Lecture #6

Chapter 2 Linearized GR

* approximation in which gravity is weak ie space time is nearly flat
* metric (gravitational field) when
  (1) far away from an isolated body (source): identify mass, angular momentum, ...
  (2) degrees of freedom of a weak gravitational field in the absence of sources
  → gravitational waves

Recall: (GR1)

* <u>Einstein field equations</u>

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$$ <u>EFEq</u>

→ relates the metric g, the gravitational field, to its sources

$$\nabla^a T_{ab} = 0$$ <u>conservation law</u>

is a consequence of EFEq

problem sheet 1: one can prove that
$$\nabla^a \left(R_{ab} - \frac{1}{2} g_{ab} R\right) = 0$$
So EFEq implies $\nabla^a T_{ab} = 0$

* <u>Examples of sources:</u>

<u>dust</u>: $T_{ab} = \rho u_a u_b$

↳ distribution of matter with rest density ρ and 4-velocity u

<u>perfect fluid</u>: $T_{ab} = (\rho + p) u_a u_b + p g_{ab}$

<u>EM field</u>: $T_{ab} = F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd}$

$F_{ab} = 2 \partial_{[a} A_{b]}$ EM field strength

* In <u>vacuum</u>: take trace of EFEq

$$R - \frac{1}{2} \cdot 4R = 8\pi G T, \qquad T = \mathrm{tr}\,T = T_{ab} g^{ab}$$

$$\Rightarrow \qquad R = -8\pi G T$$

Using this in EFEq:

$$R_{ab} - \frac{1}{2} g_{ab} R = R_{ab} - \frac{1}{2} g_{ab}(-8\pi G T) = 8\pi G T_{ab}$$

so

$$\underline{R_{ab} = 8\pi G \left(T_{ab} - \frac{1}{2} g_{ab} T \right)}$$

In vacuum: $T_{ab} = 0$ so
no sources

$$\underline{R_{ab} = 0}$$

equation for the gravitational field in space <u>exterior</u> to all sources

2.1 Linearizing the field equation

Assume: $g_{ab} = \eta_{ab} + \epsilon h_{ab}$ where

η_{ab} Minkowski metric

h_{ab}
- symmetric $(0,2)$ tensor
- regard as a smooth field on Minkowski space

ϵ "small parameter" ie we neglect terms ϵ^n, $n \geq 2$

linearized GR: theory of a symmetric tensor h_{ab} on Minkowski space

$\rightsquigarrow$ raise and lower indices with η

Next: use $g_{ab} = \eta_{ab} + \epsilon h_{ab}$

into EFEq to obtain an equation for h (linearized field equations)

Need: the inverse of the metric, g^{ab}, the Christoffel symbols, and the curvature.

(5)

Inverse of g: g^{ab} st $g^{ab}g_{bc} = \delta^a_c$

Let $g^{ab} = \eta^{ab} - \epsilon S^{ab} + O(\epsilon^2)$
for some symmetric tensor S^{ab}. Then

$$\delta^a_b = g^{ac}g_{cb} = (\eta^{ac} - \epsilon S^{ac} + \cdots)(\eta_{bc} + \epsilon h_{bc} + \cdots)$$

$$= \delta^a_b + \epsilon(\eta^{ac}h_{bc} - S^{ac}\eta_{bc}] + \cdots$$

$$\therefore \eta^{ac}h_{bc} = S^{ac}\eta_{bc} \qquad (\text{contract with } \eta^{bd})$$

$$\Rightarrow S^{ad} = \eta^{ac}\eta^{bd}h_{bc} \equiv h^{ad}$$

$$\therefore \underline{g^{ab} = \eta^{ab} - \epsilon h^{ab} + O(\epsilon^2)}$$

Christoffel symbols:

$$\Gamma^c_{ab} = \frac{1}{2}g^{cd}(\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab})$$

$$\underline{\Gamma^c_{ab} = \frac{1}{2}\epsilon\eta^{cd}(\partial_a h_{bd} + \partial_b h_{ad} - \partial_d h_{ab}) + \cdots}$$

Curvature:

$$R^d{}_{cab} = \partial_a \Gamma^d{}_{bc} - \partial_b \Gamma^d{}_{ac} + \underbrace{\Gamma^d{}_{ae}\Gamma^e{}_{bc} - \Gamma^d{}_{be}\Gamma^e{}_{ac}}_{\mathcal{O}(\epsilon^2)}$$

$$= \partial_a \Gamma^d{}_{bc} - \partial_b \Gamma^d{}_{ac} + \mathcal{O}(\epsilon^2)$$

$$\partial_a \Gamma^d{}_{bc} = \frac{1}{2}\epsilon\,\eta^{de}\partial_a\left(\partial_b h_{ce} + \partial_c h_{be} - \partial_e h_{bc}\right) + \dots$$

$$\Rightarrow \quad R^d{}_{cab} = \frac{1}{2}\epsilon\,\eta^{de}\left(\partial_a\partial_c h_{be} - \partial_a\partial_e h_{bc} - \partial_b\partial_c h_{ae} + \partial_b\partial_e h_{ac}\right) + \dots$$

$$R^d{}_{cab} = \epsilon\,\eta^{de}\left(\partial_c\partial_{[a}h_{b]e} - \partial_e\partial_{[a}h_{b]c}\right) + \dots$$

