{B}_i \models DLO$. By Cantor's theorem there exists an order preserving bijection $f_i : C_i \to B_i$. Note that any element a of M_1 lies in a unique interval C_i, or else equals $c_i^{M_1}$ for a unique i. Define $F(a) = f_i(a)$ for $x \in C_i$, and $F(a) = c_i^{\underline{B}}$ if $a = c_i^{M_1}$. Verify that F is an embedding of M_1 in $\underline{B}$. As both $\underline{B}$ and M_1 are models of DLO, and DLO eliminates quantifiers, F is an elementary embedding of M_1 in $\underline{B}$.
Both M_2, M_3 are universal. This can be shown similarly to the above, finding an elementary embedding $\underline{B} \to M_2$ and $\underline{B} \to M_3$; but it will also follow from the proof of (3) below, so we postpone it.
M_1 is not saturated as it omits p. M_2 is not saturated as it omits the partial type $p(x) \bigcup \{x < 0\}$, in a language in which a constant for 0 has been added. M_3 is the saturated one.

3. (optional). Show that any countable model of $DLO_{\mathbb{N}}$ is isomorphic to one of the above three. (Thus $I(DLO_{\mathbb{N}}, \aleph_0) = 3$; in general $I(T, \aleph_0)$ denotes the number of isomorphism classes of models of T of cardinality $\aleph_0$.)
Solution. For any countable model $\underline{B}$ of T, denote by B' the subset of B consisting of all elements below some $c_i^{\underline{B}}$, i.e. $B' = \bigcup_{n=0}^{\infty} \phi_i(\underline{B})$ with ϕ_i as in (2) above. Let $B'' := B \setminus B'$. Then B' is the universe of a substructure of $\underline{B}$. This is not true of B'', but still we can (and do) consider it as a substructure of $(B, <)$.

Now let $\underline{B}$ be a countable model of T. Let $f : M_1 \to \underline{B}$ be the embedding constructed in (2) above; by construction, the image of f is B'. We distinguish three cases.
Case 1: $B' = B$. Then we have an isomorphism $f : M_1 \to \underline{B}$.
When Case 1 fails, $B'' \neq \emptyset$; it is clearly a dense linear ordering with no greatest point; it may or may not have a smallest point. This gives two cases:
Case 2: B'' has no smallest point. Then $B'' \models DLO$. Note that this is the case for M_2. Thus $M_2'' \models DLO$, and so there exists an order-isomorphism $g : B'' \to M_2''$. Let $F = f \bigcup g$. It is easy to verify that F is an isomorphism $\underline{B} \to M_2$.
Case 3: B'' has a smallest point, b_0. This is also the case for M_1, where the smallest point of M_1' is 0. Thus there exists an order-isomorphism $g : B'' \setminus \{b_0\} \to M_1'' \setminus \{0\}$. Let $F = f \bigcup \{(b_0, 0)\} \bigcup g$. It is easy to verify that F is an isomorphism $\underline{B} \to M_1$.

4. (optional.) Find a theory with exactly four isomorphism types of countable models. (Hint: consider the theory in the language $(<, P)$ of a dense linear ordering with a unary predicate P, and additional axioms asserting that both P and the complement of P are dense, and cofinal, i.e. $(\forall x)(\exists y)(P(y) \& x < y)$, $(\forall x)(\exists y)(\neg P(y) \& x < y)$, $(\forall x)(\exists y)(P(y) \& y < x)$, $(\forall x)(\exists y)(\neg P(y) \& y < x)$, and similarly for density. Prove that this theory is $\aleph_0$-categorical. Then add constants for elements $c_1 < c_2 < \cdots$ of P.)

8 Countable saturated models

Let T be a complete theory in a countable language L. $\mathrm{S}_n(T)$ the set of all complete n-types of T.

Definition A structure $\underline{A}$ is called $\aleph_0$-*saturated* if, for any expansion $\underline{A}'$ of $\underline{A}$ by finitely many constant symbols, every 1-type in $\mathrm{Th}(\underline{A}')$ is realised in $\underline{A}'$.

