This problem sheet is for the (TA led) shorter class in week 2. Only background material and material from the first two lectures should be needed.

Estimates and asymptotics, union bound and first-moment method

1. Prove the following inequalities:

   (a) $1 + x \leqslant e^x$ for all real x.

   (b) $(1 + a)^n \leqslant e^{an}$ for $a > -1$, $n \geqslant 0$.

   (c) $k! \geqslant k^k/e^k$ for $k \geqslant 1$.

   (d) $\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \frac{n^k}{k!} \leqslant \left(\frac{en}{k}\right)^k$ for $1 \leqslant k \leqslant n$.

2. For the following functions $f(n)$ and $g(n)$, decide whether $f = o(g)$ or $g = o(f)$ or $f = \Theta(g)$ as $n \to \infty$:

   (a) $f(n) = \binom{n}{k}$, $g(n) = n^k$, first for k fixed and then for the case where $k = k(n) \to \infty$ as $n \to \infty$;

   (b) $f(n) = (\log n)^{1000}$, $g(n) = n^{1/1000}$;

3. In lectures we saw that the kth diagonal Ramsey number satisfies

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}},$$

   for each integer n. By considering $n = \lfloor e^{-1}k2^{k/2} \rfloor$, deduce that

$$R(k,k) \geqslant \big(1 - o(1)\big)e^{-1}\, k2^{k/2}.$$

4. Let F be a collection of binary strings ("codewords") of finite length, where the ith codeword has length c_i. Suppose that no member of F is a prefix of another member (so you can decode any string made up by concatenating codewords as you go along, without looking ahead). Show that $\sum_i 2^{-c_i} \leqslant 1$ (the *Kraft inequality* for prefix-free codes).

5. *Harder - think about what random choice to make!* Let G be a bipartite graph with n vertices. Suppose each vertex v has a list $S(v)$ of more than $\log_2 n$ colours associated to it. Show that there is a proper colouring of G in which each vertex v receives a colour from its list $S(v)$.

6. Show that for $r \geqslant 2$, any graph G contains an r-partite subgraph H with $e(H) \geqslant \frac{r-1}{r}e(G)$.

PTO

7. *[If time permits.]* Let G be a graph with n vertices, and let d_v denote the degree of vertex v.

   (i) Consider a random ordering of $V = V(G)$ (chosen uniformly from all $n!$ possibilities). What is the probability that v precedes all its neighbours in the ordering?

   (ii) Show that G has an independent set of size at least $\sum_{v \in V} \frac{1}{d_v+1}$.

   (iii) Deduce that any graph with n vertices and m edges has an independent set of size at least $\frac{n^2}{2m+n}$.

Bonus question (for MFoCS students, optional for others). May not be covered in classes – if not, contact me with any questions!

A (finite, or infinite and convergent) sum $S = \sum_{i \geqslant 0} a_i$ is said to *satisfy the alternating inequalities* if the partial sum $\sum_{i=0}^{t} a_i$ is at least S for all even t and at most S for all odd t; that is, the partial sums alternately over- and under-estimate the final result.

8. Let $I_1, \ldots, I_n$ be the indicator functions of n events $E_1, \ldots, E_n$. For $0 \leqslant r \leqslant n$ let $S_r = \sum_{A \subseteq [n],\, |A|=r} \prod_{i \in A} I_i$, where $[n] = \{1, 2, \ldots, n\}$. Show that

$$\prod_{i=1}^{n} (1 - I_i) = \sum_{r=0}^{n} (-1)^r S_r, \tag{0.1}$$

and that the sum satisfies the alternating inequalities. [Both sides are random; the statement is that the relevant inequalities *always* hold. You may want to consider different cases according to how many of the events E_i hold.] Deduce that

$$\mathbb{P}(\text{no } E_i \text{ holds}) = \sum_{r=0}^{n} (-1)^r \sum_{A \subseteq [n],\, |A|=r} \mathbb{P}\left(\cap_{i \in A} E_i\right), \tag{0.2}$$

and that the sum satisfies the alternating inequalities. [This is a form of the inclusion–exclusion formula.]

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk