$\boxed{2}$

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)dt^2 - \frac{4Mar}{\Sigma}\sin^2\theta\, d\phi\, dt$$
$$+ \frac{1}{\Sigma}\sin^2\theta\left(\Delta\Sigma + 2Mr(r^2+a^2)\right)d\phi^2$$
$$+ \Sigma\left(\frac{1}{\Delta}dr^2 + d\theta^2\right)$$

Θs WL tangent vector u is TL-KV

* observer's worldline
  = integral curves of u
  (which is a TL-Killing vector)

$\boxed{\Theta \text{ is an stationary observer}}$

u is a KV $\Rightarrow$ metric does not change along its integral curves

* a) u is a KV then it must be a linear combination of
  $K = \partial_t$ and $L = \partial_\phi$

$$u = u^\mu \partial_\mu = u^t \partial_t + u^\phi \partial_\phi$$
(with $u^\theta = 0$ and $u^r = 0$)

b) u is timelike

$\Rightarrow$ Lectures: there is no TL Killing vector in the region $r_- < r < r_+$

$\Rightarrow$ observer with u a TL-KV cannot exist in this region. | no stationary observers

$\omega = \frac{d\phi}{dt}$
$= \frac{d\phi/d\tau}{dt/d\tau}$

* angular velocity of observer in region $r > r_+$

$$\omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t}$$

$$g(u,u) = g_{\mu\nu} u^\mu u^\nu < 0$$

with $u = u^t \partial_t + u^\phi \partial_\phi$

(ie a linear combination of the KVs $K = \partial_t$ and $L = \partial_\phi$)

$$g(u,u) = g_{tt} (u^t)^2 + 2 g_{t\phi} u^t u^\phi + g_{\phi\phi} (u^\phi)^2$$
$$= (u^t)^2 \underbrace{\left(g_{tt} + 2 g_{t\phi}\,\omega + g_{\phi\phi}\,\omega^2\right)}_{\text{quadratic in } \omega}$$

$g(u,u) < 0$ iff

$$\frac{1}{4} D = g_{t\phi}^2 - g_{tt} g_{\phi\phi} > 0$$

lectures: u exists outside horizon $r > r_+$ and the observer will have angular velocity $\omega = \frac{u^\phi}{u^t} = \frac{d\phi/d\tau}{dt/d\tau}$

* Bounds for ω

$$g(u,u) = (u^t)^2 (g_{\phi\phi})\left(\omega^2 + 2\frac{g_{t\phi}}{g_{\phi\phi}}\omega + \frac{g_{tt}}{g_{\phi\phi}}\right)$$

$$= (u^t)^2 \underset{@\ +ve}{(g_{\phi\phi})} \underbrace{(\omega-\omega_+)(\omega-\omega_-)}_{\text{needs to be } <0}$$

where

$$\omega_\pm = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \frac{1}{2g_{\phi\phi}}\sqrt{D}$$

$$= -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

+ve

Let $$\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mar}{\Delta\Sigma + 2Mr(r^2+a^2)}$$

Ω is always positive, has same sign as $J = Ma$ (BH's angular momentum)

Then $$\omega_\pm = \Omega \pm \sqrt{\Omega^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

(note that $g_{\phi\phi}$ is always +ve)

and $g(u,u) < 0$ for

$$\omega_- < \omega < \omega_+$$

ω_+ is always +ve

ω_- is

+ve when $g_{tt} > 0$

ie $\Sigma > 2M$ ie $r_+ < r < \tilde{r}_+$

ie inside the ergosphere

−ve when $g_{tt} < 0$

ie $\Sigma < 2M$ ie $r > \tilde{r}_+$

ie outside the ergosphere

and vanishes when $g_{tt} = 0$

ie $\Sigma = 2M$ ie on $r = \tilde{r}_+$

Thus inside the ergosphere the observer has angular velocity $0 < \omega_- < \omega < \omega_+$ which is always positive and so it has the same sign as $J = Ma$

Angular velocity of BH