If $|A| = \aleph_0$, and $\aleph_0$-saturated, we simply that $\underline{A}$ is *saturated*.
Recall also that a countable model $\underline{A}$ of T is called $\aleph_0$-*universal* if, for any countable model $\underline{B}$ of T, there is an elementary embedding $\pi : \underline{B} \to \underline{A}$.

Theorem 8.1. *(i) Any countable saturated model of a complete theory T is $\aleph_0$-universal. (Definition 7.)*
(ii) Any two countable $\aleph_0$-saturated models of T are isomorphic.

Proof Exercise. Use an inductive construction similar to the one in the proof of Proposition 7.2
For (i), or (ii)) and Proposition 7.3 (for (i)).

But when does T have a countable saturated model?

Lemma 8.2. *Let $T' = T(c_1, \ldots, c_m)$ be a complete theory extending T in the language $L(c_1, \ldots, c_m)$, the extension of L by finitely many extra constants symbols $c_1, \ldots, c_m$, and suppose T is small. Then T' is small too.*

Proof Fix n. For each $p \in \mathrm{S}_n(T')$ define

$$p^* = \{\phi(v_1, \ldots, v_{n+m}) \in F_{n+m} : \ \phi(v_1, \ldots, v_n, c_1, \ldots, c_m) \in p\}.$$

It follows from the definition that $p^* \in \mathrm{S}_{n+m}(T)$, and if $p_1 \neq p_2$ then $p_1^* \neq p_2^*$. Hence we have mapping $\mathrm{S}_n(T') \to \mathrm{S}_{m+n}(T)$, which is injective. Since card $\mathrm{S}_{m+n}(T) \leq \aleph_0$, by the hypothesis, we have $\mathrm{S}_n(T') \leq \aleph_0$. $\square$

Theorem 8.3. *T has a countable $\aleph_0$-saturated model iff it is small.*

Proof. Let $\underline{A}$ be a countable model of T. Enumerate $\{a_1, \ldots, a_n, \ldots\}$ elements of $\underline{A}$. Let $C = \{c_1, \ldots, c_n, \ldots\}$ be a set of new constant symbols, $\underline{A}_C$ the structure in the language L_C obtained by assigning a_i to c_i, T_C the theory of

the structure, and $T_{\{c_1,\ldots,c_m\}}$ the fragment of the theory containing formulas with at most the first m constants symbols of C.
By the lemma above, the set of 1-types $\bigcup_m \mathrm{S}_1(T_{\{c_1,\ldots,c_m\}})$ is countable. By Lemma 6.3 we can construct a countable $\underline{B}_C \succ \underline{A}_C$ which realises all the types of $\bigcup_m \mathrm{S}_1(T_{\{c_1,\ldots,c_m\}})$. Clearly $\underline{B}$ has the property that any 1-type of an expanded theory $\mathrm{Th}(\underline{A}_{\{c_1,\ldots,c_m\}})$ is realised in $\underline{B}_C$.
Repeating this construction we get an elementary chain

$$\underline{A}^{(0)} \preccurlyeq \underline{A}^{(1)} \preccurlyeq \ldots \preccurlyeq \underline{A}^{(n)} \ldots$$

of countable models of T with $\underline{A}^{(0)} = \underline{A}$ and the property that any 1-type in $\mathrm{Th}(\underline{A}^{(n)}_{\{c_1,\ldots,c_m\}})$ is realised in $\underline{A}^{(n+1)}_{c_1,\ldots,c_m}$ for any assignment of constant symbols $c_1, \ldots, c_m$, any m.
Then the union $\underline{A}^* = \bigcup_n \underline{A}^{(n)}$ of the elementary chain, by Exercise 4.12, is an elemenary extension of $\underline{A}$ and indeed of each $\underline{A}^{(n)}$. It follows that $\underline{A}^*$ is a countable saturated model of T. The converse direction follows from the exercise below. □

Exercise 8.4. Assume T has a countable universal model. Show that T is small.

