GR II Part C Math
(also MMathPhys, MSc MTP)

Outline (roughly!)

* Mathematical techniques from differential geometry

* Field equations with matter

* Linearized GR
  - gravitational field of an isolated body at large distances
  - gravitational waves

* Black holes
  - (revision of) Schwarzschild solution
  - Penrose diagrams
  - Kerr solution
  - time permitting: charged black holes

[For MMathPhys/MScMTP: watch for topics for oral presentations]

Chapter 1: Mathematical background

1.1 Mathematical model of GR

In Einstein's GR, the model for spacetime (space of events) is a 4 dim differentiable manifold M with

- a metric g with signature $(-+++)$

  $\leftarrow$ symmetric $(0\ 2)$ tensor which is differentiable and non-degenerate (ie $\det g \neq 0$ or equivalently
  $$g_{ab} X^a Y^b = 0 \quad \forall Y \Rightarrow X = 0)$$

(g defines an inner product on $TM \cong \mathbb{R}^4$;
line element = infinitesimal distance between neighboring events: $ds^2 = g_{ab}(x)\, dx^a\, dx^b$]

AND

- a connection ∇ which is metric $(\nabla g = 0)$ and torsion free

$M \rightsquigarrow$ Riemannian geometry

Why do we want M to be a differentiable manifold?

Ⓐ Informal discussion of manifolds

set of points + neighbourhood for each point

manifold: (topological) space which locally "looks like" (homeomorphic to) $\mathbb{R}^n$

That is:

M can be covered by a collection of coordinate charts (U_i, x_i^a) $i=1,\dots,N$ (atlas) where

$U_i \subset M$ open sets in M

"coordinate patch"
"local coordinate neighborhood"

"coordinate"

$$\begin{tikzcd} x_i^a : U_i \arrow[r] & \mathbb{R}^n \\ p \arrow[r, mapsto] \arrow[u, phantom, "\in" rotate=90] & (x_i^1, x_i^2, \dots, x_i^n) \end{tikzcd}$$

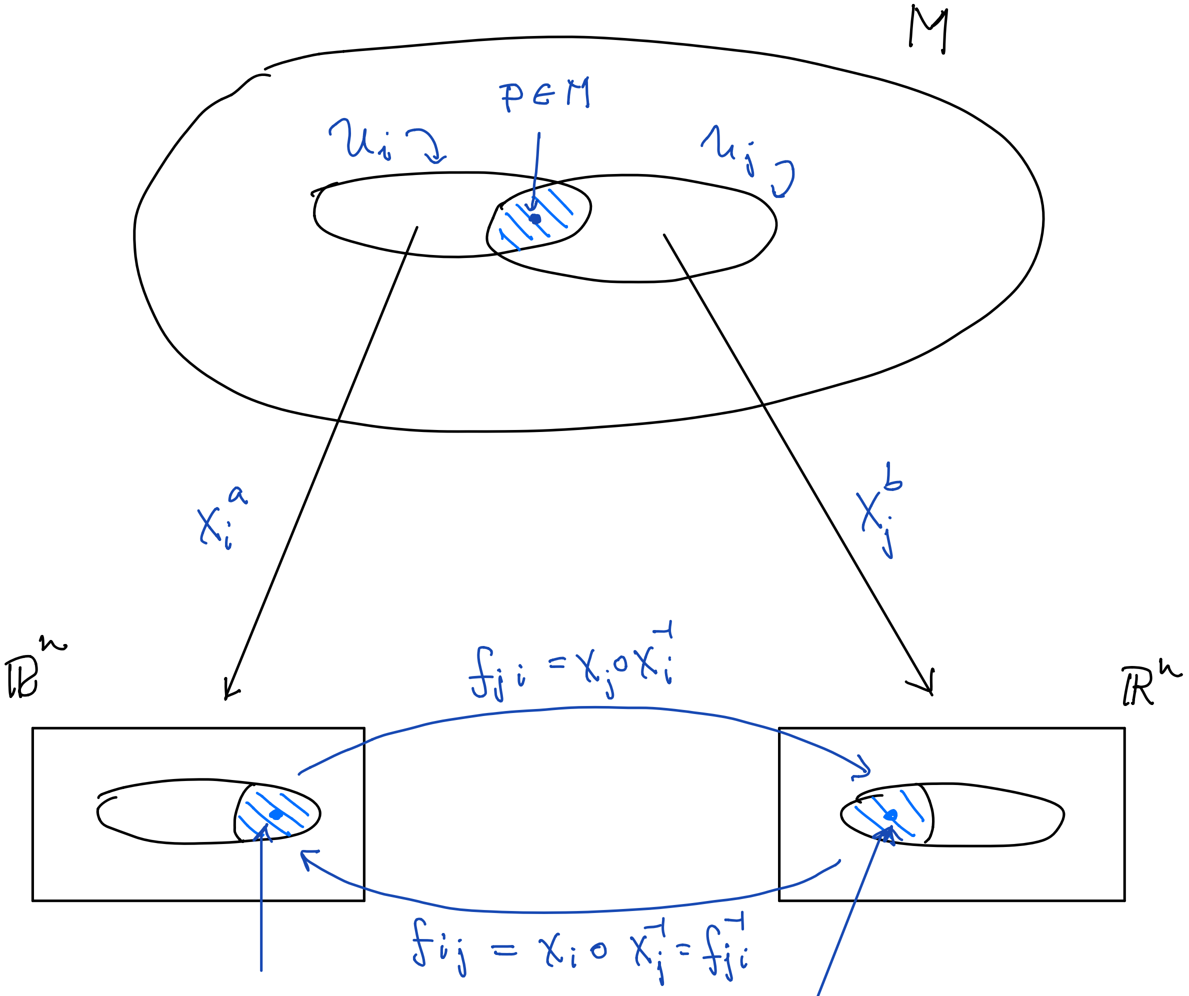
with the requirement that on

$$U_{ij} = U_i \cap U_j$$

we have transition functions

$$x_i^a = f_{ij}^a(x_j^b)$$

which are continuous



$$X_i^a(p) \longmapsto X_j^b(p) = f^b_{ji}(X^a_i(p))$$

f, f^{-1} continuous, defined only on the overlap. $\mathcal{U}_i \cap \mathcal{U}_j$

composition of f_{ji} and X_i same as X_j

$$f_{ji} \circ X_i = X_j$$

differentiable manifold :

transition functions are smooth
(ie differentiable to all orders)

For GR : we want a differentiable manifold because of

Einstein's equivalence principle

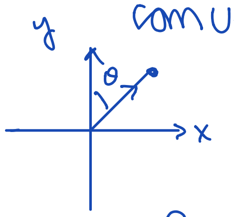
ie special relativity holds at small distances

- coordinate patches $\sim \mathbb{R}^4$
- transition functions (change of coordinates) are smooth functions

Examples (see GR1)

1) $\mathbb{R}^n$ one coordinate patch

2) S^1 $(x, y) \in \mathbb{R}^2$ st $x^2 + y^2 = 1$



conventional coord system: polar angle

$\theta : S^1 \longrightarrow \mathbb{R}$

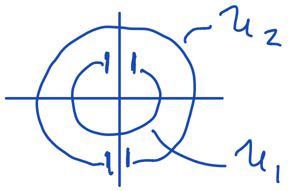
$x = \sin\theta$
$y = \cos\theta$

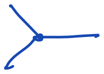
θ defined up to an integral multiple of 2π

$\theta \simeq \theta + 2\pi n \qquad n \in \mathbb{Z}$

but $0 \leq \theta \leq 2\pi$ is not an open interval

Need two patches to cover the circle



3)  ,  are not manifolds

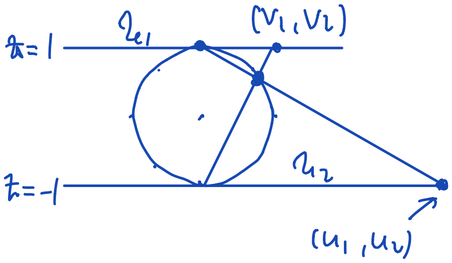
4) Sphere S^2: $(x, y, z) \in \mathbb{R}^3$ with $x^2+y^2+z^2=1$

$\mathcal{U}_1$: $S^2 \setminus \{\text{north pole}\}$ use stereographic projection

$\mathcal{U}_2$: $S^2 \setminus \{\text{south pole}\}$

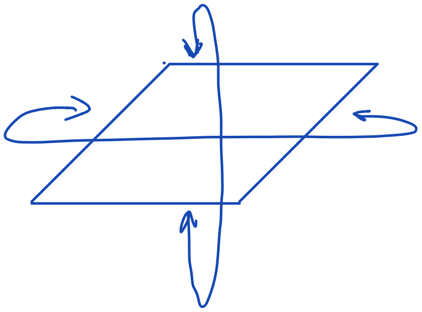
$$\mathcal{U}_1 : p \longmapsto (u_1, u_2) = \left(\frac{2x}{1-z}, \frac{2y}{1-z}\right)$$

$$\mathcal{U}_2 : p \longmapsto (v_1, v_2) = \left(\frac{2x}{1+z}, \frac{2y}{1+z}\right)$$



$\mathcal{U}_1 \cap \mathcal{U}_2 \quad -1 < z < 1$ $\qquad v_i = \dfrac{4u_i}{u_1^2+u_2^2}$ is C^∞

5) T^2



4 patches is enough.

