Lecture #10

Chapter 3: The Schwarzschild black hole

3.1 The Schwarzschild metric (1916)

Recall (GR 1)

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\,d\Omega^2$$

where

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

round metric on the 2-sphere

coordinates:

$$r > 2M$$

$$0 \leq \theta < \pi \quad , \quad 0 \leq \phi < 2\pi$$

This is the unique solution of EFEqs for the exterior gravitational field of a <u>static spherically symmetric</u> body st as $r \to \infty$ $g \to \eta$

ie g is asymptotically flat.

[Note: exact solns of $R_{ab}=0$ are <u>very</u> hard to find in general]

* <u>static</u>: there is a HSO, TL-Killing vector

That is

- there is a Killing vector $K = \partial_t$ which is TL ($g(K,K) < 0$)

  $\rightsquigarrow$ g_{ab} is independent of t

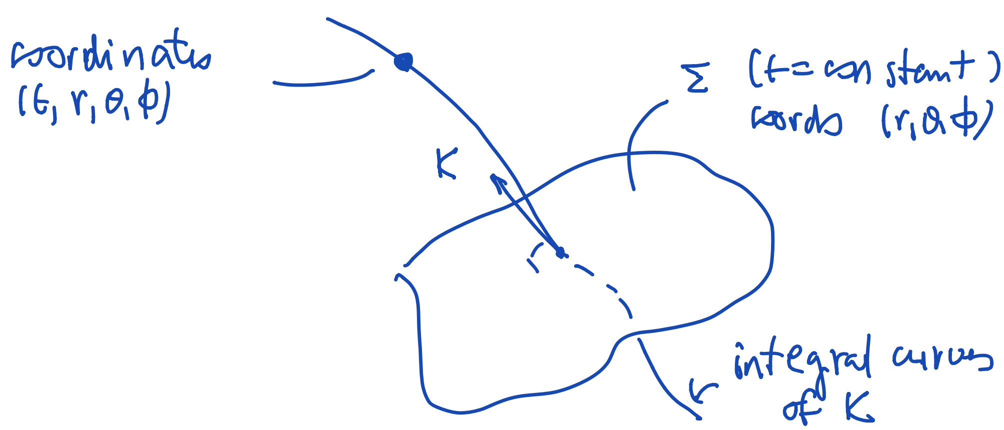
  (KV ~ time translation symmetry)

  $\rightsquigarrow$ <u>stationary</u> solution

- K is HSO: there are hypersurfaces Σ (t = constant surfaces) which are orthogonal to the integral curves of K

  $\rightsquigarrow$ metric has no g_{0i} terms

$\therefore$ spacetime is a family of 3-surfaces (space) and K is perpendicular to these



* spherically symmetric

g_{ab} is invariant under 3-space rotations

Equivalently: there are 3 Killing vectors K_i with algebra

$$[K_i, K_j] = \epsilon_{ijk} K_k$$

ie the isometry group contains $SO(3)$ and the orbits are 2-spheres

Hypersurfaces Σ (t = constant) with coords (r, θ, ϕ) have surfaces r = constant which are 2-spheres.

1923: Birkhoff's theorem (see Carroll)

The Schwarzschild metric is the unique spherically symmetric solution to vacuum EFEqs.

Spherically symmetric solns of vacuum eqs must be static and asymptotically flat.

What is M?

Analize SM far away from the body

Rewrite SM in isotropic coordinates:

$$ds^2 = -A(\rho)^2 dt^2 + B(\rho)^2 \underbrace{(d\rho^2 + \rho^2 d\Omega^2)}_{dx^2 + dy^2 + dz^2}$$

with $\quad \rho = \rho(r) \qquad \rho^2 = x^2 + y^2 + z^2$

$x = \rho \sin\theta \cos\phi$ etc

Then $\quad A(\rho)^2 = 1 - \frac{2M}{r}\quad , \quad B(\rho)^2 \rho^2 = r^2$

and $\quad B(\rho)^2 \left(\frac{d\rho}{dr}\right)^2 = \left(1 - \frac{2M}{r}\right)^{-1}$

exercise $\Longrightarrow \quad r = \rho\left(1 + \frac{M}{2\rho}\right)^2 \quad , \quad B = \left(1 + \frac{M}{2\rho}\right)^2$

and $\quad A = \dfrac{1 - M/2\rho}{1 + M/2\rho}$

[Exercise:

$$B(\rho)^2 = \left(\frac{r}{\rho}\right)^2 \quad \text{and} \quad B(\rho)^2 \left(\frac{d\rho}{dr}\right)^2 = \left(1-\frac{2M}{r}\right)^{-1}$$

$$\Rightarrow \left(\frac{r}{\rho}\right)\left(\frac{d\rho}{dr}\right) = \pm\left(1-\frac{2M}{r}\right)^{-1/2}$$