Ricci tensor:

$$R_{ab} = R^c{}_{acb}$$

$$R_{cb} = \epsilon\,\eta^{ae}\left(\partial_c\partial_{[a}h_{b]e} - \partial_e\partial_{[a}h_{b]c}\right) + \dots$$

$$= \frac{1}{2}\epsilon\left(\partial_a\partial_c h_b{}^a - \partial_b\partial_c h - \Box h_{bc} + \partial_b\partial_a h_c{}^a\right) + \dots$$

where $h = \eta^{ab}h_{ab}$, $\quad \Box = \eta^{ab}\partial_a\partial_b$

$$R_{ab} = -\frac{1}{2}\epsilon\left(\Box h_{ab} + \partial_a\partial_b h - \partial_c\left(\partial_a h_b{}^c + \partial_b h_a{}^c\right)\right) + \dots$$

Write in a more convenient form (7)

Let $H_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab} h$ trace reversing $(H = -h)$

Consider the second and third terms of R_{ab}

$$\partial_a \partial_b h - \partial_c(\partial_a h_b{}^c + \partial_b h_a{}^c)$$

$$= -\partial_c \partial_a \left(h_b{}^c - \frac{1}{2}\delta^c_b h\right)$$

$$- \partial_c \partial_b \left(h_a{}^c - \frac{1}{2}\delta^c_a h\right)$$

$$= -\partial_c\left(\partial_a H_b{}^c + \partial_b H_a{}^c\right)$$

Then

$$\boxed{R_{ab} = +\frac{1}{2}\epsilon\left(-\Box h_{ab} + \partial_c\left(\partial_a H_b{}^c + \partial_b H_a{}^c\right)\right)}$$

Scalar curvature

$$R = g^{ab} R_{ab} = \eta^{ab} R_{ab} + O(\epsilon^2)$$

$$R = \epsilon\left(-\Box h + \partial_a \partial_b h^{ab}\right)$$

Field equations are still very complicated.

$$R_{ab} = +\frac{1}{2}\epsilon\left(-\Box h_{ab} + \partial_c\left(\partial_a H_b{}^c + \partial_b H_a{}^c\right)\right)$$

$$= 8\pi G\left(T_{ab} - \frac{1}{2} g_{ab} T\right)$$

However: we will see that we have some freedom to choose a convenient coordinate systems so R_{ab} and the field equations simplify.

Change coordinates

Consider: $\quad x^a \longrightarrow \hat{x}^a = x^a + \epsilon\, y^a(x)$

Then $\quad \partial_b \hat{x}^a = \delta^a_b + \epsilon\, \partial_b y^a(x)$

Change in the metric:

$$g_{ab}(x) = \hat{g}_{ab}(x) + \epsilon \, \mathcal{L}_y g_{ab} + \mathcal{O}(\epsilon^2)$$

In fact we can think of the coordinate change $\hat{x}^a = x^a + \epsilon y^a$ as a "small" coordinate change induced by a 1-parameter group of diffeomorphisms corresponding to a vector Y with components y^a.

Then: $h_{ab}(x) = \hat{h}_{ab}(x) + \mathcal{L}_y g_{ab} + \cdots$

$$\mathcal{L}_y g_{ab} = y^c \partial_c g_{ab} + g_{ca} \partial_b y^c + g_{cb} \partial_a y^c$$

$$= \mathcal{L}_y \eta_{ab} + \mathcal{O}(\epsilon)$$

$$= 2 \partial_{(a} y_{b)} + \mathcal{O}(\epsilon)$$

$$\Longrightarrow \boxed{h_{ab}(x) = \hat{h}_{ab}(x) + 2 \partial_{(a} y_{b)} + \mathcal{O}(\epsilon)}$$

trace: $\qquad h(x) = \hat{h}(x) + 2\partial_a y^a$

Change in R_{ab}

$$R_{ab}(h) = +\frac{1}{2}G\left(-\Box h_{ab} + \partial_c\left(\partial_a H_b{}^c + \partial_b H_a{}^c\right)\right) + \dots$$

$$H_{ab}(x) = h_{ab}(x) - \frac{1}{2}\eta_{ab}h$$

$$= \hat{h}_{ab}(x) + 2\partial_{(a}y_{b)} - \frac{1}{2}\eta_{ab}\left(\hat{h}(x) + 2\partial_c y^c\right)$$

$$= \hat{H}_{ab}(x) + 2\partial_{(a}y_{b)} - \eta_{ab}\partial_c y^c$$

$$\partial_a\partial_c H_b{}^c = \partial_a\partial_c\left(\hat{H}_b{}^c + \cancel{\partial_b y^c} + \eta^{cd}\partial_d y_b - \cancel{\delta_b^c}\partial_d y^d\right)$$

$$= \partial_a\partial_c\hat{H}_b{}^c + \Box\partial_a y_b$$

$$\partial_c\left(\partial_a H_b{}^c + \partial_b H_a{}^c\right) =$$

$$R_{ab}(h) = \frac{1}{2}G\left(-\Box\left(\hat{h}_{ab} + 2\partial_{(a}y_{b)}\right) + \partial_c\left(\partial_a\hat{H}_b{}^c + \partial_b\hat{H}_a{}^c\right) + 2\Box\partial_{(a}y_{b)}\right)$$

$$= R_{ab}(\hat{h}) + \dots$$

$\therefore$ h and $\hat{h}$ give the same EFEq

The EFEq have a large indeterminacy.