Solution. Let N be a countable universal model of T. Any n-type p of T is realized by some n-tuple b in some model M of T. By Löwenehim-Skolem, there exists a countable elmentary submodel M_0 of M containing the entries of b. Since N is universal, there exists an elementary embedding $f : M \to N$. Thus $p = tp_M(b) = tp_{M_0}(b) = tp_N(f(b))$. So p is realized in N.
Thus the map from n-tuples of N to n-types of T, defined by

$$a \mapsto tp_N(a)$$

is a surjective map. So the number of n-types of T is at most $|N^n| \leq \aleph_0^n \leq \aleph_0$. This proves that T is small.
When T is not small, we can conclude it has many non-isomorphic models.

Proposition 8.5. *Suppose* card $\mathrm{S}_n(T) = \kappa > \aleph_0$. *Then T has a set of κ pairwise non-isomorphic countable models.*

Proof For any n-type there is a countable model that realises the type, and in a countable model at most countably many complete types can be realized. □

Exercise 8.6. Let T be a complete theory in a countable language. Assume some countable model M of T is both prime and universal. Prove that T is $\aleph_0$-categorical.

Theorem 8.7 (Vaught's never-two theorem)**.** *Let T be a complete theory in a countable language. Then the number of isomorphism types of countable models of T is not equal to* 2.

Proof. Suppose for the sake of contradiction that T has precisely two countable models, up to isomorphism. In particular it has $\leq \aleph_0$ of them; so by Proposition 8.5, T is small.
Thus T has a prime model, and a saturated model. If they are isomorphic, since the saturated model is universal and the prime model is atomic, *every* countable model of T is atomic. But in this case any two countable models are isomorphic, so the number is 1 and not 2.
Thus we have identified the two models of T: the prime and the saturated one. We will now construct a third.
Let p be a non-principal n-type of T, say in the variables x. Let c be a new tuple of constant symbols, and let T' be the theory $p(c)$. Then T' is also small, so it has a prime model M'. Let $a = c^{M'}$, and let M be the reduct of M' to L. Note that M' realizes a non-principal type (namely p) so it is not the atomic model. But $F_n(T')$ contains $F_n(T)$ and so is infinite, so T' is not $\aleph_0$-categorical, hence has a non-prinicipal type q. This type is *not* realized in M'. Thus M is not saturated. We have found a third model of T. $\square$

Exercise 8.8. Go through the above corollary and justify each statement using a previous result.

A model M can be called *finitely universal* if it realizes all types.

Exercise 8.9. Assume T has finitely many isomorphism types of countable models. Show that T has a countable finitely universal model, that is not saturated.

I do not know whether in this situation the finitely universal model must be universal.
Question. If T has finitely many countable models, up to isomorphism, must it have a universal model that is not saturated?
Remark There exist examples of complete countable theories whose number of countable models is $3, 4, 5, \cdots$ as well as $1, \aleph_0$ and $2^{\aleph_0}$. Morley has proved

that the number cannot be strictly between $\aleph_1$ and $2^{\aleph_0}$. It remains unknown whether when $\aleph_1 < 2^{\aleph_0}$ there can be a theory with precisely $\aleph_1$ isomorphism types of countable models.
Vaught conjectured that the answer is no; Shelah conjectured yes in a strong form. Vaught's conjecture led to a considerable amount of research; many cases were proved, notably for ω-stable theories, i.e. theories T such that T_M is small for all countable models M.

8.10 The perfect set theorem

Theorem 8.11. *Suppose* $\mathrm{S}_n(T)$ *is uncountable. Then* card $\mathrm{S}_n(T) = 2^{\aleph_0}$.

Proof Let F_n be the algebra of formulas of T in n variables. Call a formula $\varphi(x_1, \ldots, x_n)$ *small* if

$$U_\varphi = \{p \in S_n(T) : \ \varphi \in p\}$$

is countable. Otherwise, say φ is big.

Lemma 8.12. *For any big* φ *there are big* φ_0 *and* φ_1 *such that* $\varphi \equiv \varphi_0 \vee \varphi_1$ *and there is no* n*-type containing both of the formulas, that is* $T \vDash \neg\exists\bar{v}(\varphi_0 \wedge \varphi_1)$.

