Q3

K is a Killing vector and S a Killing horizon for K.

The vector K satisfies

$$K^b \nabla_b K_a = \sigma K_a$$

where σ = surface gravity

(I am labelling the surface gravity by σ instead of the usual κ [ie kappa] so that K is not confused with κ in this soln)

K is hypersurface orthogonal on S:

$$K_{[a} \nabla_b K_{c]} = 0$$

so

$$0 = K_a \nabla_{[b} K_{c]} + K_b \nabla_{[c} K_{a]} + K_c \nabla_{[a} K_{b]}$$

As K is Killing $\nabla_a K_b = - \nabla_b K_a$.

Then

$$2 K_c \nabla_a K_b = -2 K_a \nabla_b K_c + 2 K_b \nabla_a K_c$$

so

$$K_c \nabla_a K_b = -2 K_{[a} \nabla_{b]} K_c$$

Contracting this equation with $\nabla^a K^b$ we have

$$K_c (\nabla^a K^b)(\nabla_a K_b)$$
$$= -2 (\nabla^a K^b) K_{[a} \nabla_{b]} K_c$$
$$= -2 (\nabla^{[a} K^{b]}) K_a \nabla_b K_c$$

K is Killing

$$= -2 (\nabla^a K^b) K_a \nabla_b K_c$$
$$= -2 (K^a \nabla_a K_b) \nabla^b K_c$$
$$= -2 (\sigma K_b) K^b K_c$$
$$= -2 \sigma^2 K_c$$

Hence

$$\underline{\sigma^2 = -\frac{1}{2} (\nabla^a K^b)(\nabla_a K_b)}$$

on the horizon S

$\boxed{Q4}$

K TL-Killing st at infinity

$$g(K, K) \to -1$$

S Killing horizon for K

Let u be the 4-velocity of a stationary observer, then

$$u^0 = 1 \qquad u^i = 0,$$

and moreover, it moves on an orbit of K so

$$u = \alpha^{-1}(x)\, K$$

for some function α. Then

$$g(u, u) = -1 = \alpha^{-2}\, g(K, K)$$

so $g(K, K) = -\alpha^2$

with $\alpha^2 \to 1$ at infinity.

Then $\alpha^2 = (-g(K, K))$

The proper acceleration of the observer is

$$A = \frac{d}{d\tau} u = u^a \nabla_a u$$

$$= \alpha^{-1} K^a \nabla_a (\alpha^{-1} K)$$

$$A^a = \alpha^{-1} K^b \nabla_b (\alpha^{-1} K^a)$$

$$= \alpha^{-1} K^b (K^a \nabla_b \alpha^{-1} + \alpha^{-1} \nabla_b K^a)$$

First term: To compute $\nabla_b \alpha^{-1}$ note that α is a function of $\sqrt{-g(K,K)}$ only. Then

$$K^b \nabla_b \alpha^{-1} = -\alpha^{-2} K^b \alpha' \nabla_b \sqrt{-g(K,K)}$$

$$= -\alpha^{-2} K^b \alpha' \frac{1}{2\sqrt{-g(K,K)}} \nabla_b (-g(K,K))$$

$$= -\alpha^{-2} \quad \alpha' \frac{K^b K^c}{\sqrt{-g(K,K)}} \nabla_b K_c = 0$$

as K is a Killing vector.

Then

$$A^a = \alpha^{-2} K^b \nabla_b K^a$$

$$= -\alpha^{-2} K^b (\nabla_c K_b) g^{ca}$$

$$= -\alpha^{-2} \frac{1}{2} \nabla_c (K^b K_b) g^{ca}$$

$$= +\alpha^{-2} \frac{1}{2} (\nabla_c \alpha^2) g^{ca}$$

$$= \alpha^{-1} (\nabla_c \alpha) g^{ca}$$

so $\underline{A^a = g^{ab} \nabla_b \log \alpha}$

Let A be the magnitude of the observer's acceleration. Then

$$A^2 = g(A, A)$$

$$= (\nabla_a \log \alpha)(\nabla^a \log \alpha)$$

$$= \alpha^{-2} (\nabla_a \alpha)(\nabla^a \alpha)$$

so $A = \alpha^{-1} \left((\nabla_a \alpha)(\nabla^a \alpha) \right)^{1/2}$

Particle is held along an stationary orbit by a "massless string" with one end held by an observer at large r.

