Problem Sheet 3, General Relativity 2, HT 2019

1. Let K^a be the null vector with components $(1,0,0,1)$, and let e_{ab} such that

$$\left(e_{ab} - \frac{1}{2}\eta_{ab}e\right)K^b = 0$$

where $e = \eta^{ab}e_{ab}$ (as in the lectures for chapter 2). Show that a 4-vector λ^a can be found so that $\tilde{e}_{ab} = e_{ab} + K_a\lambda_b + \lambda_a K_b$ is given by

$$\tilde{e}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & B & -A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

[Hint: first show that λ^a can be chosen to make $\tilde{e}_{0a} = 0$, then deduce that $\tilde{e}$ must vanish, etc....]

2. A massive particle crosses $r = 2M$ in the Schwarzschild metric following some ingoing radial time-like path. Show that it arrives at $r = 0$ after an elapsed proper time Δs which can be no greater that πM.

[*As a first step, what class of paths maximise proper time?*].

3. How to recognize a null hypersurface: A null hypersurface is a surface Σ with normal n^μ which is null. In some space-time, the scalar function S has the property that $S = 0$ is a null hypersurface. S is used as a coordinate, $S = x^0$, in a coordinate system (x^0, x^1, x^2, x^3). Show that the component g^{00} of the contravariant metric vanishes at $S = 0$. Decompose the covariant metric into blocks as

$$g_{ab} = \begin{pmatrix} V & \mathbf{Y}^t \\ \mathbf{Y} & A \end{pmatrix}$$

where A is a (symmetric) 3×3 matrix, $\mathbf{Y}$ a 3 component column vector and V a function.

Show that $\det A = 0$ at $S = 0$.

[*So this is one (quick) way to recognize a null hypersurface. For the proof consider the identity $g^{ac}g_{cb} = \delta^a_b$.*]

For the Schwarzschild metric, use this to show that the surface $S = r - 2M = 0$ is null in the Eddington- Finkelstein metrics.

4. Static versus Stationary: A space-time with a time-like Killing vector K^a is said to be *static* is K^a is hypersurface-orthogonal (recall that this means $K_{[a}\nabla_b K_{c]} = 0$) and stationary otherwise. The Kerr metric in the Boyer-Lindquist coordinates is

$$\begin{aligned}
\mathrm{d}s^2 = &-\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mar}{\Sigma}\sin^2\theta\,\mathrm{d}\phi\mathrm{d}t \\
&+ \frac{1}{\Sigma}\sin^2\theta\left(\Delta\Sigma + 2Mr(r^2+a^2)\right)\mathrm{d}\phi^2 + \Sigma\left(\frac{1}{\Delta}\mathrm{d}r^2 + \mathrm{d}\theta^2\right)
\end{aligned}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. Show that this metric is not static unless $J = Ma = 0$ (when of course it reduces to the Schwarzschild metric).

5. Show that a vector field $\mathbf{K}$ in flat 3-space is hypersurface orthogonal (HSO) if and only if

$$\mathbf{K}\cdot\nabla\wedge\mathbf{K} = 0\ .$$

If $\mathbf{K} = (y, -x, f(r))$ where $r^2 = x^2 + y^2$, for what f is $\mathbf{K}$ HSO? What do the integral curves (equivalently 'streamlines') of $\mathbf{K}$ look like?

If $\mathbf{K} = (\cos g(z), \sin g(z), 0)$ where $g' \neq 0$, show that $\mathbf{K}$ is never HSO and that the integral curves of $\mathbf{K}$ are straight lines.

6. The Kerr metric in Kerr-Eddington coordinates is

$$\begin{aligned}
\mathrm{d}s^2 = &-\mathrm{d}T^2 + \mathrm{d}r^2 - 2a\sin^2\theta\mathrm{d}r\mathrm{d}\Phi + \Sigma\mathrm{d}\theta^2 + (r^2+a^2)\sin^2\theta\mathrm{d}\Phi^2 \\
&+ \frac{2Mr}{\Sigma}\left(\mathrm{d}T - a\sin^2\theta\mathrm{d}\Phi + \mathrm{d}r\right)^2
\end{aligned}$$

where T and Φ are defined by the equations

$$\mathrm{d}T = \mathrm{d}t + \frac{2Mr}{\Delta}\mathrm{d}r\ , \qquad \mathrm{d}\Phi = \mathrm{d}\phi + \frac{a}{\Delta}\mathrm{d}r\ ,$$

and

$$\Delta = r^2 - 2Mr + a^2\ , \qquad \Sigma = r^2 + a^2\cos^2\theta\ .$$

Use the method in question 4 with this metric to show that $r = r_+$ and $r = r_-$ are null hypersurfaces in the Kerr metric (recall that $r_\pm$ are the values of r for which $\Delta = 0$).