Proof Suppose not. Define

$$q_\varphi = \{\psi \in F_n : (\psi \wedge \varphi) \text{ is big }\}.$$

This is a complete type. Indeed, (i) of the definition of type follows from the fact that every ψ in q_φ belongs to a type, since ψ is big .
(ii) follows from the assumption that φ can not be divided into two big parts: $\psi_1 \wedge \psi_2 \wedge \varphi$ is big , if $\psi_1, \psi_2 \in q_\varphi$.
(iii) is immediate from the same assumption.
Now notice that

$$U_\varphi = \{q_\varphi\} \cup \bigcup \{U_{\neg\psi\wedge\varphi} : \psi \in q_\varphi\}.$$

By assumptions $U_{\neg\psi\wedge\varphi}$ is at most countable, for every $\psi \in q_\varphi$, contradicting the fact that φ is big . □

Proof of the theorem. Notice first that the number of n-types is not greater than $2^{\aleph_0}$ since each type is just a subset of the countable set F^n. So we want to show that the number is not less than $2^{\aleph_0}$.

Let $\mathcal{M} = \{\mu : \mathbb{N} \to \{0,1\}\}$ be the set of all $\{0,1\}$-sequences. For each μ and $n \in \mathbb{N}$ define $\mu_{|n}$, the initial n-cut of μ, to be the reduction of μ to $\{1, \ldots, n\}$.
Define a big formula $\varphi_{\mu,n}$ by induction on n :
For $n = 0$ let it be the formula $v_1 = v_1$.
If $\varphi_{\mu,n}$ is defined then $\varphi_{\mu,n+1}$ is either one of the two big formulas that divide $\varphi_{\mu,n+1}$, as given by the lemma above, depending on whether $\mu(n+1)$ is 0 or 1. So if $\mu_{|n} = \nu_{|n}$ and $\mu_{|n+1} \neq \nu_{|n+1}$, then $\varphi_{\mu,n} = \varphi_{\nu,n}$, and $\varphi_{\mu,n+1}$ but $\varphi_{\nu,n+1}$ can not belong to a common type. Also $T \vDash \forall \bar{v}(\varphi_{\mu,n+1} \to \varphi_{\mu,n})$.
Let now for each μ

$$q_\mu = \{\varphi_{\mu,i_1} \wedge \ldots \wedge \varphi_{\mu,i_n} : i_1, \ldots, i_n \in \mathbb{N}\}.$$

This, by definition, is a type. So, there is an extension $p_\mu \supseteq q_\mu$ which is a complete type. If $\mu \neq \nu$, say n is the first number such that $\mu(n) \neq \nu(n)$, then $\varphi_{\mu,n} \in p_\mu$, $\varphi_{\nu,n} \in p_\nu$ are the two mutually inconsistent formulas dividing $\varphi_{\mu,n}$, and so $p_\mu \neq p_\nu$.
Thus the number of complete types is not less than the number of infinite $\{0,1\}$-sequences, which is $2^{\aleph_0}$. □

Remark The theorem is a special case of the classical topological fact: *A complete separable metric space is either countable or contains a perfect set (a nonempty closed set with no isolated points); the latter has cardinality continuum.* , or a similar theorem in a topological version for compact Hausdorff spaces. This was proved by Cantor for closed subsets of the real line. Our U_ϕ's form a basis of such a topology on $\mathrm{S}_n(T)$.

Applying Theorem 8.5 and taking into account that, given a countable language L, there is at most $2^{\aleph_0}$ countable L-structures, we have:

Corollary 8.13. *Suppose for some n, $\mathrm{S}_n(T)$ is uncountable. Then T has exactly $2^{\aleph_0}$ non-isomorphic countable models.*

9 Saturated models

We continue to assume that L is countable for notational simplicity; but by contrast to the results on atomic models that depended on on the omitting types theorem, this is no longer really essential.

Definition A structure $\underline{A}$ is called $\aleph_1$-**saturated** if, for any expansion $\underline{A}'$ of $\underline{A}$ by countably many contant symbols, every 1-type in $\mathrm{Th}(\underline{A}')$ is realised in $\underline{A}'$.