Ⓑ **Tensors** geometric objects on M
defined naturally by the manifold structure

In GR: we write equations in terms of tensors (eg g, the metric, R, the curvature of the metric, etc)

((laws of nature in terms of tensors!))

Principle of relativity ↔ general covariance

laws of nature are covariant ie independent of the choice of a coordinate system.

(p,q) tensor $T^{a_1 - a_p}{}_{b_1 \cdots b_q}$

defined by its transformation properties under coordinate transformations

indices: book-keeping of transformation laws

Under a coordinate transformation $X^a \to \tilde{X}^a(X)$

$$\tilde{T}^{a_1 \cdots a_p}{}_{b_1 \cdots b_q} = (\partial_{c_1} \tilde{X}^{a_1} \cdots \partial_{c_p} \tilde{X}^{a_p})(\tilde{\partial}_{b_1} X^{d_1} \cdots \tilde{\partial}_{b_q} X^{d_q})\, T^{c_1 \cdots c_p}{}_{d_1 \cdots d_q}$$

Notation: $\partial_a = \partial / \partial x^a$

Examples:
- scalar $(0,0)$ $\tilde{\phi}(\tilde{x}) = \phi(x)$
- vector $(1,0)$ (contravariant vector)
- $(0,1)$ 1-form, co-vector, covariant vector
- the metric is a $(0,2)$ symmetric tensor

Tensor operations (part 1)

(operations on tensors or between tensors to get other tensors)

1) addition: $T+S$; T, S same type

2) contraction: $(p,q) \longrightarrow (p-1, q-1)$

sum over one upper index and one lower index

3) raise and lower indices with the metric

$(p,q) \longrightarrow (p-1, q+1)$ "lowering"

$(p,q) \longrightarrow (p+1, q-1)$ "raising"

(Needs a metric g on M. for example $TM \simeq (TM)^*$ is an isomorphism given by $V_a = g_{ab} u^b$ $\quad u^a = g^{ab} V_b$)

4) exterior derivative & p-forms

- f a scalar: $\partial_a f$ is a 1-form (covector)
- let $V = V_a dx^a$ be a 1-form

want derivation which is a tensor

$\partial_a V_b$ is not a tensor

The same is true more generally: if T is (p,q)-tensor, $\partial_a T$ is not a tensor

p forms

- However: if A is a p-form ie a totally antisymmetric $(0,p)$ tensor

exterior derivative

Then $dA = \frac{1}{p!} \partial_{[b} A_{a_1 \dots a_p]} dx^{b a_1 \dots a_p}$ is a $p+1$ form.

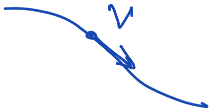
(Will discuss derivations dx, $[\,,\,]$, ∇ later)

Ⓒ Vector fields, diffeomorphisms and integral curves

Key ideas:

Vector field ⟷ integral curves of V ⟷ 1-parameter group of diffeomorphisms

V



$\varphi : M \to M$

A diffeomorphism between two differentiable manifolds M & M' is a C^∞ bijective map

$$\varphi : M \longrightarrow M'$$

which preserves the manifold structure

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, phantom, "\cup"] & M' \\
\mathcal{U}_i \arrow[r, "\varphi"] \arrow[d, "X_i^a"'] & \mathcal{U}'_j \arrow[d, "Y_j^a"] \\
\mathbb{R}^n \arrow[r, "\tilde{\varphi} = Y_j \circ \varphi \circ X_i^{-1}", "\tilde{\varphi} \text{ is } C^\infty"'] & \mathbb{R}^n
\end{tikzcd}$$

We think of two diffeomorphic manifolds as "equivalent"

Examples

1) S^2 with coordinates (θ, ϕ)

⇝ a rotation around the z-axis

2) T^2

3) $S^2 \longrightarrow$ ellipse

$$(x, y, z) \longleftrightarrow \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$$

$$x^2+y^2+y^2=1$$

4) S^2 & T^2 are not diffeomorphic

(End of lecture 1)

Lecture #2

A one-parameter group of diffeomorphisms of M is a differentiable map

$$\varphi : \mathbb{R} \times M \longrightarrow M$$

$$(t, x^a) \longmapsto \varphi^a(t, x) = \varphi_t^a(x) = x^a(t)$$

$t \in (a,b) \subset \mathbb{R}$

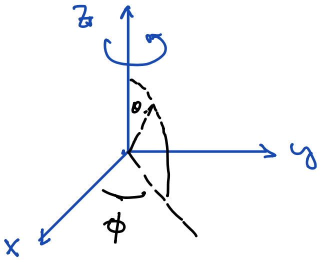
where $\varphi_t : M \longrightarrow M$ is a diffeomorphism

with $\varphi_0 = \text{identity}$, and

$\varphi_{s+t} = \varphi_s \circ \varphi_t$ group law

Example: S^2 with coordinates (θ, ϕ)

$\varphi_t(\theta, \phi) = (\theta, \phi + t)$ rotation by t around z-axis



orbits → circles $\theta = \text{constant}$

Remark: $\varphi^a(t, x) = x^a(t)$ defines a curve on M parametrized by t

Theorems: on the relation between vector fields and 1-parameter group of diffeomorphisms

$\varphi \to V$ (i) To each 1 parameter group of diffeomorphisms φ_t we can associate a vector field V by

$$V^a(x) = \frac{d}{dt} \varphi^a_t(x)$$

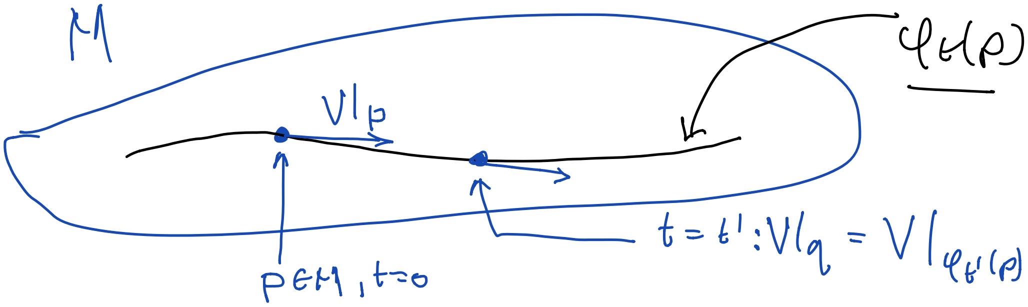
$V \to \varphi$ (ii) Conversely: let V be a smooth vector field on M. Then, there is a family of curves associated to $V|_p$ for each $p \in M$, such that one and only one curve passes through $p \in M$, and the tangent to this curve at p is V_p.

These curves are called integral curves of V

Proof of (i) what happens to $p \in M$ under the action of φ?

$\varphi \to V$

Fix $p \in M$: $\varphi_t(p)$ is a unique curve through p at $t=0$ $(\varphi_0(p) = p)$



φ_t traces a path on M (orbit of φ_t through p)

The components of the tangent vector V to the curve at $\varphi_t(p)$ are given by

$$V^a(x(t)) = \frac{d}{dt}\varphi_t^a(x) \quad \text{(differentiate } \varphi)$$

in particular, at $t=0$ we have $V|_p$, the tangent vector to the path at p.

$V \to \varphi$

Proof of (ii)

let x^a be coordinates in a neighborhood $p \in M$.

To find a curve through p with tangent vector $V|_p$ at p given φ, we solve the differential equation

$$V^a(x(t)) = \frac{d}{dt}\varphi_t^a(x) = \frac{d}{dt}\varphi^a(x,t) \qquad \text{(integrate)}$$

for the components of V.

The solution exists and it is unique given the initial condition at $t=0$

$$\varphi_0(p) = p$$

(by Picard's theorem on existence and uniqueness of 1st order differential equations with initial conditions)

Thus for every smooth vector field V, there is a family of integral curves on M.

