C8.2: Stochastic analysis and PDEs
Problem sheet 4

The questions on this sheet are divided into two sections. Those in the first section are compulsory and should be handed in for marking. Those in the second are extra practice questions and should not *be handed in.*

Section 1 (Compulsory)

1. Let r satisfy the stochastic differential equation

$$dr_t = -\beta r_t dt + \sigma\sqrt{r_t}dW_t,$$

   where $\{W_t\}_{t\geq 0}$ is standard $\mathbb{P}$-Brownian motion and $r_0 > 0$.

   Suppose that $\{u(t)\}_{t\geq 0}$ satisfies the ordinary differential equation

$$\frac{du}{dt}(t) = -\beta u(t) - \frac{\sigma^2}{2}u(t)^2, \quad u(0) = \theta,$$

   for some constant $\theta > 0$. Fix $T > 0$. For $0 \leq t \leq T$ find the stochastic differential equation satisfied by

$$\exp\left(-u(T-t)r_t\right).$$

   Hence find the moment generating function for r_T. Calculate the mean and variance of r_T and $\mathbb{P}[r_T = 0]$.

2. Use the Feynman-Kac stochastic representation formula to solve

$$\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2\frac{\partial^2 F}{\partial x^2}(t,x) = 0,$$

   subject to the terminal value condition

$$F(T,x) = x^4.$$

3. We can use the Feynman-Kac representation to find the partial differential equation solved by the transition densities of solutions to stochastic differential equations.

   Suppose that

$$dX_t = \mu(t,X_t)dt + \sigma(t,X_t)dW_t. \tag{1}$$

   For any set B let

$$p_B(t,x;T,y) \triangleq \mathbb{P}\left[X_T \in B\middle|\, X_t = x\right] = \mathbb{E}\left[\mathbf{1}_B(X_T)\middle|\, X_t = x\right].$$

Use the Feynman-Kac representation (assuming integrability conditions are satisfied) to write down an equation for

$$\frac{\partial p_B}{\partial t}(t, x; T, y)$$

Deduce that the transition density of the solution $\{X_s\}_{s \geq 0}$ to the stochastic differential equation (1) solves

$$\begin{aligned} \frac{\partial p}{\partial t}(t, x; T, y) + Ap(t, x; T, y) &= 0 \\ p(t, x; T, y) &\to \delta_y(x) \quad \text{as } t \to T. \end{aligned} \tag{2}$$

Equation (2) is known as the *Kolmogorov backward equation* (it operates on the 'backward in time' variables (t, x)).

There is also a Kolmogorov forward equation acting on the *forward* variables (T, y). In the above notation,

$$\frac{\partial p}{\partial T}(t, x; T, y) = A^* p(t, x; T, y)$$

where

$$A^* f(T, y) = -\frac{\partial}{\partial y}\left(\mu(T, y) f(T, y)\right) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\left(\sigma^2(t, Y) f(T, y)\right).$$

4. Suppose that $\{X_t\}_{t \geq 0}$ solves

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where $\{W_t\}_{t \geq 0}$ is a $\mathbb{P}$-Brownian motion. For $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ given deterministic functions, find the partial differential equation satisfied by the function

$$F(t, x) \triangleq \mathbb{E}\left[\exp\left(-\int_t^T k(s, X_s)ds\right)\Phi(X_T)\middle| X_t = x\right],$$

for $0 \leq t \leq T$.

5. Suppose that for $0 \leq s \leq T$,

$$dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s, \quad X_t = x,$$

where $\{W_s\}_{t \leq s \leq T}$ is a $\mathbb{P}$-Brownian motion, and let $k, \Phi : \mathbb{R} \to \mathbb{R}$ be given deterministic functions. Find the partial differential equation satisfied by

$$F(t, x) = \mathbb{E}\left[\Phi(X_T) \middle| X_t = x\right] + \int_t^T \mathbb{E}\left[k(X_s) \middle| X_t = x\right] ds.$$

6. Let B be a Brownian motion in $\mathbb{R}$ and consider $A_t = \int_0^t I_{\{B_u > 0\}} du$, the amount of time that Brownian motion spends in the positive half line up to time t. Let $F(t, x) = \mathbb{E}(\exp(-\theta A_t)|B_0 = x)$, the Laplace transform of A_t given that the Brownian motion starts from x. By setting the dissipation term to be $r(t, B_t) = -\theta I_{\{B_t > 0\}}$ and the initial condition to be 1 and using a time reversed version of the Feynman-Kac formula, show the PDE satisfied by F, is

$$\frac{\partial F}{\partial t} = \begin{cases} \frac{1}{2}\frac{\partial^2 F}{\partial x^2} - \theta F & x > 0, t > 0 \\ \frac{1}{2}\frac{\partial^2 F}{\partial x^2} & x \leq 0, t > 0. \end{cases}$$

specifying the initial conditions and, carefully, the continuity conditions at 0. By taking Laplace transforms, $\hat{F}(\lambda, x) = \int_0^\infty \exp(-\lambda t) F(t, x) dt$ and solving the resulting ODE, show that

$$\hat{F}(\lambda, 0) = \frac{1}{\sqrt{\lambda}\sqrt{\lambda + \theta}}. \tag{3}$$

From this we can derive Levy's arcsine law,

$$P(A_t \leq s | X_0 = 0) = \int_0^s \frac{1}{\pi\sqrt{u(t-u)}} du = \frac{2}{\pi} \arcsin(\sqrt{\frac{s}{t}}), \quad 0 \leq s \leq t.$$

To see this compute the Laplace transform of the arcsine law by suitably integrating to show that the transform is as given in (3).

Section 2 (Extra practice questions, not for hand-in)

A. Suppose we start a three dimensional Brownian motion at the origin. Fix a radius $0 < r < 1$. In which of the annuli

$$A[a] = \{x \in \mathbb{R}^3 \, : \, a - r \leq |x| \leq a\} \text{ for } a \in [r, 1]$$

is the expected occupation time maximal?

B. Suppose that $v(t, x)$ solves

$$\frac{\partial v}{\partial t}(t, x) + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2}(t, x) - rv(t, x) = 0, \quad 0 \leq t \leq T.$$

Show that for any constant θ,

$$v_\theta(t, x) \triangleq \frac{x}{\theta} v\left(t, \frac{\theta^2}{x}\right)$$

is another solution.

C. *(The Ornstein-Uhlenbeck process).* Let $\{W_t\}_{t \geq 0}$ denote standard Brownian motion under $\mathbb{P}$. The Ornstein-Uhlenbeck process, $\{X_t\}_{t \geq 0}$, is the unique solution to *Langevin's equation*,

$$dX_t = -\alpha X_t dt + dW_t, \qquad X_0 = x.$$

This equation was originally introduced as a simple idealised model for the velocity of a particle suspended in a liquid. Verify that

$$X_t = e^{-\alpha t} x + e^{-\alpha t} \int_0^t e^{\alpha s} dW_s,$$

and use this expression to calculate the mean and variance of X_t.

D. The process usually known as *Geometric Brownian motion* solves the s.d.e.

$$dS_t = \mu S_t dt + \sigma S_t dW_t.$$

Find the forward and backward Kolmogorov equations for geometric Brownian motion and show that the transition density for the process is the lognormal density given by

$$p(t, x; T, y) = \frac{1}{\sigma y \sqrt{2\pi(T-t)}} \exp\left(-\frac{\left(\log(y/x) - \left(\mu - \frac{1}{2}\sigma^2\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right).$$