When $|\underline{A}| = \aleph_1$, we simply say that $\underline{A}$ is *saturated* if it is $\aleph_1$-saturated.

Proposition 9.1. *Let T be a complete theory in a countable language. T has an $\aleph_1$-saturated model N of cardinality $2^{\aleph_0}$. Any model of T of cardinality $\leq \aleph_1$ embeds elementarily into N.*

Proof. This is very similar to the countable case, but with a different type count:
Claim Let $M \models T$, $|M| \leq 2^{\aleph_0}$. There are then $\leq 2^{\aleph_0}$ countable subsets $A \subset M$; for each such A, the number of (even partial) types over A is $\leq 2^{\aleph_0}$.

Proof. The first statement: $(2^{\aleph_0})^{\aleph_0} =^{\aleph_0 \cdot \aleph_0}= 2^{\aleph_0}$. For the second, note that $L(A)$ is countable so the set of formulas is countable; a (partial) type is a subset of the set of formulas, so there are at most $2^{\aleph_0}$. □

From this it follows that there exists $M^* \succ M$, realizing every type over a countable subset of M.
We now build a model N as a limit of an elementary chain $M_\alpha, \alpha < \omega_1$, with $M_\alpha \leq 2^{\aleph_0}$. If α is a limit ordinal, we let $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. If $\alpha = \beta + 1$, let $M_\alpha = M^*_\beta$.
Any countable subset of N is contained in some M_α, so N is $\aleph_1$-saturated. We have $|N| \leq \aleph_1 2^{\aleph_0} = 2^{\aleph_0}$.

□

Proposition 9.2. *Let T be a complete theory. Any two $\aleph_1$-saturated models of T of cardinality $\aleph_1$ are isomorphic.*

Proof. Let M, N be two $\aleph_1$-saturated models of T of cardinality $\aleph_1$. We seek to construct an isomorphism $F : M \to N$. An *approximation* is a partial elementary map $f : A \to B$ with A a countable subset of M, and B a countable subset of N. Such an approximation can always be extended to another, $f' : A' \to B'$, with a given element $a \in A$ lying in A'. We now set up a back-and-forth construction of approximations $f_\alpha : A_\alpha \to B_\alpha$ for $\alpha < \omega_1$; at the α'th (back-and-forth) step, we add the α'th element of a pre-determined enumeration of M to the domain, and the α'th element of N to the range; at limit steps, we take unions. □

The proof also shows that any approximation can be extended to an isomorphism; in the special case $M = N$, we obtain, as in Corollary 7.5:

Proposition 9.3. *Any $\aleph_1$-saturated model of T of cardinality $\aleph_1$ is homogeneous.*

Corollary 9.4. *Assume $\aleph_1 = 2^{\aleph_0}$, and let T be a theory with no finite models. Then T is complete if and only if any two saturated model of cardinality $\aleph_1$ are isomorphic.*

Proof. One direction was proved above. In the other, if T is not complete, it has two satisfiable extensions $T \bigcup \{\sigma\}$, $T \bigcup \{\neg\sigma\}$. Let T' be a complete theory containing $T \bigcup \{\sigma\}$, and T'' a complete theory containing $T \bigcup \{\neg\sigma\}$; let $M' \models T'$ and $M'' \models T''$ be saturated models of cardinality $\aleph_1$. Then M', M'' cannot be isomorphic. □

Let M be a structure of cardinality $\aleph_1$. Say M is qf-homogoeous if any isomorphism $f : A \to A'$ between countable substructures extends to an automorphism of M.
Corollary 9.4 can serve as a substitute for the Łoś-Vaught test; it works in principle for any theory. Let us extend it to formulas.