In summary: For each $p \in M$

$$V \xleftrightarrow{1-1} \text{integral curves } x(t)$$

smooth vector field on M

$\qquad\qquad\qquad\qquad\qquad ||$

orbits of a 1-parameter

(*) group of diffeomorphisms $\varphi^a(t,x) = \varphi_t^a(x) = x^a(t)$

given by

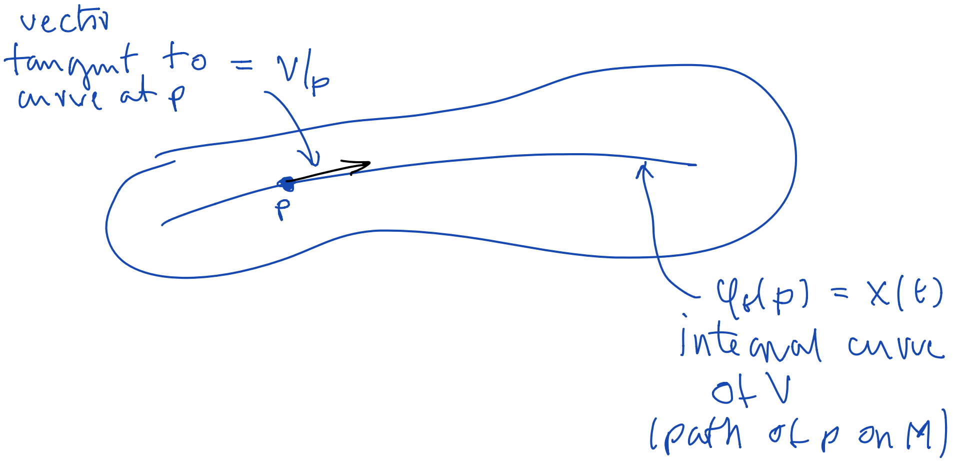
$$V^a(x(t)) = \frac{d}{dt}\varphi_t^a(x), \quad \varphi_0(p) = p$$

(initial conditions)

($\longleftarrow$ differentiation

integration $\longrightarrow$)

* there is a unique curve $\varphi_t(p)$ through each $p \in M$ at $t=0$ st $V|_{\varphi_t(p)}$ is tangent to the point $\varphi_t(p)$



Examples

① $M = \mathbb{R}^2$ coordinates (x,y)

Find the integral curves of $X = \frac{\partial}{\partial x}$

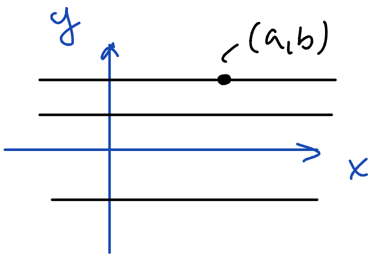
Solve $V^a = \frac{d}{dt}\varphi^a_t$, $V^1 = 1$, $V^2 = 0$

$\varphi^a_t = x^a(t)$: $\varphi^1_t = x(t)$, $\varphi^2_t = y(t)$

and solve $1 = \frac{dx}{dt}$, $0 = \frac{dy}{dt}$

Integrating: $x(t) = t + a$, $y(t) = b$ $\Big\}$ a, b constants

so $\varphi_t = (t+a, b)$ is a unique curve through each point $(a,b) \in \mathbb{R}^2$



Integral curves are lines parallel to the x-axis

$X = \frac{\partial}{\partial x}$ ⟿ translations in the x-direction

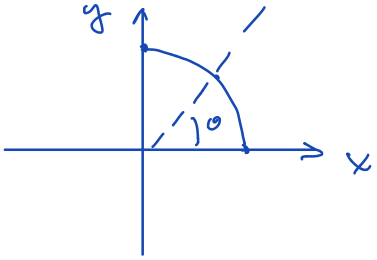
② $M = \mathbb{R}^2$

(i) Find a vector V st its integrable curves are circles centered at the origin

Consider a circle of radius r

$x = r\cos\theta$

$y = r\sin\theta$



$$V^a = \frac{dx^a}{d\theta} \longrightarrow \begin{array}{l} V^1 = -r\sin\theta = -y \\ V^2 = r\cos\theta = x \end{array}$$

so $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{\partial}{\partial \theta}$

↳ generates rotations around the origin

(ii) Conversely: given $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$

what are the integral curves of V?

Solve: $-y(t) = \frac{dx}{dt}$, $x(t) = \frac{dy}{dt}$ (coords (x,y))

so $x\frac{dx}{dt} + y\frac{dy}{dt} = 0$

so $x^2 + y^2 = \text{constant}$ circles centered at the origin

Ⓓ Tensor operations (part 2]

(derivations)

5) Lie bracket of two vector fields
fields U, V is another vector field

$$Z = [U, V]$$

with components

$$Z^a = U^b \partial_b V^a - V^b \partial_b U^c$$

or equivalently

$$Z(f) = Z^a \partial_a f = U(V(f)) - V(U(f))$$

(one can prove that in fact Z is a vector)

Properties: Let X, Y, Z be vector fields

- $[X, Y] = -[Y, X]$, $[X, X] = 0 \ \forall X$
- bilinearity

$$[X, \alpha Y + \beta Z] = \alpha [X, Y] + \beta [X, Z]$$

for any constants α, β

- Jacobi identity

$$0 = [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]$$

(vectors & the binary operation $[,]$ form a Lie-algebra)

(geometric interpretation later]

want to generalize the notion of directional derivative along a vector X

6) **Definition:** Lie derivative

Let X be a vector field

(i) the Lie derivative of a function f along X is

$$\mathcal{L}_X f = X(f) = X^a \partial_a f$$

ie the directional derivative of f along X

(ii) The Lie derivative of a vector field Y along X is

$$\mathcal{L}_X Y = [X, Y]$$

(iii) The Lie derivative $\mathcal{L}_X T$ of a (p,q) tensor T along X is defined by requiring the Lie derivative to satisfy the Leibnitz rule over tensor products and contractions ie

$$\mathcal{L}_X (T^{\cdots}{}_{\cdots} S^{\cdots}{}_{\cdots}) = T^{\cdots}{}_{\cdots} \mathcal{L}_X S + S^{\cdots}{}_{\cdots} \mathcal{L}_X T^{\cdots}{}_{\cdots}$$

Note: $\mathcal{L}_X T$ is of same type as T

One finds (by induction)

$$\mathcal{L}_X T^{a_1 \dots a_p}{}_{b_1 \dots b_q} = X^c \partial_c T^{a_1 \dots a_p}{}_{b_1 \dots b_q}$$
$$- \sum_{i=1}^{p} T^{a_1 \dots a_{i-1} c a_{i+1} \dots a_p}{}_{b_1 \dots b_q} \partial_c X^{a_i}$$
$$+ \sum_{i=1}^{q} T^{a_1 \dots a_p}{}_{b_1 \dots b_{i-1} c b_{i+1} \dots b_q} \partial_{b_i} X^c$$

For example: let S be a 1-form. Then

$$\mathcal{L}_X S_a = X^b \partial_b S_a + S_b \partial_a X^b$$

Proof: Let Q be a vector field

Then $$\mathcal{L}_X(\underbrace{Q^a S_a}_{\text{scalar}}) = X^b \partial_b (Q^a S_a)$$

so $$\mathcal{L}_X(Q^a S_a) = X^b (\partial_b Q^a) S_a + Q^a X^b \partial_b S_a \quad ①$$

On the other hand: requiring that $\mathcal{L}_X$ satisfies the Leibnitz rule we have

$$\mathcal{L}_X(Q^a S_a) = (\mathcal{L}_X Q^a) S_a + Q^a \mathcal{L}_X S_a$$
$$\overset{(ii)}{=} (X^b (\partial_b Q^a) - Q^b \partial_b X^a) S_a + Q^a \mathcal{L}_X S_a \quad ②$$

Equating ① & ②

$$Q^a \mathcal{L}_X S_a = Q^a (X^b \partial_b S_a + S_b \partial_a X^b)$$ must be true $\forall Q$

$$\Longleftrightarrow \quad \mathcal{L}_X S_a = X^b \partial_b S_a + S_b \partial_a X^b$$

Property:

$$\mathcal{L}_{[X,Y]} = \mathcal{L}_X \mathcal{L}_Y - \mathcal{L}_Y \mathcal{L}_X$$

(sheet 1, and what is the meaning of this)

Later:

$\mathcal{L}_X T \sim$ change of T along X
(change along curves generated by X)
$\sim$ directional derivative of T along X

Remark: all operations defined so far (except (5)) are independent of a metric on M or a connection on M (depend only on the manifold structure)

7) Covariant differentiation ∇ (GR1)

$\partial_a T$ is not in general a tensor

Want: differentiation st

(p,q) tensor $\longrightarrow$ $(p,q+1)$ tensor

Need extra structure on M

Definition: a covariant derivative ∇ on M is a linear ($\nabla(T+S) = \nabla T + \nabla S$) map st for any (p,q) tensor T, ∇T is a $(p,q+1)$ tensor which satisfies

(i) $\nabla_a f = \partial_a f$ $(0,1)$ tensor $\nabla_X f = X(f)$

(ii) $\nabla_a V^b = \partial_a V^b + \Gamma^b{}_{ac} V^c$ where

Γ (connection coefficients) transform under coordinate transformations st

$\nabla_a V^b$ is a $(1,1)$ tensor

Γ is not a $(1,2)$ tensor!