Let F = force on the particle applied by the string

$$= m A$$

$$F = m \alpha^{-1} (\nabla_a \alpha \nabla^a \alpha)^{1/2}$$

Moving the particle from one stationary orbit to another by δx^μ at large r, changes the energy by

$$\delta E_\infty = (F_\infty)_a \quad \delta x^a$$

energy at infinity

$$= \delta(-m K^a u_a) = \delta(-m\alpha)$$

$$= -m (\nabla_a \alpha) \, \delta x^a$$

So the magnitude of the force at infinity is

$$F_\infty = m\, A_\infty$$

$$= m\, \left((\nabla_a \alpha)(\nabla^a \alpha) \right)^{1/2}$$

so

$$A_\infty = \left((\nabla_a \alpha)(\nabla^a \alpha) \right)^{1/2}$$

$$= \alpha A$$

From Q3, the surface gravity is

$$\sigma^2 = -\frac{1}{2} (\nabla^a K^b)(\nabla_a K_b) \Big|_S$$

Consider

$$3 \left(K^{[a} \nabla^b K^{c]} \right) K_{[a} \nabla_b K_{c]}$$

$$= 3 \left(K^a \nabla^b K^c \right) K_{[a} \nabla_b K_{c]}$$

$$= \left(K^a \nabla^b K^c \right) \big(K_a \nabla_b K_c +$$

$$+ 2 K_{[b} \nabla_{c]} K_a \big)$$

$$3\left(K^{[a}\nabla^{b}K^{c]}\right)\left(K_{[a}\nabla_{b}K_{c]}\right)$$

$$= -\alpha^2 \left(\nabla^a K^b\right)\left(\nabla_a K_b\right)$$

$$- 2\, K^a\left(\nabla^b K^c\right) K_b \nabla_a K_c$$

As $\nabla_a\left(K_b K^b\right) \neq 0$ on $r_H = 0$

where $r_H = 0$ defines S

and $K_{[a}\nabla_b K_{c]} = 0$ on S

Then using L'Hospital rule

$$\lim_{r \to r_H} \frac{3\left(K^{[a}\nabla^{b}K^{c]}\right)\left(K_{[a}\nabla_{b}K_{c]}\right)}{K^a K_a}$$

$$= 0 = + \lim_{r\to r_H} \left(\nabla^a K^b\right)\left(\nabla_a K_b\right)$$

$$+ 2 \lim_{r\to r_H} \frac{K^a\left(\nabla^b K^c\right) K_b \nabla_a K_c}{\alpha^2}$$

Then

$$\sigma^2 = \lim_{r \to r_H} \frac{(K^b \bar{\nabla}_b K^c)(K^a \bar{\nabla}_a K_c)}{\alpha^2}$$

$$= \lim_{r \to r_H} \frac{(\alpha^2 A^c)(\alpha^2 A_c)}{\alpha^2}$$

where we have used

$$A^a = \alpha^{-2} K^b \bar{\nabla}_b K^a$$

Then $$\sigma = \lim_{r \to r_H} (\alpha A)$$

$\boxed{\text{Q5}}$

de Sitter:

$$-T^2+x^2+y^2+z^2+w^2=a^2$$

in 4+1 Minkowski space, M_{4+1}

The metric on M_{4+1} is

$$ds^2=-dT^2+dx^2+dy^2+dz^2+dw^2$$

$$dx+i\,dy=ae^{i\phi}\Big(d\big(\cosh\tfrac{t}{a}\sin\chi\sin\theta\big)$$
$$+i\big(\cosh\tfrac{t}{a}\sin\chi\sin\theta\big)d\phi\Big)$$