Corollary 9.5. *Assume $\aleph_1 = 2^{\aleph_0}$. Let T be a complete theory in a countable language. Then T admits quantifier elimination if and only if any saturated model of cardinality $\aleph_1$ is qf-homogeneous.*

Proof. This follows from Corollary 9.4, Proposition 9.2 and our characterization of quantifier elimination in terms of completeness of the theories T_A. But let us give a direct proof. Assume first that T admits quantifier elimination, and let M be a saturated model of cardinality $\aleph_1$. Let $f : A \to B$ be an isomorphism between countable substructures. Since every formula is equivalent to a quantifier-free one, f preserves all formulas so $f : A \to B$ is an approximation in the sense of the proof of Proposition 9.2; hence it extends to an isomorphism $F : M \to M$.
If T does not admit quantifier elimination, there exists a formula $\alpha(x)$ in variables $x = (x_1, \dots, x_n)$ not equivalent to a qf formula. So the following partial type is satisfiable:

$$\{\alpha(x), \neg\alpha(y)\} \bigcup \{\phi(x) \leftrightarrow \phi(y) : \phi \in F_L\}$$

Let M be any saturated model of cardinality $\aleph_1$, in which this partial type is realized, say by (a, b). Let A, B be the substructures generated by a, b. Then there exists an isomorphism $f : A \to B$ with $f(a) = b$. Since $M \models \alpha(a) \wedge \neg\alpha(b)$, it is impossible to extend f to an automorphism of M.

□

Remark. In many situations, we can harmlessly assume the continuum hypothesis; for instance, when we would like to prove the completeness of a given theory T in a countable language. This is explained in the axiomatic set theory class: using Gödel's constructible sets, a proof of finitary statements using CH can be converted to a proof without it. Beth's theorem below, as well as the proof of completeness of RCF using Corollary 9.4, are examples of this.

9.6 Beth's definability theorem

Let T be a theory in a language L, and T' a theory in a bigger language $L' = L \bigcup \{R\}$. We say that R is *implicitly defined* by T' if any model of T can be expanded in at most one way to a model of T'. Note that this implies, in particular, that if $M' \models T'$ and $M = M'|L$, then any automorphism σ of M must preserve R and hence be an automorphism of M' (since R,$\sigma(R)$ are two expansions of M to models of T', we have $R = \sigma(R)$.)
We say that R is *explicitly defined* if for some formula ϕ of L, $T' \models \phi \leftrightarrow R$.

Lemma 9.7. *R is explicitly defined if and only if there exist formulas $\phi_1, \ldots, \phi_m$ of L such that*

$$T' \models (\forall x, y)(\bigwedge_i \phi_i(x) \leftrightarrow \phi_i(y)) \to (R(x) \leftrightarrow R(y))$$

Proof. For a function $\nu : \{1, \ldots, m\} \to \{0, 1\}$, let $\psi_\nu = \bigwedge_{i=1}^m \neg^{\nu(i)}\phi_i$ (a conjunction of the ϕ_i and their negations.) Then for each ν, either $T' \models \psi_\nu \to R$ or $T' \models \psi_\nu \to \neg R$. Let Z be the set of ν such that the first case holds. Then $T' \models R \leftrightarrow \bigwedge_{\nu \in Z} \psi_\nu$. So R is explicitly definable. □

Theorem 9.8 (Beth's definability theorem)**.** *If R is implicitly definable, it is explicitly definable.*

Proof. (This proof assumes CH.) Consider the set of formulas

$$\Gamma = \{R(x), \neg R(y)\} \bigcup \{\phi(x) \leftrightarrow \phi(y) : \phi \in F_L\}$$

Claim $T \bigcup \Gamma$ is not satisfiable.
For let M' be an $\aleph_1$-saturated model of T' of size $\aleph_1$, and let M be the restriction to L. Note that M is also saturated. If Γ were satisfiable then by saturation of M', it would be realized by some (a, b) from M. Then $tp_L(a) = tp_L(b)$, so by homogeneity of M, there exists an automorphism $\sigma : M \to M$ with $\sigma(a) = b$. By the implicit definability of R, we have $\sigma(R) = R$. But $R(a)$ and $\neg R(b)$, a contradiction.
By compactness, there exist $\phi_1, \ldots, \phi_m$ such that

$$T' \models (\bigwedge_i \phi_i(x) \leftrightarrow \phi_i(y)) \to (R(x) \leftrightarrow R(y))$$

By the lemma, R is explicitly definable. □

9.9 The theory of the real field.

We will apply our completeness and QE criterion to the ordered field of real numbers.