[under $x^a \to \tilde{x}^a(x)$

$$\tilde{\Gamma}^c{}_{ab} = \partial_f \tilde{x}^c \tilde{\partial}_a x^d \tilde{\partial}_b x^e \Gamma^f{}_{de} + (\partial_d \tilde{x}^c) \tilde{\partial}_a \tilde{\partial}_b x^d \ ]$$

(iii) Extend definition to other tensors by requiring that ∇ satisfies the Leibnitz rule

$$\nabla_a (T^{\cdots}{}_{\cdots} S^{\cdots}{}_{\cdots}) = (\nabla_a T^{\cdots}{}_{\cdots}) S^{\cdots}{}_{\cdots} + T^{\cdots}{}_{\cdots} \nabla_a S^{\cdots}{}_{\cdots}$$

Examples

$$\nabla_a W_b = \partial_a W_b - \Gamma^c_{ab} W_c \qquad (0,1) \text{ tensor}$$

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d{}_{ca} g_{db} - \Gamma^d{}_{cb} g_{ad}$$

(0,3) tensor

Definition: the torsion T of ∇ is defined as

$$T^c{}_{ab} = 2\Gamma^c{}_{[ab]} .$$

T is $(1,2)$-tensor

GR: M is a differentiable manifold endowed with a torsion free $(T=0)$ covariant derivative ∇

Then $T=0 \iff \Gamma^c{}_{ab} = \Gamma^c{}_{ba}$

(Einstein-Cartan studied effects of $T \neq 0$)

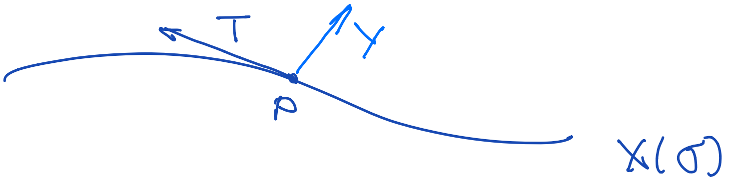
Geometrically: ∇ induces the notion of parallel transport along a curve $X^a(\sigma)$.

Suppose the curve has tangent vector T^a

Then

$$\frac{DY}{D\sigma} = \nabla_T Y = T^a \nabla_a Y$$

parallel transport of Y along the curve determined by T



- allows for the identification of tangent spaces along T

- geodesics $\nabla_T T = 0$ (affine parametrization)

  curve that parallel transports its own tangent vector

GR: TL geodesics ⇝ trajectory of a a free falling body in the gravitational field g

Lecture #3

(E) Going back to the mathematical model of GR

metric $\updownarrow$ grav. field

M is a differentiable manifold with a torsion free connection ∇ and a metric g st $\nabla g = 0$

Theorem: Fundamental theorem of Riemannian geometry

Let g be a metric on a differentiable manifold M. Then there is a unique torsion free covariant derivative ∇ on M st

$$\nabla g = 0$$

This is the Levi-Civita connection

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} (\partial_b g_{cd} + \partial_c g_{bd} - \partial_d g_{bc})$$

Proof: GR 1

(see also Hughston & Tod for proof using $\mathcal{L}_X$)

Remark: $\nabla g = 0$ implies that the operation of "raising" and "lowering" indices with the metric commutes with ∇.

Definition Curvature of ∇

Assuming that Γ is once differentiable on each coordinate patch, we define the (1,3) tensor

$$(\nabla_b \nabla_c - \nabla_c \nabla_b) V^a = -\underline{R_{bcd}{}^a}\, V^d$$

for any vector field V on M

$R_{bcd}{}^a$: Riemann curvature tensor

($R_{bcd}{}^a \leftrightarrow$ gravitational field analogue to F_{ab} in Maxwell's theory of EM)

One finds:

$$-R_{abc}{}^d = \partial_a \Gamma^d_{bc} + \Gamma^d_{ae} \Gamma^e_{bc} - (a \leftrightarrow b)$$

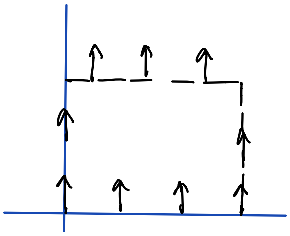
(clearly $R_{abc}{}^d = -R_{bca}{}^d$)

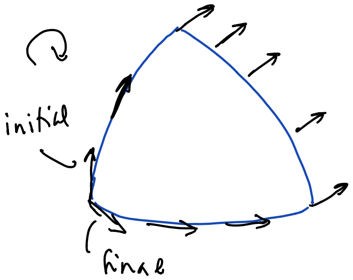
For a torsion free connection (GR 1)

(i) $R_{[abc]d} = 0$

(ii) $\nabla_{[a} R_{bc]d}{}^e = 0$ (Bianchi identity)

The curvature is a "measure" of the change of a vector field when parallel transported around a loop in M


flat space



sphere

The geodesic deviation is an important
GR1 interpretation of curvature as the gravitational field.

(we will use this when studying gravitational waves)

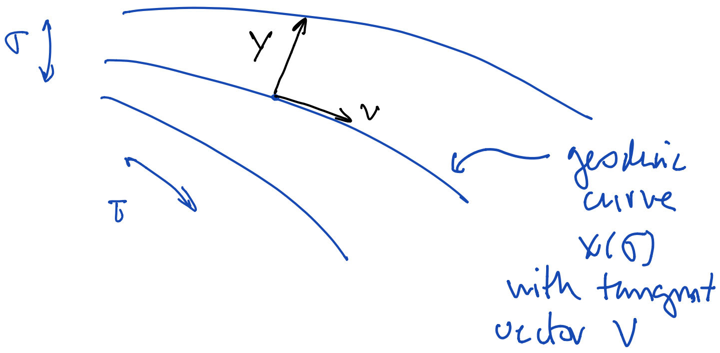
Describes the relative motion of a pair of nearby geodesics

Consider a one-parameter family of (affinely parametrised) geodesics

$$X^a(\tau, \sigma)$$

proper time ↗ (τ)

↳ labels member of family (σ)



$$V^a = \frac{dx^a}{d\tau}$$

$$0 = V^a \nabla_a V^b = \nabla_V V^b$$

Let Y be a vector field which measures the infinitesimal displacement of nearby geodesics as σ varies so Y is tangent to curves with τ = constant.

That is: Y represents relative motion

Then: $$\frac{D^2 Y^a}{D\tau^2} = \nabla_V^2 Y^a = -R_{bcd}{}^a V^b Y^c V^d$$

LHS = relative acceleration of nearby particles in free fall

Geodesic deviation: relative motion reveals presence of the gravitational field (curvature) as relative acceleration of nearby particles in free fall.

1.2 Killing vectors and isometries

From now on: M is a differentiable manifold with metric g and a torsion free connection ∇ st $\nabla g = 0$

Recall that $\mathcal{L}_X T$ is defined using only the manifold structure

However for a torsion free connection

$$\mathcal{L}_X f = X^a \partial_a f = X^a \nabla_a f$$

$$\mathcal{L}_X Y^a = [X,Y]^a = X^b \nabla_b Y^a - Y^b \nabla_b X^a$$

(extra terms cancel if $T=0$ ie $\Gamma^a_{bc} = \Gamma^a_{cb}$)

etc ..

Consider $\mathcal{L}_X g$: we have (exercise)

$$\mathcal{L}_X g_{ab} = X^c \nabla_c g_{ab} + g_{cb} \nabla_a X^b + g_{ac} \nabla_b X^c$$

If $\nabla g = 0$ then: $\mathcal{L}_X g_{ab} = 2 \nabla_{(a} X_{b)}$

Definition: A Killing vector field is a vector X st $\mathcal{L}_X g = 0$ ie $\nabla_{(a} X_{b)} = 0$

(X is a direction along which g is unchanged)

Properties:

(a) closure under the Lie bracket

If L & K are Killing vectors so is $[L,K]$

(b) $\nabla_a \nabla_b K_c = -R_{bca}{}^d K_d$

for any Killing vector K.