In fact: there are only 6 equations for 10 unknowns

* g_{ab} has 10 components (unknowns)
* $R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G T_{ab}$ EFEq : 10 eqs (in 4 dims)

but we also have 4 relations

$$\nabla^a \left(R_{ab} - \frac{1}{2} g_{ab} R \right) = 0$$

$\therefore$ there are only $10 - 4 = 6$ equations

In general: if g is a solution of EFEqs then so is $\hat{g}$ obtained by a coordinate transformation $x^a \longrightarrow \hat{x}^a$. This coordinate transformation involves precisely 4 arbitrary functions $\hat{x}^a(x)$

Thus: use this freedom to simplify EFEqs.

lecture #7

Analogy with electromagnetism

Maxwell's equations in terms of the electromagnetic potential A:

$$\Box A_a - \partial_a \partial_b A^b = J_a$$

$\leftarrow$ sources

These give 4 equations for the 4 unknowns A_a. However, there are only 3 independent equations because

$$\partial^a J_a = 0 \quad \text{(conservation eq)}$$

[Note that in fact $\partial^a (\Box A_a - \partial_a \partial_b A^b) = 0$]

Hence Maxwell's eqs do not determine A uniquely. In fact, if A is a solution, so is

$$\hat{A}_a = A_a + \partial_a \phi \quad \text{for any } \phi$$

($\hat{A}$ and A produce the same $F_{ab} = 2\partial_{[a} A_{b]}$)

This is called a <u>gauge transformation</u> and we say that F is <u>gauge invariant</u>.

To remove the ambiguity in A we "choose a gauge" (and choose st equations are simpler).

For example: <u>Lorenz-gauge</u>

Construct a solution such that

$$\partial_a \hat{A}^a = 0$$

(extra condition which fixes the ambiguity)

that is, $\partial_a A^a = -\Box \phi$

In this gauge:

$$J_a = \Box (\hat{A}_a - \cancel{\partial_a \phi}) - \partial_a \partial_b (\hat{A}^b - \cancel{\partial^b \phi})$$

$$= \Box \hat{A}_a - \cancel{\partial_a \partial_b \hat{A}^b}$$

Lorenz-gauge

$$\therefore \quad \Box \hat{A}_a = J_a$$

inhomogeneous wave equation

<u>In GR</u>: we have some freedom to find a convenient coordinate system to compute R_{ab}.

Back to GR:

want to fix the indeterminacy in R_{ab}
(like in EM: a good choice can simplify considerably)

There exists a coordinate choice st

$$\Gamma^a \equiv g^{bc}\,\Gamma^a{}_{bc} = 0$$

(problem sheet 2)

In this gauge

$$\Box \hat{x}^a = -g^{bc}\,\Gamma^a{}_{bc} = 0$$

ie the coordinate functions are <u>harmonic</u>.

When $g_{ab} = \eta_{ab} + \epsilon h_{ab}$

$$\Gamma^a = 0 \quad \text{iff} \quad \partial_b H_a{}^b = 0$$

Definition: the choice

$$\partial_a H_b{}^a = 0$$

is called the de Donder gauge and the coordinates are called harmonic (wave) coordinates.

In this gauge

so $$R_{ab}(h) = -\frac{\epsilon}{2}\Box h_{ab}$$

We use this together with EFEqs and

$H_{ab} = h_{ab} - \frac{1}{2}\eta_{ab} h$ chosen st

$$\partial_b H_a{}^b = 0$$

Exercise: Show that in fact there is a coordinate system st

$$\Box y^a = 0 \quad , \quad \partial_b H_a{}^b = 0$$

Let $\hat{x}^a = x^a + \epsilon y^a$. Then

$$h_{ab} = \hat{h}_{ab} + 2\,\partial_{(a} y_{b)}$$

$$H_{ab} = \hat{H}_{ab} + 2\,\partial_{(a} y_{b)} - \eta_{ab}\,\partial_c y^c$$

$$\partial_b H_a{}^b = \partial_b \hat{H}_a{}^b + \cancel{\partial_b \partial_a y^b} + \Box y_a - \cancel{\partial_a \partial_c y^c}$$

$$= \partial_b \hat{H}_a{}^b + \Box y_a$$

Choose coordinates y^a such that (de Donder)

$$0 = \partial_b \hat{H}_a{}^b = \partial_b \left(\hat{h}_a{}^b - \tfrac{1}{2}\,\delta_a{}^b \hat{h} \right)$$

ie such that $\quad \Box y_a = \partial_b H_a{}^b$

There is still some remaining freedom to change coordinates further while preserving the deDonder gauge.

Let $\hat{x}^a \longmapsto \hat{\hat{x}}^a = \hat{x}^a + \epsilon \hat{y}^a(\hat{x})$
where $\hat{x}^a = x^a + \epsilon y^a(x)$

Then $\hat{h}_{ab}(\hat{x}) = \hat{\hat{h}}_{ab}(\hat{x}) + 2\hat{\partial}_{(a}\hat{y}_{b)}$ and

$$R_{ab}(\hat{h}) = -\frac{\epsilon}{2}\Box \hat{h}_{ab} + \ldots = -\frac{\epsilon}{2}\Box\left(\hat{\hat{h}}_{ab}(\hat{x}) + 2\hat{\partial}_{(a}\hat{y}_{b)}\right) + \ldots$$

with $\hat{\Box}\hat{y}^a = \hat{\partial}_b(\cancel{\hat{H}_a}^{\,b} - \hat{\hat{H}}_a{}^b) = -\hat{\hat{\partial}}_b \hat{\hat{H}}_a{}^b$ $*$

Require the deDonder gauge to be preserved
ie $\hat{\partial}_b \hat{\hat{H}}_a{}^b = 0$

Then we find $\hat{\Box}\hat{y}_a = 0$

and $R_{ab}(\hat{h}) = R_{ab}(\hat{\hat{h}}) = -\frac{\epsilon}{2}\Box\hat{\hat{h}}_{ab}$

