Problem sheet 1, General Relativity 2, HT 2019.

(Problems which are marked as "**revision**" or "**optional**" will not be discussed in classes.)

1. (**revision**) Use the Bianchi identity $\nabla_{[a}R_{bc]de} = 0$ to prove the contracted Bianchi identity

$$\nabla^a \left(R_{ab} - \frac{1}{2} R\, g_{ab} \right) = 0 \ .$$

2. (**revision**) Use the definition of the commutator or Lie Bracket of vector fields to prove the Jacobi identity, that for smooth vector fields X, Y, and Z

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \ .$$

3. Show that for a torsion free connection

$$[X, Y]^a = X^b \nabla_b Y^a - Y^b \nabla_b X^a$$

for any two smooth vectors X and Y.

For any smooth vector field X define the operator $\nabla_X = X^a \nabla_a$. Show that for smooth vector fields X, Y, and Z,

$$(\nabla_{[X,Y]} - \nabla_X \nabla_Y + \nabla_Y \nabla_X) Z^a = R_{bcd}{}^a X^b Y^c Z^d \ .$$

(This is often used as a definition of curvature.)

4. (**optional**) Given smooth vector fields X and Y, define the operator D by

$$D = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X - \mathcal{L}_{[X,Y]} \ .$$

Show that D satisfies the Leibnitz property,*i.e.,* that

$$D(S^{\cdots}{}_{\ldots} T^{\cdots}{}_{\ldots}) = (D S^{\cdots}{}_{\ldots}) T^{\cdots}{}_{\ldots} + S^{\cdots}{}_{\ldots} D T^{\cdots}{}_{\ldots} \ ,$$

for tensor fields $S^{\cdots}{}_{\ldots}$ and $T^{\cdots}{}_{\ldots}$ (where we have suppressed the indices).

Show that for any smooth function f, $Df = 0$, and use problem 2 above to show that, for any smooth vector field Z^a, $DZ^a = 0$.

Deduce that $D \equiv 0$ for any X and Y. (There is no curvature for Lie derivatives.)

5. Show that for any smooth vector field X^a and any tensor R_{abcd} (not necessarily the Riemann tensor):

$$\mathcal{L}_X R_{abcd} = X^e \nabla_e R_{abcd} + R_{ebcd} \nabla_a X^e + R_{aecd} \nabla_b X^e + R_{abed} \nabla_c X^e + R_{abce} \nabla_d X^e \ .$$

If X^a is a Killing vector and R_{abcd} *is* the Riemann tensor, you might expect the above quantity to vanish. Can you prove this starting from the equation

$$\nabla_a \nabla_b X_c = -R_{bcad} X^d \ ?$$

(Recall that this identity is valid for any Killing vector X^a.) [Hint: consider $[\nabla_a, \nabla_b] \nabla_c X_d$.]

6. Use the equation

$$\nabla_a \nabla_b X_c = -R_{bcad} X^d \ ,$$

from the lectures to show that the maximum number of linearly independent Killing vector fields in a space of dimension n is $n(n+1)/2$.

[Hint: show that X_a and $\nabla_a X_b$ can be chosen freely at a point and then X is determined as a vector field; then how many choices does this represent?]

If a space has the maximum possible number of Killing vectors, what does problem 5 tell you about $\nabla_a R_{bcde}$?

And what about R_{abcd}? [this part is hard; try taking the trace]

In flat 4-dimensional Minkowski space, find explicit expressions in terms of the pseudo-Cartesian coordinates x^a and constant tensors for a 10-parameter family of Killing vectors [what does $\nabla_a \nabla_b X_c = -R_{bcad} X^d$ tell in this case?].

Can you classify these Killing vectors into translations, rotations and 'standard Lorentz transformations'?