(c) Killing vectors give rise to geodesic integrals of motion

ie let V be tangent to a geodesic.

Then, $K_a V^a$ is a constant along the geodesic.

[easier to solve: gives 1st order diff eq for $x^a(\tau)$ vs 2nd order Euler Lagrange eqs]

Proof

(a) $\mathcal{L}_{[K,L]} = \mathcal{L}_K \mathcal{L}_L - \mathcal{L}_L \mathcal{L}_K \implies \mathcal{L}_{[K,L]}\, g = 0$

(b) For any vector field

$$\nabla_{[a}\nabla_{b]} K_c = \frac{1}{2} R_{abc}{}^d K_d$$

Recall Bianchi identity : $R_{[abc]}{}^d = 0$. Then

$$0 = \nabla_{[a}\nabla_{b]} K_c + \nabla_{[b}\nabla_{c]} K_a + \nabla_{[c}\nabla_{a]} K_b$$

$$0 = \nabla_a \underline{\nabla_b K_c} - \nabla_b \nabla_a K_c + \nabla_b \nabla_c K_a - \nabla_c \nabla_b K_a + \nabla_c \nabla_a K_b - \nabla_a \underline{\nabla_c K_b}$$

using Killing equations $\rightarrow$ $\quad + \nabla_b \nabla_c K_a \qquad - \nabla_c \nabla_a K_b \qquad + \nabla_a \nabla_b K_c$

$$= 2\left(\nabla_a \nabla_b K_c + \nabla_b \nabla_c K_a - \nabla_c \nabla_b K_a \right)$$

$$\implies \nabla_a \nabla_b K_c = -2 \nabla_{[c}\nabla_{b]} K_a = -R_{bca}{}^d K_d$$

✓

(c) GR 1

Let V be tangent to an affinely parametrised geodesic $x^a(\tau)$. Then

$$\frac{d}{d\tau}(K_a V^a) = \frac{dx^b}{d\tau}\partial_b(K_a V^a) = V^b \nabla_b (K_a V^a)$$

$$= V^b(\nabla_b K_a)V^a + \underbrace{K_a V^b \nabla_b V^a}\rightarrow 0 \text{ by geodesic equation}$$

$$= V^a V^b \nabla_b K_a = 2 V^a V^b \nabla_{(b} V_{a)}$$

$$= 0 \quad \text{for } V \text{ a Killing vector}$$

so $K_a V^a$ constant along the geodesic. //

Definition: an isometry is a diffeomorphism of M which leaves the metric invariant

$$\mathcal{L}_V g = 0$$

where V is the (Killing) vector which generates the diffeomorphism.

Note: extensive discussion of isometries in problem sheets.

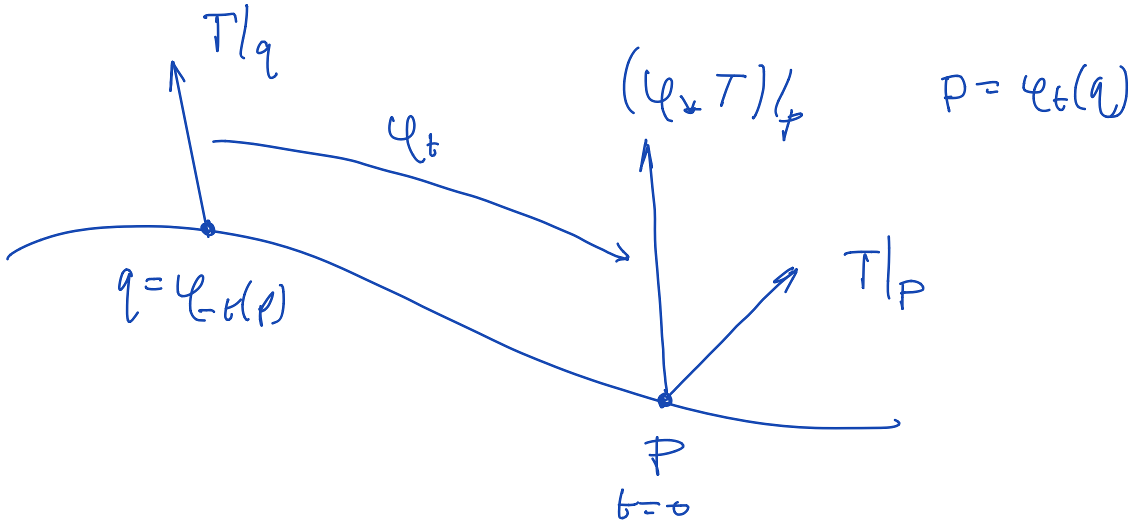
Let T be a tensor type (p,q) and V a smooth vector field on M.

We know that at each point $p \in M$ V defines a 1-parameter group of diffeomorphisms φ_t (φ_t takes a point $p \in M$ a parameter distance t along the integral curves of V)

The diffeomorphism φ_t induces a linear map φ_{t*} on tensors (push forward)

$$\varphi_{t*} : T|_q \longmapsto \varphi_{t*}T\,|_{\varphi_t(q)}$$

where $\varphi_{t*}T\,|_{\varphi_t(q)}$ is the tensor T pushed along the integral curve (flow of V) a distance t (φ_* preserves type)



Definition: The Lie derivative of T along a vector V, denoted $\mathcal{L}_X T$, is the rate of change of T along the integral curves of V, that is, it is the rate of change of T under φ_t

$$\mathcal{L}_V T\,\big|_p = \lim_{t\to 0} \frac{1}{t}\left(T|_p - \varphi_{t*} T|_p\right)$$

Of course this needs to be equivalent to our previous definition. (see page 19)

Lecture #4

Properties

✓ ① $\mathcal{L}_v T$ maps (p,q) tensors into (p,q) tensors

② $\mathcal{L}_v T$ is linear: ∀ constants α, β

✓ $$\mathcal{L}_v(\alpha T+\beta S) = \alpha \mathcal{L}_v(T) + \beta \mathcal{L}_v(S)$$

③ $\mathcal{L}_v$ obeys the Leibnitz rule

✓ $$\mathcal{L}_v(TS) = T\mathcal{L}_v S + (\mathcal{L}_v T)S$$

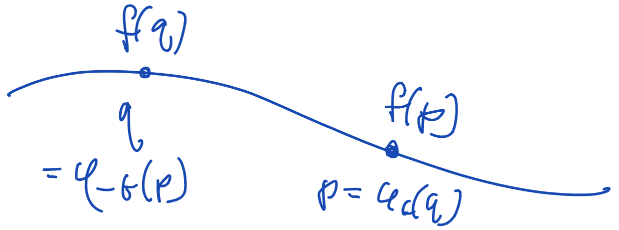
④ $\mathcal{L}_v f = V(f)$

⑤ $\mathcal{L}_V U = [U, V]$

①, ②, ③ : proofs as in calculus

Proof of ④

$$\mathcal{L}_v f\,\big|_p = \lim_{t\to 0}\frac{1}{t}\left(f|_p - \varphi_{t*} f|_p\right)$$



In this case $(\varphi_{t*} f)(p) \equiv f(q)$

$$= f(\varphi_{-t}(p))$$

ie $\varphi_{t*}\cdot f = f\circ\varphi_{-t}$

ie $\varphi_{t*} f$ = function on M whose value at $p=\varphi_t(q)$ is the value of the function at p

Hence

$$\mathcal{L}_v f\,\big|_p = \lim_{t\to0}\frac{1}{t}\Big(f(\underset{\substack{\uparrow\\ p\sim q+t}}{p}) - f(\underbrace{\varphi_{-t}(p)}_{q})\Big) = \frac{df}{dt}\Big|_p$$

[Compare : let $\{x^a\}$ be local coords in a neighborhood of p st and let V be the integral curve through p with $\frac{dx^a}{dt} = V^a(x(t)) \Rightarrow \mathcal{L}_v f = V^a\partial_a f = \frac{df}{dt} = V(t)$] //

Proof of ⑤

Let u be a vector on M and f a smooth function on M

Recall notation $u(f] = u^a \partial_a f$

$$\mathcal{L}_V(u(f)) = V(u(f]) \quad \text{by } ④$$

Using the Leibnitz rule

$$\mathcal{L}_V(u(f]) = (\mathcal{L}_V u)(f) + u(\mathcal{L}_V f)$$

Then

$$V u(f) = [\mathcal{L}_V u](f) + u V(f)$$

Hence $\quad \mathcal{L}_V u = [V, u]$

Remark: One can also prove this directly from the definition. In this case

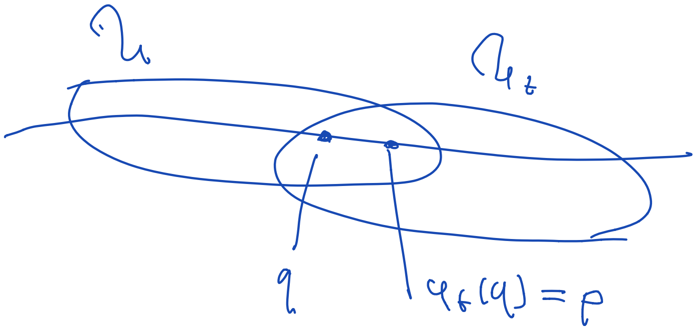
$$\begin{array}{cccc} \varphi_{t*} : & TM|_q & \longrightarrow & TM|_{\varphi(q)=p} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & u_q & \longmapsto & (\varphi_* u]_{\varphi_t(q)} \end{array}$$

where $(\varphi_{t*}u)_{\varphi_t(q)}$ is the vector tangent to the integral curves of u at $\varphi_t(q) \in M$.