$$dx^2+dy^2=a^2\Big(d\big(\cosh\tfrac{t}{a}\sin\chi\sin\theta\big)^2$$
$$+\big(\cosh\tfrac{t}{a}\sin\chi\sin\theta\big)^2d\phi^2\Big)$$
$$=a^2\Big(\Big(d\big(\cosh\tfrac{t}{a}\sin\chi\big)\sin\theta$$
$$+\cosh\tfrac{t}{a}\sin\chi\cos\theta\,d\theta\Big)^2$$
$$+\underbrace{\big(\cosh\tfrac{t}{a}\sin\chi\sin\theta\big)^2}_{f^2}d\phi^2\Big)$$

5.2

$$dx^2+dy^2+dz^2$$
$$= a^2\Big(\Big(d\big(\cosh\tfrac{t}{a}\sin\chi\big)\sin\theta + \cosh\tfrac{t}{a}\sin\chi\cos\theta\, d\theta\Big)^2 +$$
$$+ \Big(d\big(\cosh\tfrac{t}{a}\sin\chi\big)\cos\theta - \big(\cosh\tfrac{t}{a}\sin\chi\big)\sin\theta\, d\theta\Big)^2 + f^2 d\phi^2\Big)$$
$$= a^2\Big(d\big(\cosh\tfrac{t}{a}\sin\chi\big)^2 + \big(\cosh\tfrac{t}{a}\sin\chi\big)^2 d\theta^2 + f^2 d\phi^2\Big)$$

$$dx^2+dy^2+dz^2+dw^2$$
$$= a^2\Big(\Big(d\big(\cosh\tfrac{t}{a}\big)\sin\chi - \cosh\tfrac{t}{a}\cos\chi\, d\chi\Big)^2 + \big(\cosh\tfrac{t}{a}\sin\chi\big)^2 d\theta^2 + f^2 d\phi^2$$
$$+ \Big(d\big(\cosh\tfrac{t}{a}\big)\cos\chi + \cosh\tfrac{t}{a}\sin\chi\, d\chi\Big)^2\Big)$$

5.3

$$dx^2+dy^2+dz^2+dw^2-dT^2$$

$$= a^2\Big(d(\cosh\tfrac{t}{a})^2 + \cosh^2\tfrac{t}{a}\, d\chi^2$$

$$+ (\cosh\tfrac{t}{a}\sinh\chi)^2 d\theta^2 + f^2 d\phi^2$$

$$- d(\sinh\tfrac{t}{a})^2 \Big)$$

$$= a^2\Big(-\frac{1}{a^2}dt^2 + \cosh^2\frac{t}{a}\, d\chi^2$$

$$+ (\cosh\tfrac{t}{a}\sin\chi)^2 d\theta^2$$

$$+ (\cosh\tfrac{t}{a}\sin\chi\sin\theta)\, d\phi^2 \Big)$$

So

$$ds^2 = -dt^2 + \cosh^2\frac{t}{a}\, d\Omega_3^2$$

where $d\Omega_3^2 = d\chi^2$

$$+ \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Define λ such that

$$d\lambda^2 = \frac{1}{a^2 \cosh^2 \frac{t}{a}} \, dt^2 \qquad (*)$$

Then the metric becomes

$$ds^2 = a^2 \cosh^2 \frac{t}{a} \left(- d\lambda^2 + d\Omega_3^2 \right)$$

Equation $(*)$ can be integrated out :

