C8.2: Stochastic analysis and PDEs
Problem sheet 2

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The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

Section 1 (Compulsory)

1. Show that a Markov pregenerator has the property that for $f \in \mathcal{D}(A)$, $\lambda \geq 0$ and $f - \lambda A f = g$, then $\|f\| \leq \|g\|$. Deduce that, in particular, g determines f uniquely.

2. Calculate the infinitesimal generator of the pure jump process X_t modelling the motion of a particle which, if it is currently at location x, it will wait an exponentially distributed amount of time with parameter $\alpha(x)$ before jumping to a new location determined by the probability measure $\mu(x, dy)$. You may assume that $\alpha(x)$ is uniformly bounded.

3. Check that each of the following is a Markov generator:

   (a) $A = G - I$ where G is a positive operator defined on all of $C(E)$ such that $G1 = 1$.

   (b) $E = [0,1]$ and $Af(\eta) = \frac{1}{2} f''(\eta)$ with
   $$\mathcal{D}(A) = \{f \in C(E) : f'' \in C(E), f'(0) = 0 = f'(1)\}.$$

   (c) $E = [0,1]$ and $Af(\eta) = \frac{1}{2} f''(\eta)$ with
   $$\mathcal{D}(A) = \{f \in C(E) : f'' \in C(E), f''(0) = 0 = f''(1)\}.$$

4. Let $E = [0,1]$ and consider the operator $\mathcal{L}$ defined by $\mathcal{L}f(x) = f'(0)$ with
   $$\mathcal{D}(\mathcal{L}) = \{f \in C([0,1]) : f'(0) \text{ exists}\}.$$
   Show that the closure of the graph of $\mathcal{L}$ does not correspond to the graph of a linear operator.

5. (Discrete time martingale problem, Ethier & Kurtz, Chapter 4, Exercise 16)

   (a) Let E be a compact (or locally compact) space and $B(E)$ the bounded Borel measurable functions on E. Let $\mu(x, \Gamma)$ be a transition function on $E \times \mathcal{B}(E)$ and let $\{X(n)\}_{n \in \mathbb{N}}$ be a sequence of E-valued random variables. Define $A : B(E) \to B(E)$ by
   $$Af(x) = \int_E f(y)\mu(x, dy) - f(x),$$

and suppose that, for each $f \in B(E)$,

$$f(X(n)) - \sum_{k=0}^{n-1} Af(X(k))$$

is a martingale with respect to the natural filtration generated by X. Show that X is a Markov chain with transition function $\mu(x, \Gamma)$.

(b) Let $X(n)$, $n = 0.1. \ldots$, be a sequence of $\mathbb{Z}$-valued random variables such that for each $n \geq 0$, $|X(n+1) - X(n)| = 1$. Let $g : \mathbb{Z} \to [-1, 1]$ and suppose that

$$X(n) - \sum_{k=0}^{n-1} g(X(k))$$

is a martingale with respect to the natural filtration generated by X. Show that X is a Markov chain and calculate its transition probabilities in terms of g.

6. The Wright-Fisher diffusion, which takes values in $[0, 1]$ has generator

$$Af(x) = \frac{1}{2} x(1-x) f''(x),$$

when restricted to an appropriate subset of the twice continuously differentiable functions on $[0, 1]$. By considering the martingale problem, that

$$f(X_t) - \int_0^t Af(X_s) ds,$$

will be a $\mathbb{P}$-local martingale, with suitable functions f:

(a) Show that $X_\infty = \lim_{t \to \infty} X_t$ exists and find its expectation;

(b) Show that $\mathbb{P}[X_\infty \in \{0, 1\}] = 1$ and (using your previous calculation) find $\mathbb{P}[X_\infty = 1]$;

(c) Find $\mathbb{E}[\int_0^\infty X_s(1 - X_s) ds]$.

Now take $f(x) = 2x \log x + 2(1-x) \log(1-x)$. Although f isn't in the domain of the generator, we can find a twice continuously differentiable function which equals f on $[\epsilon, 1-\epsilon]$ and is in the domain. Taking this on trust, find an expression for the expected hitting time of $\{\epsilon, 1-\epsilon\}$ and hence of $\{0, 1\}$.