//

Equivalently: φ_t (or any diffeomorphism) induces a coordinate transformation

$$\begin{tikzcd}
\varphi_t : M \arrow[r] & M \\
\cup & \cup \\
\mathcal{U} \arrow[r] & \mathcal{U}_t = \varphi_t(\mathcal{U}) \\
\cup & \cup \\
q \arrow[r, mapsto] \arrow[d, mapsto] & \varphi_t(q) = p \arrow[d, mapsto] \\
x^a(q) \arrow[r, mapsto, "\tilde{x} \circ \varphi_t \circ \tilde{x}^{-1}"'] & \tilde{x}^a(\varphi_t(q)) = \tilde{x}^a(p)
\end{tikzcd}$$



Consider an infinitesimal coordinate change induced by a 1-parameter group of diffeomorphisms φ_t:

$$\underset{\text{coords of } p}{\tilde{x}^a} = \underset{\text{coords of } q}{x^a} + t\, V^a(x)|_p + \cdots$$

$$V^a|_p = \frac{dV^a}{dt}\bigg|_p$$

Thm: for a function f

$$(\varphi_{t*} f)(p) = f(q)$$

$$= f(\tilde{x} - t V^a(x))$$

$$= f(\tilde{x}) - t V^a(x) \partial_a f + \dots$$

$$= f(p) - t V^a(x) \partial_a f + \dots$$

so $$V^a(x) \partial_a f = \delta_V f = \frac{df}{dt}$$

Remark: one can do this for other tensors

In particular:

$$(\varphi_{t*} T)_p = \tilde{T}(q) = \tilde{T}(\tilde{x} - t V^a(x))$$

In summary:

Killing vector ⟷ isometry

$\nabla_{(a}V_{b)}=0$ $\mathcal{L}_V g = 0$

"metric invariant"

- Extensive discussion of isometries in problem sheets, with examples!

- Important consequence for g: allows for an expression for g which is independent of one of the coordinates

Corollary of theorem page 13

Let V be a non-zero vector. Then there exist coordinates on M $\{x^a\} = \{t, y^i\}$

$a = 0,1,\ldots,n-1$ $\qquad$ $i = 1,\ldots,n-1$

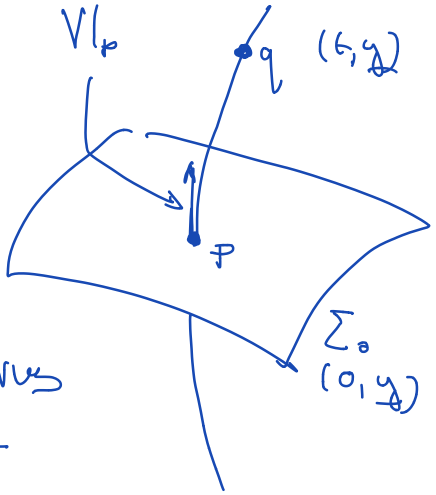
st locally $V = \partial/\partial t$.

Proof: Consider a hypersurface $\Sigma_0 \subset M$ defined by a function $f(\tilde{x}) = 0$ where $\tilde{x}^a$ are coordinates on M.

WLOG: take Σ_0 & V st V is not tangent to Σ_0)

Choose $x^0 = t = f(\tilde{x})$ so Σ_0 is the hypersurface $t=0$ with coordinates $(0, \underline{y})$ where $y = (y_1, y_2, \ldots, y_{n-1})$

Given $V = \partial_t$: there is unique integral curve of V through each point $p \in \Sigma_0$ (with tangent $V|_p$ at that point) which is the unique soln of



$$\frac{dx^0(t)}{dt} = V^0 = 1 \; ; \quad \frac{dy^i}{\partial t} = V^i = 0$$

with initial condition

$$x^a(p) = (0, y)$$

Then y = constant on these curves and the coordinates of q are (t, y). //

[Remark: Span{V} is tangent to a foliation by curves y = constant (the leaves which are the integral curves of V with parameter t)
$V \to$ tangent to the leaves]

Important property of the metric when there are killing vectors:

Let $K \neq 0$ be a vector field on M. Then there are coordinates (x^0, x^1, x^2, x^3) st

$$K = K^a \partial_a = \partial_0 \quad \text{ie} \quad K^0 = 1, \quad K^i = 0 \quad i = 1,2,3$$

$$K \text{ a Killing vector: } \partial_0 g_{ab} = 0$$

that is, the metric does not depend on the coordinate x^0 (ie g does not change along the integral curves of V)

Proof: K is a killing vector iff $\mathcal{L}_K g_{ab} = 0$

iff $0 = K^c \partial_c g_{ab} + \cancel{g_{ca} \partial_b K^c} + \cancel{g_{cb} \partial_a K^c}$

iff $0 = K^0 \partial_0 g_{ab} = \partial_0 g_{ab}$ //

<u>Next</u>: One can generalize the notion of integral curves.

Let $$W = \text{span}\{\underbrace{X_1, \dots, X_n}_{\text{set of linearly independent vector fields}}\}$$

Then $\forall\, p \in M$, $W|_p \subset TM|_p$

<u>Question</u>: Let $p \in M$. ¿Is there

$S \subset M$ (embedded submanifold)

through p st if $TS|_p = W|_p$, then for each $q \in S$ we have

$$TS|_q = W|_q \quad ?$$

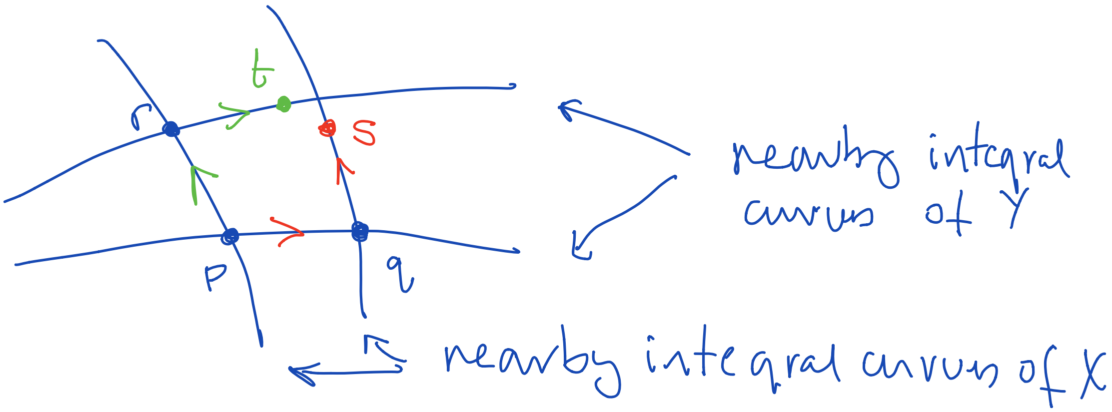
$n = 1$ $S \rightarrow$ integral curves

$n > 1$ <u>NEED</u> $[X_i, X_j] \in W$

$\forall\, i, j = 1, \dots, n$

$\boxed{1.3}$ Geometry of $[X, Y]$

Let X, Y be smooth vector fields on M
Consider the integral curves of X & Y



Let $p \in M$: compare $YX(p)$ & $XY(p)$

XY: let $q = p + \epsilon Y(p)$
$s = q + \epsilon X(q)$

$$s = p + \epsilon Y(p) + \epsilon X(p + \epsilon Y(p))$$
$$= p + \epsilon Y(p) + \epsilon X(p) + \epsilon^2 XY(p)$$

YX: let $r = p + \epsilon X(p)$
$t = r + \epsilon Y(p)$

$$t = p + \epsilon X(p) + \epsilon Y(p) + \epsilon^2 YX(p)$$

$$\therefore \quad s - t = \epsilon^2 [X, Y]\big|_p$$

[Notation: $X(p) \to$ action of 1-parameter group of diffeomorphisms on p ie $\varphi_t(p)$]

