Problem sheet 4, General Relativity 2, HT 2019

1. Reproduce the figure in Section 3.4 for the Kruskal-Szekeres spacetime. Show a radial space–like geodesic with $E = 0$, and a radial time–like geodesic with $E < 1$. What does this condition tell you about $\dot{r}$, and therefore r? Where on this figure are there radial time–like geodesics with $E = 0$?

2. The Kerr metric in Boyer-Lindquist coordinates is

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 - \frac{4Mar}{\Sigma}\sin^2\theta\,\mathrm{d}\phi\mathrm{d}t \\ + \frac{1}{\Sigma}\sin^2\theta\left(\Delta\Sigma + 2Mr(r^2+a^2)\right)\,\mathrm{d}\phi^2 + \Sigma\left(\frac{1}{\Delta}\mathrm{d}r^2 + \mathrm{d}\theta^2\right) \ .$$

Consider an observer moving in this gravitational field. The observer's worldline has a tangent vector which is a timelike Killing vector U. Explain why such an observer does not see any changes in the metric.
Show that such an observer cannot exist in the region between the inner and outer event horizons (that is $r_- < r < r_+$), and that such an observer exists outside the horizon $r > r_+$ for values of its angular velocity ω in the region $\omega_- \le \omega \le \omega_+$ where

$$\omega_\pm = \Omega \pm \sqrt{\Omega^2 - \frac{g_{tt}}{g_{\phi\phi}}} \ ,$$

where Ω is a quantity that you should find. Prove that inside the ergosphere, that is $r_+ < r < \tilde{r}_+$, the observer co-rotates with the black hole. (See lecture notes for the definition of the various quantities in this problem. See for example Sean Carroll's book if you get stuck.)

3. Let K be a timelike Killing vector such that at infinity $g(K,K) \to -1$, and let S be the Killing horizon for K. Prove that the surface gravity of S, which is defined by

$$K^a\,\nabla_a K_b = \kappa\,K_b \ ,$$

satisfies

$$\kappa^2 = -\frac{1}{2}\left[(\nabla_a K_b)(\nabla^a K^b)\right]\Big|_S \ .$$

(Hint: you can use the fact that K is hypersurface orthogonal on the horizon, or see R Wald's book for help.)

4. Let K be a timelike Killing vector such that at infinity $g(K,K) \to -1$, and let S be the Killing horizon for K. Explain why a stationary observer has four velocity U which satisfies

$$K^a = \alpha(x)\,U^a \ ,$$

for some function $\alpha(x)$. Show that the acceleration of the observer is

$$A^a = g^{ab}\,\nabla_b(\log \alpha)\ ,$$

and that the acceleration of the observer measured by an stationary observer at infinity A_∞ is

$$A_\infty = \alpha A\ .$$

where A is the magnitude of the acceleration (see S. Carroll for help if necessary). Show also that the surface gravity of S is given by

$$\kappa = \lim_{r\to r_H}(\alpha A)\ ,$$

where $r = r_H$, and r_H is a constant, is the equation that defines the horizon S (Hint: use problem 3).

5. The deSitter space-time is a solution of Einstein's field equations with positive cosmological constant. We can think of deSitter space time as the four dimensional hyperboloid

$$-T^2 + x^2 + y^2 + z^2 + w^2 = a^2\ ,$$

embedded in 4+1 dimensional Minkowski space. Consider the coordinate system (T, χ, θ, ϕ) with

$$\begin{aligned}
x &= a\,\cosh(t/a)\,\sin\chi\,\sin\theta\,\cos\phi\\
y &= a\,\cosh(t/a)\,\sin\chi\,\sin\theta\,\sin\phi\\
z &= a\,\cosh(t/a)\,\sin\chi\,\cos\theta\\
w &= a\,\cosh(t/a)\,\cos\chi\\
T &= a\,\sinh(t/a)\ .
\end{aligned}$$

Show that the deSitter metric can be written as

$$\mathrm{ds^2 = -dt^2 + a^2\,\cosh^2(t/a)\,d\Omega_3^2}\ ,$$

where $\mathrm{d}\Omega_3^2$, is the round metric on the three-sphere

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\chi^2 + \sin^2\chi(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\ \ ,$$

so the spatial sections are three-spheres with radius $a\,\cosh(t/a)$.
Write the metric in the form

$$\mathrm{ds}^2 = \frac{\mathrm{a}^2}{\cos^2\lambda}\,(-\mathrm{d}\lambda^2 + \mathrm{d}\Omega_3^2)\ ,$$

where λ is defined by

$$\mathrm{d}\lambda^2 = \frac{1}{\mathrm{a^2\,\cosh^2(t/a)}}\,\mathrm{dt}^2\ .$$

Draw the Penrose diagram for this space-time. Mark the surfaces corresponding to the past and future null infinity. Explain why a given observer cannot observe the entire space–time.