Section 2 (Extra practice questions, not for hand-in)

A. Show that if a Markov pregenerator is everywhere defined and is a bounded operator, then it is automatically a Markov generator. [*Hint: A bounded operator is automatically closed. To check that $\mathcal{R}(I - \lambda A) = C(E)$ for sufficiently small λ, it suffices to solve $f - \lambda Af = g$, for which you can try a 'geometric series' $f = \sum_{n=0}^\infty \lambda^n A^n g$, just as we did on the previous problem sheet.*]

B. (Brownian Motion with sticky boundary.) Show that $Af = \frac{1}{2} f''$ on

$$\mathcal{D}(A) = \{f \in C([0, \infty)) : f', f'' \in C([0, \infty)), f'(0) = cf''(0)\}$$

for a fixed $c > 0$ defines a Markov pregenerator.

The corresponding stochastic process is called *sticky* Brownian motion. It interpolates between absorbing and reflecting Brownian motion on the half line. In particular, unlike reflecting Brownian motion, the Lebesgue measure of $\{t : X_t = 0\}$ is positive. Indeed one can check that

$$\mathbb{E}_0 \left[\int_0^\infty \alpha e^{-\alpha t} \mathbf{1}_{X_t > 0} dt \right] = \frac{1}{1 + c\sqrt{2\alpha}}. \tag{1}$$

If you want to try to show this, use the fact that since the semigroups are known explicitly in the absorbing and reflecting cases, in an obvious notation we can solve the equations

$$f_a - \lambda A_a f_a = g, \qquad f_r - \lambda A_r f_r = g$$

explicitly. Since the form of the generators is the same (only the domains differ), one can solve $f - \lambda Af = g$ by taking f to be a constant multiple of $f_r''(0) f_a(x) + c f_a'(0) f_r(x)$. That provides an expression for

$$\mathbb{E} \left[\int_0^\infty \alpha e^{-\alpha t} g(X_t) dt \right].$$

C. Let X be a strong Markov process with Markov generator A on a compact set E. Let $P_t^\lambda f(x) = \mathbb{E}^x \exp(-\lambda t) f(X_t)$.

   (a) Show by the strong Markov property that for a stopping time τ we have

   $$P_\tau^\lambda P_t^\lambda = P_{\tau+t}^\lambda,$$

   and hence that we have Dynkin's formula for the resolvent $R_\lambda = \int_0^\infty e^{-\lambda t} P_t dt$; for $g \in C(E), \lambda > 0, x \in E$ that

   $$R_\lambda g(x) = \mathbb{E}^x \int_0^\tau e^{-\lambda t} g(X_t) dt + P_\tau^\lambda R_\lambda g(x).$$

   (b) Apply this to $g = (\lambda - A)f$, for $f \in \mathcal{D}(A)$, to obtain

   $$\mathbb{E}^x e^{-\lambda \tau} f(X_\tau) - f(x) = \mathbb{E}^x \int_0^\tau e^{-\lambda s} (A - \lambda) f(X_s) ds. \tag{2}$$

   Now let $\lambda \to 0$ to obtain for x such that $E^x \tau < \infty$,

   $$\mathbb{E}^x f(X_\tau) - f(x) = \mathbb{E}^x \int_0^\tau Af(X_s) ds.$$

   (c) Let X be a Brownian motion and define $T_a = \inf\{t : X_t = a\}$ to be the first hitting time of the point $a > 0$. Working over $C_0(\mathbb{R})$ and applying formula (2) to $f(x) = \exp(\theta x) I_{x \le a}$ for a suitably chosen θ show that

   $$\mathbb{E}^0 e^{-\lambda T_a} = e^{-a\sqrt{2\lambda}}, \ \ \forall \lambda \ge 0.$$

D. Show that almost sure convergence implies convergence in distribution.

E. Prove the Portmanteau Theorem:

   Theorem 0.1 (Portmanteau Theorem)**.** *Let $(X_n)_{n \ge 1}$ be a sequence of random variables taking values in S. The following are equivalent.*

(i) $X_n \to X$ *in distribution.*

(ii) *For any closed set* $K \subseteq S$, $\limsup_{n\to\infty} \mathbb{P}[X_n \in K] \leq \mathbb{P}[X \in K]$.

(iii) *For any open set* $O \subseteq S$, $\liminf_{n\to\infty} \mathbb{P}[X_n \in O] \geq \mathbb{P}[X \in O]$.

(iv) *For all Borel sets* $A \subseteq S$ *such that* $\mathbb{P}[X \in \partial A] = 0$, $\lim_{n\to\infty} \mathbb{P}[X_n \in A] = \mathbb{P}[X \in A]$.

(v) *For any bounded function* f, *denote by* D_f *the set of discontinuities of* f. *Then for any* f *such that* $\mathbb{P}[X \in D_f] = 0$, $\mathbb{E}[f(X_n)] \to \mathbb{E}[f(X)]$ *as* $n \to \infty$.