$[X,Y]=0$ (or more generally $[X,Y] \in \text{Span}\{X\,Y\}$ as we will see later)

$\Rightarrow$ the set of all points that can be reached along integral curves of X & Y from a given point $p \in M$ forms a 2-dim submanifold $S \subset M$ through p

The tangent space to S at any $q \in S$ is the <u>same</u> as the tangent space to S at p:

let $W = \text{Span}\{X,Y\}$ & $TS|_p = W|_p \subset TM|_p$

then $TS|_q = W|_q \quad \forall\, q \in S$.

lecture #5

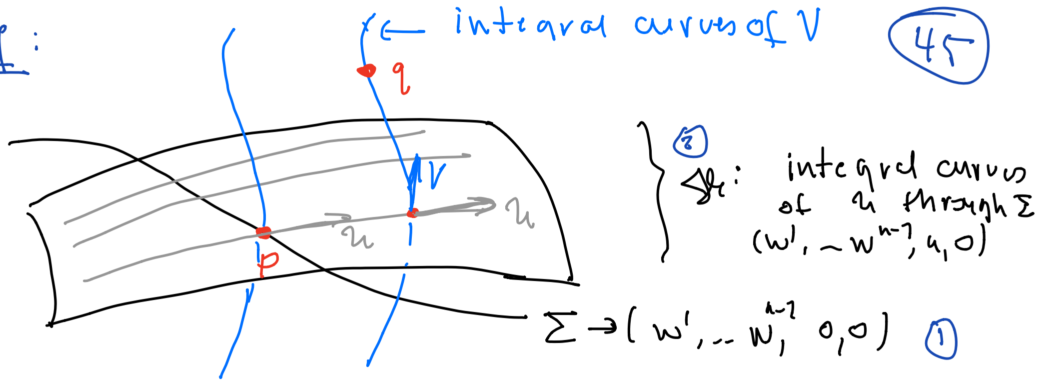
<u>Theorem</u>: (Frobenius for $k=2$)

Let U, V be linearly independent smooth vector fields on M. Then there are coords

$$\{w^1, \ldots, w^{n-2}, u, v\}$$

st $\quad U = \frac{\partial}{\partial u}, \quad V = \frac{\partial}{\partial v}$ iff $\quad [U,V]=0$

Proof:



① Let Σ be a $(n-2)$-surface. Then
$\forall\ p \in \Sigma$ <u>choose</u> coords $(w^1, .., w^{n-2}, 0, 0)$

② Let $\mathcal{H}$ be a hypersurface consisting of integral curves of $\mathcal{U}$ through Σ.
Let u : parameter length wrt $\mathcal{U}$ for the integral curves of $\mathcal{U}$ through Σ
Then: coords of points on $\mathcal{H}$ $(w^1, \dots w^{n-2}, u, 0)$

③ Let q be a point near $\mathcal{H}$ on an integral curve of V through $\mathcal{H}$.
Let v : parameter length wrt V for the integral curves of V through $\mathcal{H}$
coords of q : $(w^1, w^2, \dots w^{n-2}, u, v)$

Tangent vectors <u>at q</u>:

$$V = \partial_v \quad , \quad \mathcal{U} = a\,\partial_u + b\,\partial_v + s^i \frac{\partial}{\partial w^i}$$

st $\underline{\text{on } \mathcal{N}_0}$ $\quad s^i = 0 \quad \forall\, i = 1, \dots, n-2$
$\uparrow$ $v=0$ $\quad b = 0 \;,\; a = 1$ $\Big\}$ $\mathcal{U}$ tangent to its integral curves

Let f be a function on M

$$[\mathcal{U}, V](f) = \mathcal{U}(V(f)) - V(\mathcal{U}(f))$$
$$= \mathcal{U}(\partial_v(f)) - \partial_v(a\,\partial_u f + b\,\partial_v f + s^i \partial_i f)$$
$$= \mathcal{U}(\cancel{\partial_v}(f)) - (\partial_v a)\partial_u f - (\partial_v b)\partial_v f - (\partial_v s^i)\partial_i f - \cancel{\partial_v(\mathcal{U}(f))}$$
$$= -(\partial_v a)\partial_u f - (\partial_v b)\partial_v f - (\partial_v s^i)\partial_i f$$

Thus $[\mathcal{U}, V](f) = 0$ for any arbitrary differentiable function f

iff $\partial_v a = 0 \;,\; \partial_v b = 0 \;,\; \partial_v s^i = 0 \quad \forall\, i$

(because $\partial_u, \partial_v, \partial_i$ are linearly independent)

iff $a = a(u, w^i) \;,\; b = b(u, w^i) \;,\; s^i = s(u, w^i)$

ie <u>a, b, s^i independent of v</u>

On $\mathcal{N}$, where $v = 0$: $\quad \mathcal{U} = \partial_u$

ie $\quad a = 1 \;,\; b = 0 \;,\; s^i = 0$

$\therefore \quad b = 0 \;,\; s^i = 0 \;\; \forall i \;,$ and $a = 1$ & $\mathcal{U} = \partial_u$

Remark:

Lemma: Let $\hat{u}$ and $\hat{v}$ be linearly independent vector fields such that

$$[\hat{u}, \hat{v}] \in \text{Span}\{\hat{u}, \hat{v}\}$$

Thm: there exist u, v with

$$\text{Span}\{u, v\} = \text{Span}\{\hat{u}, \hat{v}\}$$

and $$[u, v] = 0$$

Proof: Let $u = \lambda \hat{u}$, $v = \mu \hat{v}$

$$[u,v](f) = [\lambda \hat{u}, \mu \hat{v}](f) = \lambda \hat{u}(\mu \hat{v}(f)) - \mu \hat{v}(\lambda \hat{u}(f))$$

$$= \lambda \hat{u}(\mu)\, \hat{v}(f) + \lambda \mu\, \hat{u} \hat{v}(f) - \mu \hat{v}(\lambda) \hat{u}(f) - \mu \lambda\, \hat{v} \hat{u}(f)$$

$$= \lambda \hat{u}(\mu)\, \hat{v}(f) - \mu \hat{v}(\lambda) \hat{u}(f) + \lambda \mu\, [\hat{u}, \hat{v}](f)$$

But $[\hat{u}, \hat{v}] \in \text{Span}\{\hat{u}, \hat{v}\}$, then for some a, b

$$[\hat{u}, \hat{v}](f) = a \hat{u} + b \hat{v}$$

Thus:

$$[u,v](f) = \lambda \mu \left\{ \frac{1}{\mu} \hat{u}(\mu)\, \hat{v}(f) - \frac{1}{\lambda} \hat{v}(\lambda) \hat{u}(f) + a \hat{u}(f) + b \hat{v}(f) \right\}$$

$$= \lambda \mu \left\{ \left(\frac{1}{\mu} \hat{u}(\mu) + b \right) \hat{v} + \left(-\frac{1}{\lambda} \hat{v}(\lambda) + a \right) \hat{u} \right\}(f)$$

For $[u,v] = 0 \ \forall f$: solve for λ, μ

$\hat{u}(\mu) = -\mu b \qquad \hat{v}(\lambda) = \lambda a$

ie $[u,v] = 0$ for λ, μ $(u = \lambda \hat{u},\ v = \mu \hat{v})$

st $\hat{u}(\log \mu) = -b \qquad \hat{v}(\log \lambda) = a$ //

Corollary:

$[u,v] \in \text{span}\{u,v\}$ iff the span$\{u,v\}$ is tangent to a family of 2-surfaces (integral surfaces)

On the surfaces $\{w^1,\dots,w^{n-2}\}$ is constant and $\{u,v\}$ parametrize the surfaces

(We say that the span$\{u,v\}$ is integrable)

Proof: by lemma & theorem.

Theorem: Frobenius

Let $W = \text{span}\{\underbrace{u_1,\dots,u_k}_{\text{set of linearly independent smooth vector fields.}}\}$

$[u_i,u_j] \in W$ iff W is tangent to a family of k-dim surfaces (integral surfaces). ←

The integral surfaces are parametrized by coordinates $(u_1,\dots,u_k)$ and given by $(w_1,\dots,w^{n-k})$ = constant.