$$\lambda = \pm 2 \tan^{-1} e^{t/a} + \text{constant}$$

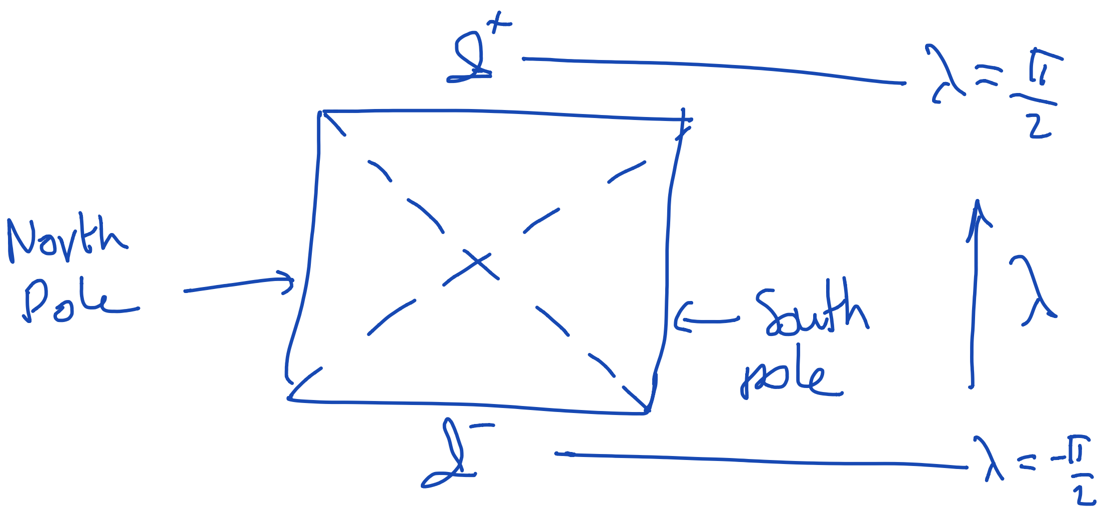
Choose the +ve sign and the constant equal to $-\frac{\pi}{2}$ so that λ is an increasing function and it takes values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

Then

$$ds^2 = \frac{a^2}{\cos^2\lambda}\left(-d\lambda^2 + d\Omega_3^2\right)$$

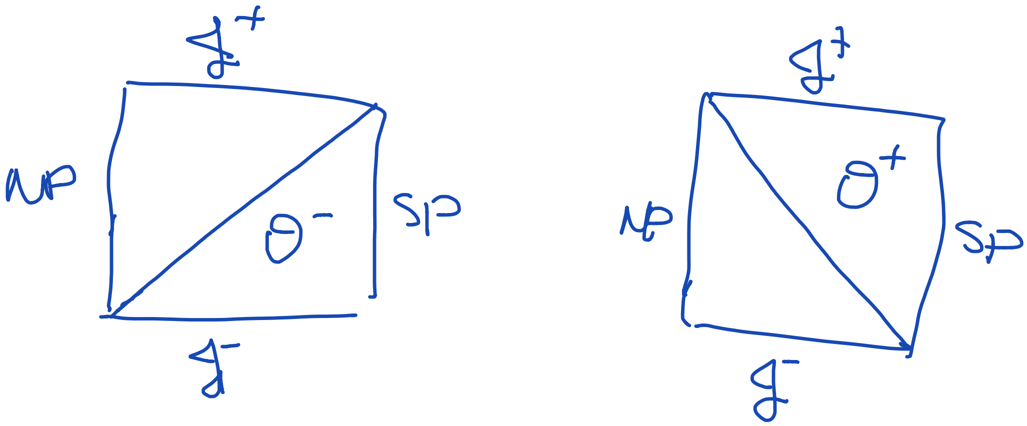
This metric is conformal to Einstein static universe.

Penrose diagram



- North & south poles: TL lines
- every point in the interior: S^2
- horizontal slice: S^3
- dashed lines: past and future horizons of an observer at the south pole
- $\mathcal{I}^\pm$ future & past null infinity

5.6



region $\mathcal{O}^{\pm}$: region that can be observed by an observer at the NP (SP)