(end of exercise) //

Linearized field equations

Let

$$\hat{T}_{ab} = T_{ab} - \frac{1}{2} g_{ab} T, \qquad T = g^{ab} T_{ab}$$

Then EFEqs are $R_{ab} = 8\pi G \hat{T}_{ab}$

The linearized field equations (in the de Donder gauge are)

$$R_{ab}(h) = -\frac{\epsilon}{2} \Box h_{ab} = 8\pi G \hat{T}_{ab}$$

weak internal gravity source

ie $$\boxed{\epsilon \Box h_{ab} = -16\pi G \hat{T}_{ab}}$$

inhomogeneous wave eq for h with source $\hat{T}$ for a <u>weak</u> gravitational field

together with $$\underline{\partial_b \left(h_a{}^b - \frac{1}{2} \delta_a{}^b h \right) = 0}$$

2.2 Asymptotic metric of an isolated body

Consider a stationary (time independent) solution of the linearized field eqs

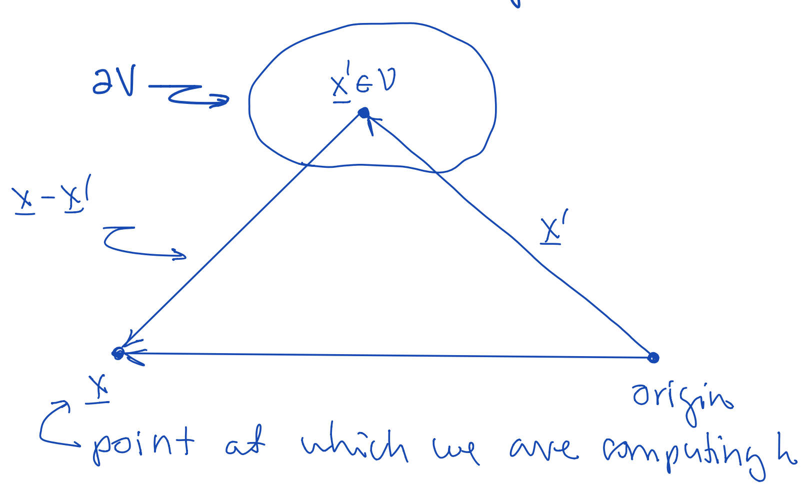
$$\Box \cdot = \eta^{ab} \partial_a \partial_b \cdot = \eta^{ij} \partial_i \partial_j \cdot = \nabla \cdot$$

Then : $\quad \epsilon \nabla^2 h_{ab} = -16 \pi G \hat{T}_{ab}$

Solve using Green's functions

$$\epsilon h_{ab}(\underline{x}) = 4\pi \int_V \frac{\hat{T}_{ab}(\underline{x}')}{|\underline{x} - \underline{x}'|} d^3x'$$

where we are assuming that $\hat{T}_{ab}$ has compact support V (ie sources are restricted to a compact space-time region)



<u>Want</u>: $h_{ab}(\underline{x})$ in a region of space-time <u>far</u> outside the source

So we expand the integrand in powers of $r = |\underline{x}|$

We will need the following identities

0) $\partial_i T^{i0} = 0 \qquad \partial_i T^{ij} = 0$

(from $\nabla_a T^{ab} = 0$: $\partial_a T^{ab} = 0$
$\Rightarrow \cancel{\partial_0} T^{0a} + \partial_i T^{ia} = 0 \Rightarrow \partial_i T^{ia} = 0$)

1) $$\frac{1}{|\underline{x} - \underline{x}'|} = \frac{1}{r} + \frac{1}{r^3}\,\underline{x}\cdot\underline{x}' + O\left(\frac{1}{r^3}\right)$$

2) $(T^{lm} x^i x^j)_{,lm} = 2T^{ij}$,

(Thus: $$2\int_V T^{ij}\, d^3x = \int_V (T^{lm} x^i x^j)_{,lm}\, d^3x = \int_{\partial V} (T^{lm} x^i x^j)_{,l}\, dS_m = 0$$

Also: $(T^{ji} x^i)_{,j} = T^{ii}$ $\left(\Rightarrow \int_V T^{ii}\, d^3x = 0\right)$
(no sum over i)

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3) $\left(T^{0i} x^i x^k\right)_{,i} = 2\, T^{0(j} x^{k)}$

$$\left(\Rightarrow \int_V T^{0(j} x^{k)}\, d^3x = 0 \right)$$

4) $\left(T_i^{\ k} x^i x^j - \frac{1}{2} T^{jk} x_i x^i\right)_{,k} = T_i^{\ i} x^j$

$$\left(\Rightarrow \int_V T_i^{\ i} x^j\, d^3x = 0 \right)$$

$\underline{h_{00}(\underline{x})}$

$$\hat{T}_{00} = T_{00} + \frac{1}{2} T$$

$$T = \eta^{ab} T_{ab} = -T_{00} + T_i^{\ i}$$

$$\hat{T}_{00} = \frac{1}{2}\left(T_{00} + T_i^{\ i}\right)$$

$$\epsilon\, h_{00}(\underline{x}) = 4G \int_V \frac{\hat{T}_{00}(\underline{x}')}{|\underline{x}-\underline{x}'|}\, d^3x'$$

$$\stackrel{(1)}{=} \frac{2G}{r} \int_V \left(1 + \frac{1}{r^2}\, \underline{x}\cdot\underline{x}' + \cdots\right)\left(T_{00} + T_i^{\ i}\right) d^3x'$$

$$\epsilon h_{00}(\underline{x}) = \frac{2G}{r}\int_V \left(T_{00} + \underbrace{T_i{}^i}_{\to 0 \text{ by } (2)} + \frac{1}{r^2}\,\underline{x}\cdot\underline{x}'\,T_{00} + \underbrace{\frac{1}{r^2}\,\underline{x}\cdot\underline{x}'\,T_i{}^i}_{\to 0 \text{ by } (4)} \right) d^3x'$$