Theorem 9.10. $Th((\mathbb{R}, +, \cdot, <))$ *is decidable, and admits quantifier elimination.*

Example 9.11. Let $X \subset \mathbb{R}^3$ be a set defined by an algebraic equation. Consider also a curve $c = (c_1(t), c_2(t), c_3(t))$ in $\mathbb{R}^3$ parameterized by a variable t, for instance $c_i(t) = t^i$. Let $\delta(t)$ be the *distance* from $c(t)$ to X. It is definable, as a function of t: we have

$$c(t) \leq d \iff (\exists (x_1, x_2, x_3) \in X)(\sum_{i=1}^{3}(c_i(t) - x_i)^2 = d^2)$$

and $c(t) = d$ iff $c(t) \leq d \& (\forall u < d)(\neg c(t) \leq u)$. It follows from quantifier-elimination that $\delta(t)$ is defined by a *quantifier-free* formula from t, and hence is an *algebraic function*, i.e. there exists a polynomial $H(t, u)$ such that $H(t, \delta(t)) = 0$.

To prove the theorem, we first write a theory T that holds in $Th((\mathbb{R}, +, \cdot, <))$. We then show that T is complete and admits QE, using our criteria above.
Definition A *real closed field* is an ordered field K satisfying the following property: for every $f \in K[t]$ and every $a, b \in K$ with $a < b$ and $f(a)f(b) < 0$, there exists $c \in (a, b)$ with $f(c) = 0$.

This is not the standard definition, but it is equivalent to it and is convenient for our purposes. It is easy to see that the class of real closed fields is axiomatizable. Asides from the ordered field axioms, for each n, we have an axiom asserting that for any polynomial $f(t) = a_0 + a_1 t + \cdots + a_{n-1}t^{n-1} + a_n t^n$ of degree $\leq n$, and every $a, b \in K$ with $a < b$ and $f(a)f(b) < 0$, there exists $c \in (a, b)$ with $f(c) = 0$. Let RCF be the theory generated by these axioms.

Proposition 9.12. *RCF is complete and has QE.*

To prove this, let $M, N \models RCF$ and let A, B be countable subrings, and $f : A \to B$ an isomorphism. We may assume $A = B$ and f is the identity. We have to extend f so that the domain includes a new element $m \in M$. We define the *degree* of m/A to be the degree of the least polynomial in $A[X]$ of which m is a root; or ∞ if m is not algebraic over A.
We use the following strategy: first extend $f : A \to B$ by adding, if possible, an algebraic element to A or to B, and of least possible degree. Only after this has been exhausted, proceed to add an arbitrary ('next') $m \in M$ or $n \in N$.
Assume first that no element of $M \setminus A$ or of $N \setminus A$ is algebraic over A. In this case, choose any $n \in N$ with the same cut as m over A. In other words for $a \in A$, we have $a < m$ iff $a < n$. This is possible by $\aleph_1$-saturation of N. This done, we define an isomorphism $A[m] \to A[n]$: simply map $f(m) \mapsto f(n)$ for any $f \in A[X]$. We must show that if $f(m) > 0$, then $f(n) > 0$. Now f has only finitely many roots in A; let r, s be respectively the largest one below m and the smallest above it, i.e. $r < m < s$ (modify the argument using only one inequality if there is no r or no s.) Then f does not change sign in M in the interval $[r, s]$ For otherwise, it would have an additional zero in the interval, which would be in A. Similarly, f does not change sign in N. So $f(m) > 0$ iff $f((r + s)/2) > 0$ iff $f(n) > 0$.
The proof of the finite degree case is similar: we take the least possible degree on either side. Say $f(m) = 0$, $f \in A[X]$ has degree d and no smaller degree is possible, in M or in N. Find $r < s \in A$ with $r < m < s$ and such that f has no zeroes in (r, m) or in (m, s). Argue that $f(r)f(s) < 0$ so that f has a zero n in N too. Now to construct the isomorphism $A[m] \to A[n]$ we need worry only about polynomials g of degree $< d$, and this goes as before.