Proof: induction on k // not examinable

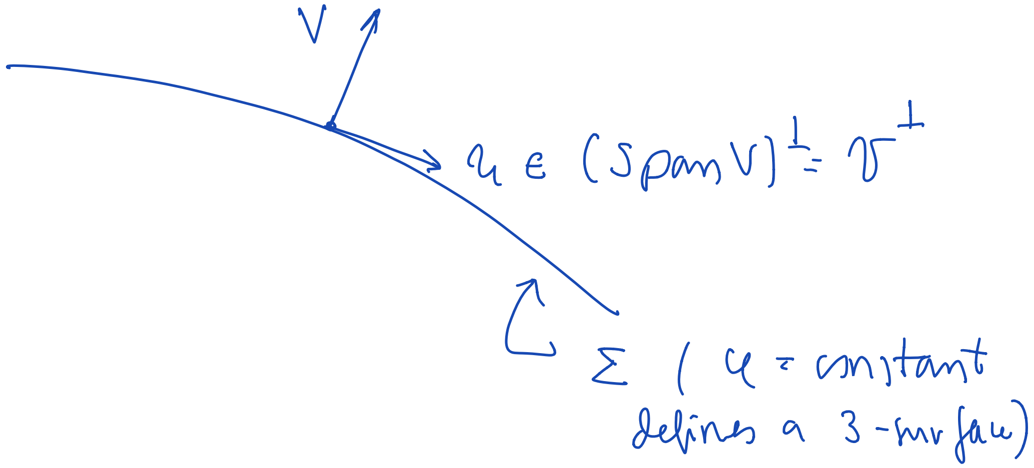
Dual versions

Theorem: Let $V_a \neq 0$ be a smooth 1-form

$V_{[a} \nabla_b V_{c]} = 0$ iff there are functions φ, ψ st $V_a = \psi \partial_a \varphi$ //

Definition: such V_a is perpendicular to the hypersurface Σ defined by $\varphi =$ constant and it is said to be hypersurface orthogonal (HSO).

Recall: V_a is perpendicular to the hypersurface Σ (φ = constant) iff $V_a u^a = 0$ $\forall$ vectors u which are tangent to Σ



Proof

(⟸) straightforward & computational

Suppose there are functions φ, ψ st

$$V_a = \psi \partial_a \varphi$$

$$V_a \nabla_{[b} V_{c]} = \psi \partial_a \varphi \nabla_{[b} (\psi \partial_{c]} \varphi)$$

$$= \psi \partial_a \varphi \left(\partial_{[b} (\psi \partial_{c]} \varphi) - \cancel{\Gamma^d_{[bc]}} \psi \partial_d \varphi \right)$$

$$= \psi \partial_a \varphi \left(\partial_{[b} \psi \partial_{c]} \varphi + \psi \cancel{\partial_{[b} \partial_{c]}} \varphi \right)$$

$$= \psi \partial_a \varphi \partial_{[b} \psi \partial_{c]} \varphi$$

$$3 \, V_a \nabla_{[b} V_{c]} = 3 \, \psi \, \partial_{[a} \varphi \, \partial_b \psi \, \partial_{c]} \varphi = 0$$

(⟹) Let $\mathcal{V}^{\perp} = (\mathrm{Span}\{V\})^{\perp}$ orthogonal complement of Span{V} in TM

Recall: a vector $W \in \mathcal{V}^{\perp}$ iff $g_{ab} W^a X^b = W_a X^b = 0$, $\forall X \in \mathrm{Span}\{V\}$

Claim: $V_{[a} \nabla_b V_{c]} = 0 \Rightarrow [X, Y] \in \mathcal{V}^{\perp}$, $\forall X, Y$ linearly indep vectors in $\mathcal{V}^{\perp}$

<u>If true</u>: by Frobenius, there is a family of (n−1)-surfaces tangent to $V^\perp$ and V is perpendicular to these surfaces.

Then we can pick φ st φ = constant on each member of this family, and

$\partial_a \varphi$ ("gradient")

is perpendicular to these surfaces ie

$V_a = \psi \, \partial_a \varphi$ for some function ψ

Proof of the claim: (TP: if V st $V_{[a}\nabla_b V_{c]}=0$ $\Longrightarrow [X,Y] \in V^\perp, \forall X, Y$ lin. indep vectors in $V^\perp$)

want to prove that this vanishes.

$$[X,Y]^a V_a = (X^b \nabla_b Y^a - Y^b \nabla_b X^a) V_a$$

$$= X^b(\nabla_b(\cancel{Y^a V_a}) - Y^a \nabla_b V_a) - Y^b(\nabla_b(\cancel{X^a V_a}) - X^a \nabla_b V_a)$$

↑ as $Y \in V^\perp$ ↑ as $X \in V^\perp$

$$= -X^a Y^b (\nabla_a V_b - \nabla_b V_a)$$

so $[X,Y]^a V_a = -2 X^a Y^b \nabla_{[a} V_{b]}$ (*)

Now: $V_{[a}\nabla_b V_{c]} = 0$ iff

$$X^b Y^c V_{[a}\nabla_b V_{c]} = 0 \quad \forall X, Y \in \mathcal{V}^\perp$$

So

$$0 = X^b Y^c V_{[a}\nabla_b V_{c]}$$

$$= \frac{1}{3} X^b Y^c \left(V_a \nabla_{[b} V_{c]} + \cancel{V_b \nabla_{[c} V_{a]}} + \cancel{V_c \nabla_{[a} V_{b]}} \right)$$

as $X^b V_b = 0$ as $Y^c V_c = 0$

(*) $\rightarrow$ $= \frac{1}{3} V_a \left(-\frac{1}{2} [X, Y]^b V_b \right)$

$\therefore$ $[X,Y]^b V_b = 0$ ie $[X,Y] \in \mathcal{V}^\perp$

and $\mathcal{V}^\perp$ (is integrable) is tangent to a family of $(n-1)$-surfaces //

not in the lectures

General: Let $\mathcal{V} = \text{Span}\{\underbrace{V^{(1)}, \dots, V^{(h)}}_{\text{set of linearly independent 1-forms}}\}$

Let $X, Y \in \mathcal{V}^\perp$. Then

$[X,Y] \in \mathcal{V}^\perp$ iff $[X,Y]^a V_a = -2 X^a Y^b \nabla_{[a} V_{b]} = 0 \quad \forall V \in \mathcal{V}$

iff $\nabla_{[a} V_{b]} = \sum_{i=1}^{h} \beta^{(i)}_{[a} V^{(i)}_{b]}$, $\forall V \in \mathcal{V}$

where $\beta^{(i)}$ are arbitrary 1-forms

iff there is a family of $(n-h)$ hypersurfaces tangent to $\mathcal{V}^\perp$

Important example

Let K be HSO (but not necessarily killing)
then $g_{0i} = 0$ $\quad i = 1, 2, 3$

Proof : $K_a = g_{ab}K^b = \psi\,\partial_a\varphi$

(K perpendicular to a 3 dim hypersurface defined by $\varphi =$ constant)

We can choose coordinates st $\varphi = x^0$ and $K = \partial_0$

Then

$$K_0 = g_{00}K^0 = g_{00} = \psi\,\partial_0\varphi = \psi$$

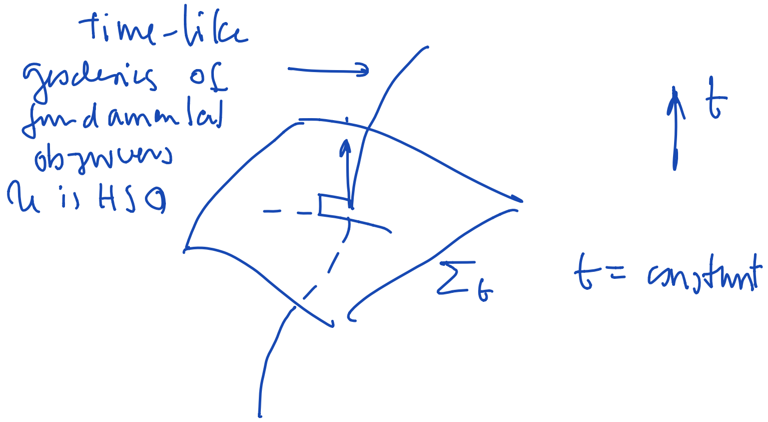
$$K_i = g_{i0}K^0 = g_{i0} = \psi\,\partial_i\underset{\hookrightarrow x^0}{\varphi} = 0$$

$$\therefore \quad ds^2 = \psi\,dx^{0\,2} + g_{ij}\,dx^i dx^j \qquad //$$

If K is also Killing then g_{ab} is independent of x^0

Applications:

- cosmology



- Schwarzschild black hole

⇝ static soln of Eintein's eqs in vacuum

↳ there is a HSO TL K-V

- stationary solns : TL K-V.

- Think about the case where

  $W = \text{Span}\{K_1, \dots K_n\}$

  K_i are Killing vectors and the fact that $[K_i, K_j]$ is also a Killing vector

End of Chapter 1