Then

$$\boxed{\epsilon h_{00}(\underline{x}) = \frac{2GM}{r} + O\left(\frac{1}{r^2}\right)}$$

where

$$M \equiv \int_V T_{00}\, d^3x'$$

<u>interpret as total mass</u>

and

$$\underbrace{\int_V T_{00}\, x^i\, d^3x' = 0}_{\text{center of mass}}$$

translate center of mass to origin (center of mass frame)

$\underline{\underline{h_{0i}(\underline{x})}}$: $\hat{T}_{0i} = T_{0i} - \frac{1}{2}\cancel{\eta_{0i}} T = T_{0i}$

$$\epsilon\, h_{0i}(\underline{x}) = 4G \int_V \frac{T_{0i}(\underline{x}')}{|\underline{x}-\underline{x}'|} d^3x'$$

$$= \frac{4G}{r} \int_V \left(1 + \frac{1}{r^2}\underline{x}\cdot\underline{x}' + \cdots\right) T_{0i}(\underline{x}')\, d^3x'$$

$$= \frac{4G}{r}\left(\int_V T_{0i}(\underline{x}')\, d^3x' + \frac{1}{r^2} x^j \int_V T_{0i}(\underline{x}')\, x'_j\, d^3x'\right)$$

$$P^i = \int_V T^{0i}\, d^3x$$ total momentum in the x^i direction

$P^i = 0$ in the rest frame of the system

Thus

$$\epsilon\, h_{0i}(\underline{x}) = \frac{4G}{r^3} x^j \int_V T_{0i}(\underline{x}')\, x'_j\, d^3x' + \cdots$$

By identity ③

$$\int_V T_{0(i} X'_{j)} \, d^3x' = 0$$

Then

$$\int_V X'_j T_{0i} \, d^3x' = -\int_V X'_{[i} T_{j]0} \, d^3x'$$

$$\equiv -\frac{1}{2} \epsilon_{ijk} J^k$$

and $\displaystyle\int \underline{x} \cdot \underline{x}' \, T_{0i} \, d^3x' = -\frac{1}{2} (\underline{r} \times \underline{J})_i$

We interpret $\underline{J}$ as the angular momentum.

$$\boxed{\epsilon h_{0i}(\underline{x}) = -\frac{2G}{r^3} (\underline{r} \times \underline{J})_i + O\left(\frac{1}{r^3}\right) + \cdots}$$

$\underline{h_{ij}(\underline{x})}$: $\hat{T}_{ij} = T_{ij} - \frac{1}{2}\eta_{ij}\left(-T_{00} + T_h{}^h\right)$

$$\epsilon h_{ij}(\underline{x}) = \frac{4G}{r}\int_V \left(1 + \frac{1}{r^2}\underline{x}\cdot\underline{x}'\right)\left[T_{ij} - \frac{1}{2}\eta_{ij}\left(-T_{00} + T_h{}^h\right)\right] d^3x'$$

$$= \frac{4G}{r}\int_V \left(\underbrace{T_{ij}}_{\to 0 \text{ by } (2)} + \frac{1}{2}\eta_{ij}\left(T_{00} - \underbrace{T_h{}^h}_{\to 0 \text{ by } (2)}\right)\right.$$

$$\left. + \frac{1}{r^2}\underline{x}\cdot\underline{x}'\left(T_{ij} - \frac{1}{2}\eta_{ij}\left(-T_{00} + T_h{}^h\right)\right)\right) d^3x'$$

$$\boxed{\epsilon h_{ij}(\underline{x}) = \frac{2G}{r}\eta_{ij} M + \Theta\left(\frac{1}{r^2}\right)}$$

Metric: recall $g_{ab} = \eta_{ab} + \epsilon h_{ab}$

$$ds^2 = (\eta_{ab} + \epsilon h_{ab})\, dx^a dx^b$$

$$= -\left(1 - \frac{2MG}{r} + O\left(\frac{1}{r^2}\right)\right) dt^2$$

$$-2\cdot\left(\frac{2G}{r^3}(\underline{r} \times \underline{J})_i + O\left(\frac{1}{r^3}\right)\right) dt\, dx^i$$

$$+\left(\left(1 + \frac{2MG}{r} + O\left(\frac{1}{r^2}\right)\right)\right) \eta_{ij}\, dx^i dx^j$$

Polar coordinates: (r, θ, ϕ)

let $\underline{J} = J\, \hat{e}_z$

$$(\underline{r} \times \underline{J})_i\, dx^i = J(\underline{r} \times \hat{e}_z)_i\, dx^i$$

$$= J\,(y\, dx - x\, dy) = -J r^2 \sin^2\theta\, d\varphi$$

Then

$$*\quad \boxed{\begin{aligned} ds^2 = &-\left(1-\frac{2MG}{r}\right)dt^2 + 2\cdot\frac{2G}{r}J\sin^2\theta\, dt\, d\varphi \\ &+\left(1+\frac{2MG}{r}\right)\left(dr^2 + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)\right)\end{aligned}}$$

where $\quad M = \int_V T_{00}\, d^3x \qquad$ total mass

and $\quad \frac{1}{2}\epsilon_{ijk}J^k = \int_V x'_i\, T_{j0}(x')\, d^3x'$

$\quad$ ↳ angular momentum

<u>In fact</u>: this is the metric far away from any gravitational field (asymptotic metric of an isolated body)

Remarks (see Misner, Thorne & Wheeler)

This metric was obtained assuming that T_{ab} is the energy-momentum tensor corresponding to a weak internal gravity source.
However: the result obtained is the metric corresponding to an isolated body far away from the source even when the gravitational source is strong (eg black holes)

Definition: The total mass-energy M of the body is the constant that appears in the metric (equation $*$) for the asymptotic space-time geometry.
Similarly, we define the (intrinsic) angular momentum of the body as the constant 3 vector $\underline{J}$ appearing in this metric.

