Second-moment method and thresholds:

1. Show that all (finite) graphs of the following types are strictly balanced: complete graphs, cycles, trees, complete bipartite graphs, connected regular graphs.

2. What is the threshold for $G(n,p)$ to contain a cycle of length k, for fixed k?

   Find a threshold function for the property of containing a cycle. [*Careful! It doesn't quite follow immediately from the first part.*]

3. Let $p = p(n) = \frac{\log n + f(n)}{n}$. Show that if $f(n) \to \infty$ then the probability that $G(n,p)$ contains an isolated vertex tends to 0, and that if $f(n) \to -\infty$ then this probability tends to 1. [*Hint. Apply the first and second moment methods to the number X of isolated vertices. We may assume (why?) that $|f(n)|$ is not too large, say $|f(n)| \leqslant \log n$. It may be useful to note that $1-p = e^{-p+O(p^2)}$ as $p \to 0$.*]

4. Let $S_{n,p}$ be a random subset of $\{1,2,\ldots,n\}$ chosen by including each element independently with probability p.

   (i) Show that $p = n^{-2/3}$ is a threshold function for the property "$S_{n,p}$ contains an arithmetic progression of length 3".

   (ii) Show that for $k \geqslant 3$ fixed, $p = n^{-2/k}$ is a threshold function for $S_{n,p}$ to contain an arithmetic progression of length k.

The Local Lemma:

5. Show that it is possible to colour the edges of K_n with $k = \lceil 3\sqrt{n}\rceil$ colours so that no triangle has all its edges the same colour.

6. Let $H = (V,E)$ be a hypergraph. Suppose the vertices are k-coloured uniformly at random; each $v \in V$ receives each colour with probability $1/k$, and the colours of different vertices are independent. For $e \in E$, let A_e be the event that the edge e is monochromatic.

   Show that if $|e \cap f| \leqslant 1$, then A_e and A_f are independent.

   Is it true that A_e is independent of the collection $\{A_f : |e \cap f| \leqslant 1\}$?

7. Let $G = (V,E)$ be a graph with maximum degree Δ, and let $V_1, V_2, \ldots, V_s$ be a collection of disjoint subsets of V, each of size $k \geqslant 2e\Delta$. Show that G has an independent set which contains a vertex from each set V_i. [There is a hint over the page.]

8. Let $G = (V,E)$ be a graph, and suppose that for each $v \in V$ there is a list $S(v)$ of at least $2er$ colours, where r is a positive integer. Suppose also that for each $v \in V$ and each $c \in S(v)$, there are at most r neighbours u of v such that $c \in S(u)$.

   Prove that there is a proper colouring of G under which each vertex v receives a colour from its list $S(v)$.

Hint for Question 7: pick one vertex X_i from each V_i. You could consider a 'bad' event E_{uv} for each edge uv with u and v in different sets V_i.

Bonus questions (compulsory for MFoCS students, optional for others). Most/all of these will not be discussed in the classes. Contact the lecturer to discuss!

The r^{th} (falling) *factorial moment* of a random variable X is defined to be

$$\mathbb{E}_r[X] = \mathbb{E}[X(X-1)\cdots(X-r+1)].$$

9. Let X be a random variable taking values in $\{0, 1, \ldots, n\}$. Show that

$$\mathbb{P}(X=0) = \sum_{r=0}^{n} (-1)^r \frac{\mathbb{E}_r[X]}{r!},$$

and that the sum satisfies the alternating inequalities. [Hint: write X as a sum of indicator variables.]

What can you say if X is unbounded, taking values in the non-negative integers?

10. Let $X_1, X_2, \ldots$ be a sequence of random variables each taking non-negative integer values. Suppose that for each $r \geqslant 0$ we have $\lim_{n\to\infty} \mathbb{E}_r[X_n] = \lambda_r < \infty$, and that $\lambda_r / r! \to 0$ as $r \to \infty$. Show that

$$\mathbb{P}(X_n = 0) \to \sum_{r=0}^{\infty} (-1)^r \frac{\lambda_r}{r!}.$$

[*Be careful exchanging sums and limits!*]

11. Let $X_1, X_2, \ldots$ and λ_r be as in the previous question, and let $k \geqslant 0$ be fixed. Assuming that $\lambda_{r+k}/r! \to 0$ as $r \to \infty$, show that

$$\mathbb{P}(X_n = k) \to \frac{1}{k!} \sum_{r=0}^{\infty} (-1)^r \frac{\lambda_{r+k}}{r!}.$$

In applications we often have a limiting distribution X in mind; these results show that under certain assumptions, if the moments of X_n tend to those of X, then $\mathbb{P}(X_n = k) \to \mathbb{P}(X = k)$.

12. Let $c > 0$ be constant. Suppose that $p = p(n)$ satisfies $n^4 p^6 \to c$ as $n \to \infty$, and let X_n denote the number of copies of K_4 in $G(n,p)$. Show that $\mathbb{P}(X_n > 0) \to 1 - e^{-c/24}$ as $n \to \infty$.

What can you say about the distribution of the number of copies of K_4? Can you generalize this to (certain) other graphs?

If you find an error please check the website, and if it has not already been corrected, e-mail riordan@maths.ox.ac.uk