This is defined regardless of the source (early 60's: Arnowitt, Deser, Misner, etc...)

The linearized theory is valid when the internal gravity source is weak and in this case

Newton approx

$$M = \int_V T_{00}\, d^3x \sim \int \rho \, d^3x$$

$$J_i = \epsilon_{ijk} \int_V T_{0i}\, x'_j \, d^3x'$$

These relations fail if the gravitational field is strong

[Problem sheets: on Minkowski space

$M \longrightarrow$ conserved quantity related to time translations

$J \longrightarrow$ conserved quantity related to spatial rotations

(recall: $J_a = T_{ab} K^b$, K a Killing vector

$\Rightarrow \int J_0 \, d^3x$ is conserved)]

Problem: definition of M

It is not obvious why M is positive!

Positive energy theorem:

$M > 0$ (except for Minkowski for which $M = 0$)

Schoen – Yau 1979

* Witten 1981 Comm Math Phys

↳ (topic for oral presentation)

In practice:

¿ How do we measure the mass M or J of an object?

one doesn't compute $\int T_{00}$!

$M \rightarrow$ study orbits of "test particles"
(eg planets around the sun,
stars around a black hole)
in the gravitational field of
the object.

Use Kepler's 3rd law

$$p^2 \propto L^3$$

p = period

L = semimajor axis of the orbit

together with Newton's laws

$$\frac{p^2}{L^3} \sim \frac{1}{M}$$

$J \longrightarrow$ precession of planetary orbits
(see MTW p 452-453)

$\boxed{2.3}$ Gravitational waves

Suppose there are no sources: $\hat{T}_{ab} = 0$

Then EFEqs become: $\Box h_{ab} = 0$
(wave equation)

together with $\partial_b \left(h_a{}^b - \frac{1}{2} \delta_a{}^b h \right) = 0$
(de Donder gauge)

Seek solutions which are unidirectional
plane wave solutions

$$h_{ab} = e_{ab} f(-t+z) \qquad \text{where}$$

$e_{ab} \to$ polarization tensor (10 constants)

Define a 4-vector $K = K^a \partial_a$ with
components $K^a = (1, 0, 0, 1)$

- K is null $\qquad K^a K_a = -1 + 1 = 0$
  with respect to η

- $K^a x^b \eta_{ab} = x^a K_a = -t + z$

<u>Claim:</u> $\Box h_{ab} = 0$, $\Box = \eta^{ab}\partial_a\partial_b$

$$\Box h_{ab} = e_{ab}\,\Box f$$

$$\partial_a f = K_a f'$$
$$\partial_a\partial_b f = K_a K_b f''$$
$$\Box f = (K_a K^a) f'' = 0 \quad \text{as } K \text{ is null}$$

<u>de Donder gauge:</u> $\partial_b H_a{}^b = 0$

$$H_{ab} = h_{ab} - \tfrac{1}{2}\eta_{ab} h = \left(e_{ab} - \tfrac{1}{2}\eta_{ab} e\right) f$$
$$e = \eta^{ab} e_{ab}$$

$$\partial_b H_a{}^b = \left(e_a{}^b - \tfrac{1}{2}\delta_a{}^b e\right) K_b f' = 0$$

Hence $\underline{e_a{}^b K_b = \tfrac{1}{2} K_a e}$

This imposes 4 constraints on e_{ab}.

We can make a further coordinate change to reduce the ambiguity further. Consider

$$x^a \longrightarrow \hat{x}^a = x^a + \epsilon y^a$$

We have

$$h_{ab} = \hat{h}_{ab} + 2\,\partial_{(a} y_{b)}$$

with

$$\Box y^a = 0$$

<u>Choose</u> coordinates st

$$y^a = \lambda^a F(-t+z) \,, \quad \lambda^a \ \ 4 \text{ constants}$$

Clearly $\quad \Box y^a = 0$

$$[\,\Box y^a = \lambda^a \Box F = \lambda^a (K^b K_b) F'' = 0\,]$$

Then

$$h_{ab} = \hat{h}_{ab} + 2\,\partial_{(a} y_{b)} \qquad \text{gives}$$

$$h_{ab} = \hat{h}_{ab} + (\lambda_a K_b + \lambda_b K_a) F' = \hat{h}_{ab} + 2\lambda_{(a} K_{b)} F'$$

so
$$e_{ab} f = \hat{e}_{ab} f + 2\lambda_{(a} K_{b)} F'$$

$\hat{f}(\hat{x}) = f(x)$ as f is a scalar

<u>Choosing</u>

$F' = f$ we obtain

$$e_{ab} = \hat{e}_{ab} + 2\lambda_{(a} K_{b)}$$
$$\text{trace:} \quad e = \hat{e} + 2\lambda_a K^a$$

[One must check that the de Donder gauge ($e_{ab} K^b = \frac{1}{2} K_a e$) is preserved

$$e_{ab} K^b = \hat{e}_{ab} K^b + \lambda_a \cancel{K_b K^b} + K_a \lambda_b K^b$$
$$\frac{1}{2} K_a e = \frac{1}{2} K_a (\hat{e} + 2\lambda_b K^b)$$

$$\Rightarrow \quad 0 = \hat{e}_{ab} K^b - \frac{1}{2} K_a \hat{e} \quad \checkmark \quad]$$

Moreover, we can choose the constants λ st $\hat{e}$ is traceless (<u>traceless gauge</u>) (see problem sheet 3)

That is we can choose λ^a

so that $\hat{e}_{ab}$ only has two independent constants.

In fact, λ^a can be found st

drop hats

$$e_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & B & -A & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

note that $e_{ab} K^b = 0$, $e = 0$

and all ambiguities are fixed.

Finally,

$$\boxed{ds^2 = \eta_{ab}\, dx^{ab} + \epsilon\, f(-t+z)\left(A\, dx^2 + 2B\, dx\, dy - A\, dy^2\right)}$$

Lecture #9

Remarks:

1) h_{ab} describes distortion of space-time geometry in the directions transverse to the direction of propagation of the gravitational wave.

2) there are exact wave-like solutions to the full non-linear EF Eqs.

Plane fronted with parallel rays (pp-waves) See eg S. Carrol

3) there are only 2 degrees of freedom in h_{ab}: why?

$$\# [h_{ab}] = \frac{1}{2} d(d-3) \underset{d=4}{=} 2$$

Curvature

Recall

$$R^d{}_{cab} = \epsilon \eta^{de} \left(\partial_c \partial_{[a} h_{b]e} - \partial_e \partial_{[a} h_{b]c} \right) + \mathcal{O}(\epsilon^2)$$

Then

$$\partial_a h_{bc} = e_{bc} \partial_a f = e_{bc} K_a f'$$

$$\partial_{[a} h_{b]e} = K_{[a} e_{b]e} f'$$

$$\partial_c \partial_{[a} h_{b]e} = K_{[a} e_{b]e} K_c f''$$

$$R^d{}_{cab} = \epsilon \eta^{de} \left(K_{[a} e_{b]e} K_c - K_{[a} e_{b]c} K_e \right) f'' + \cdots$$

<u>oscillating curvature</u> !

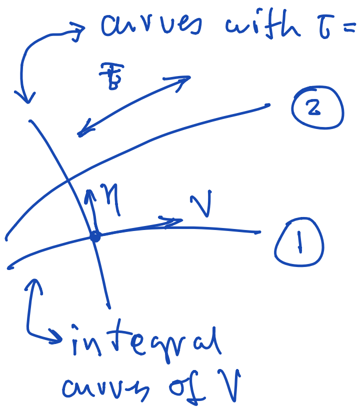
(Note that $R_{ab} = 0$)

Geodesic deviation

(determine the effect of a gravitational wave passing by: consider the influence on the relative motion of nearby particles)

Consider two neighboring geodesics, ie two free falling particles moving in this gravitational field.

The geodesic deviation equation is



$$\nabla_V^2 \eta^a = R^a{}_{bcd} V^b \eta^c V^d$$

where $\nabla_V = V^a \nabla_a$

and $V^a = \dfrac{dx^a}{d\tau} \equiv \dot{x}^a$

η: displacement along curves τ = constant

oscillating curvature produces oscillations in separation between two neighboring particles.

Consider the worldline of 1

$$x^a = (\tau, \underline{0})$$
$$\dot{x}^a = (1, \underline{0}) = V^a$$

$$\nabla_V \eta^a = V^b \nabla_b \eta^a = \dot{x}^b \partial_b \eta^a = \dot{\eta}^a$$

$$\nabla_V^2 \eta^a = \ddot{\eta}^a = R^a{}_{bcd} \dot{x}^b \eta^c \dot{x}^d$$

so $$\ddot{\eta}^a = R^a{}_{0b0} \eta^b$$

$$R_{a0b0} = \epsilon \left(K_{[b} e_{0]a} K_0 - K_{[b} e_{0]0} K_a \right) f''$$
$$= \frac{\epsilon}{2}\left(K_b \cancel{e_{0a}} K_0 - K_0 e_{ba} K_0 - K_b \cancel{e_{00}} K_a + K_0 \cancel{e_{b0}} K_a \right) f''$$

as $e_{00} = 0$ $\quad e_{a0} = e_{0a} = 0$

$$R_{a0b0} = -\frac{\epsilon}{2} e_{ab} f''$$

(recall $K = K^a \partial_b$ has components (1001)
so $K_0 = \eta_{00} K^0 = -1$)

Then

$$\ddot{\eta}^a = R^a{}_{0b0}\,\eta^b = -\frac{\epsilon}{2}\, e^a{}_b\, f''\, \eta^b$$

(tidal acceleration between two particles)

Take for example $\quad h_{ab} = \underbrace{e^{-i\omega(t-z)}}_{f(-t+z)} \epsilon_{ab}$

$$f' = +i\omega f \quad , \quad f'' = -\omega^2 f$$

$$\boxed{\ddot{\eta}^a = +\frac{\epsilon}{2}\, e^a{}_b\, \omega^2 f\, \eta^b}$$

η separation vector

let $\quad \eta^i = (x, y, z) \qquad \eta^0 = t$

$$\ddot{x} = \frac{\epsilon}{2}\omega^2 (Ax + By)\, f$$

$$\ddot{y} = \frac{\epsilon}{2}\omega^2 (Bx - Ay)\, f$$

$$\ddot{z} = 0$$

This is the equation of a 2 dim time-dependent harmonic oscillator and gives rise to oscillating movements of the test particles.

Only separation in transverse directions affected by gravitational waves: <u>transversally polarized waves</u>

For $B = 0$ (+ polarization)

$$\ddot{x} = \frac{\epsilon}{2}\omega^2 A x f \qquad \text{x separation oscillates in x direction}$$

$$\ddot{y} = -\frac{\epsilon}{2}\omega^2 B y f \qquad \text{y separation oscillates in y direction}$$

+ polarization

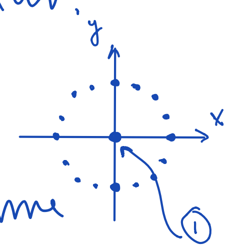
Recalling that the RHS is small, the solution to lowest order is

$$x(t) = x(0)\left(1 - \frac{\epsilon}{2} A f + \cdots\right)$$

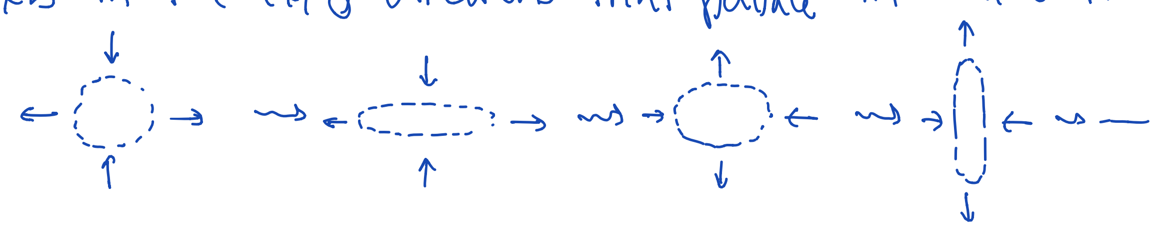
$$y(t) = y(0)\left(1 + \frac{\epsilon}{2} A f + \cdots\right)$$

$$\left(\frac{x(t)}{x(0)}\right)^2 + \left(\frac{y(t)}{y(0)}\right)^2 = 2 + O(\epsilon^2)$$

To see the effects in all directions at once, consider a circular ring of test particles on the (x,y) plane surrounding a single particle in the center.



As the gravitational waves pass through the deform the ring (measured in the reference frame of the central particle) into an ellipse with axes in the (x,y) directions that pulsate in and out.



For $A=0$ (x polarization)

$$\ddot{x} = \frac{\epsilon}{2}\omega^2 B f y$$

$$\ddot{y} = \frac{\epsilon}{2}\omega^2 B f x$$

45° rotation of + polarization

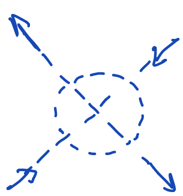
$x \to -x+y$

$y \to x+y$

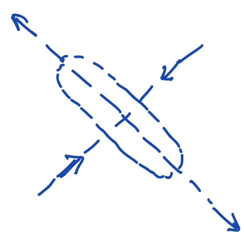
Solution to lowest order

$$x(t) = x(0) - \frac{\epsilon}{2} B f y(0) + \cdots$$

$$y(t) = y(0) - \frac{\epsilon}{2} B f x(0) + \cdots$$



$\sim$



etc.

Circular polarization

$$\epsilon_{R,L} = \frac{1}{\sqrt{2}}(A \pm i B)$$

etc

Solns : characteristic behaviour of quadrupole

Remarks

(1) effect of passing gravitational waves is to slightly perturb relative displacement of freely falling particles.

very weak effect

"equivalent to measuring one hydrogen atom diameter in the distance from Earth to sun"

- nucleus $\sim 10^{-13}$ cm
- atom $\sim 10^{-8}$ cm
- distance Earth-Sun $\sim 1.5 \times 10^{11}$ cm

2 Generation of gravitational waves

Solve time dependent linearized Einstein eqs to study how matter (sources) generate gravitational waves.

That is consider a gravitational field (radiation field) at large distances from the source.

consider retarded soln

Solve $\epsilon \,\Box H_{ab} = -16\pi G T_{ab}$ using Green's functions.

$$\Rightarrow \epsilon H_{ab}(t, \underline{x}) = 4G \int_V \left. \frac{T_{ab}}{|\underline{x} - \underline{x}'|} \right|_{t' = t_R} d^3x'$$

$t_R = t - |\underline{x} - \underline{x}'|$ retarded time

$(t_R, \underline{x}')$ lie on the past light cone whenever $(t - t', \underline{x} - \underline{x}')$ FPN.

One finds (H_{a0} are constant (time indep) $\Rightarrow$ leads to no grav. waves)

$$H_{ij}(t, \underline{x}) = \frac{2G}{r} \frac{d^2 Q_{ij}}{dt^2}$$

where $Q_{ij}(t) = \int_V x_i' x_j' \, T_{00}(t_R, x') \, d^3x'$

and $t_R = t - |\underline{x} - \underline{x}'|$ (retarded time)

Q_{ij} : quadrupole momentum tensor

$$\frac{d^2 Q_{ij}}{dt^2} = 2 \int_V T_{ij} \, d^3x \qquad \text{(problem sheet 2)}$$

quadrupole and higher moments of T_{ab} can radiate gravitationally.

[Compare with electromagnetic radiation produced by oscillating dipoles and hi[illegible]dipoles

$$A_i(t, \underline{x}) = \frac{\mu_0}{4\pi r} \frac{dP_i}{dt}$$

$$P^i(t) = \int_V x^{i\prime} J^0(t, \underline{x}') \, d^3x'$$

dipole moment]

"monopole" ~ M constant
$\Rightarrow$ no radiation

"dipole" ~ dipole radiation due to oscillation of the centre of mass violates conservation of momentum

Quadrupole moment: lowest order contribution to metric which can vary with time.

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③ <u>Ligo & Virgo</u> oral presentation

math simulations of the gravitational wave form pattern from the merge of two black holes or neutron stars

eg $M = 36$ & 27 solar masses

④ History : P. Ferreira's book (The Perfect Theory)