$$\Rightarrow \log\rho = \pm\int \frac{1}{r}\left(1-\frac{2M}{r}\right)^{-1/2} dr$$

$$= \pm \log\left(-2M + 2r + 2r\sqrt{1-\frac{2M}{r}}\right) + \text{const.}$$

$$\rho = 2c\left(r - M + r\sqrt{1-\frac{2M}{r}}\right)^{\pm 1}$$

Take $+$ sign (as $\rho \to \infty$ $r \to \infty$)

$$\rho = 2cr\left(1 - \frac{M}{r} + \sqrt{1-\frac{2M}{r}}\right)$$

Solve for $r(\rho)$:

$$\left(\frac{\rho}{2cr} - 1 + \frac{M}{r}\right)^2 = 1 - \frac{2M}{r}$$

(6)

$$r(r-2M) = \left(\frac{\rho}{2c} - r + M\right)^2$$

$$r(\cancel{r}-\cancel{2M}) = \left(\frac{\rho}{2c}+M\right)^2 - 2r\left(\frac{\rho}{2c}+\cancel{M}\right) + \cancel{r^2}$$

$$\left(\frac{\rho}{2c}+M\right)^2 = \frac{1}{c}\rho r$$

$$r = \frac{c}{\rho}\left(\frac{\rho}{2c}+M\right)^2 = \frac{\rho}{4c}\left(1+\frac{2Mc}{\rho}\right)^2$$

$$B = \frac{r}{\rho} = \frac{1}{4c}\left(1+\frac{2Mc}{\rho}\right)^2$$

$c = \frac{1}{4}$ so $B \to 1$ as $\rho \to \infty$

$$\underline{r = \rho\left(1+\frac{M}{2\rho}\right)^2}, \quad \underline{B = \left(1+\frac{M}{2\rho}\right)^2}$$

$$A^2 = 1 - \frac{2M}{\rho}\cdot\left(1+\frac{M}{2\rho}\right)^{-2} = \left(1+\frac{M}{2\rho}\right)^{-2}\left(1+\frac{M}{\rho}+\frac{M^2}{4\rho^2}-\frac{2M}{\rho}\right)$$

$$= \left(1+\frac{M}{2\rho}\right)^{-2}\left(1-\frac{M}{2\rho}\right)^2$$

$$\underline{A = \frac{1-M/2\rho}{1+M/2\rho}}$$

end of exercise //

Then

$$ds^2 = -\left(\frac{1-M/2\rho}{1+M/2\rho}\right)^2 dt^2$$
$$+\left(1+\frac{M}{2\rho}\right)^4 (dx^2+dy^2+dz^2)$$

As $\rho \to \infty$ $(r \to \infty)$

$$ds^2 = -\left(1-\frac{M}{\rho}+\cdots\right)\left(1-\frac{M}{\rho}+\cdots\right)dt^2$$
$$+\left(1+4\cdot\frac{M}{2\rho}+\cdots\right) d\underline{x}\cdot d\underline{x}$$
$$= -\left(1-\frac{2M}{\rho}+\cdots\right)dt^2+\left(1+\frac{2M}{\rho}+\cdots\right)d\underline{x}\cdot d\underline{x}$$

Comparing with metric in section 2.2.

M = total mass

$\underline{J} = 0$

Questions:

1) Want to extend the metric to
$$0 < r \leq 2M$$

2) What happens at $r=0$ & $r=2M$
   - $r=2M$ coordinate singularity
   - $r=0$ physical singularity

     ie metric is singular in any coordinate system

   This can be seen by looking at curvature invariants
$$R_{abcd} R^{abcd} = \frac{48 M^2 G^2}{r^6}$$
   (cannot use $g^{ab} R_{ab}$ as this vanishes)

   ie $r=0$ is a genuine curvature singularity as this invariant diverges.

   Note that this quantity is <u>finite</u> <u>at $r=2M$</u> (tidal forces finite at $r=2M$)

Need better understanding

→ construct a maximal extension of the Schwarzschild solution

→ event horizon at $r = 2M$

→ Penrose diagrams

3.2 Radial null geodesics (GRI)

Let $L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b$ Lagrangian for affinely parametrized geodesics

Then

$$L = \frac{1}{2}\left(-F\dot{t}^2 + F^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right)$$

where $F = 1 - \frac{2M}{r}$

L is constant along geodesics (GRI)

Choose $2L = 1, \quad 0, \quad -1$
SL N TL

Could use Lagrangian to study geodesics

geodesic eq $\Longleftrightarrow$ Euler Lagrange eq

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial x^a} = 0$$

second order differential equation

Instead: it is better to exploit symmetries

Recall: If K is a Killing vector then $Q = K_a \dot{x}^a$ is <u>constant along geodesics</u>

This gives first order differential eqs ("first integrals")

We have 4 Killing vectors. The corresponding conserved quantities are

- $K = \partial_t$ symmetry: translation in time

  L is independent of t

  conserved quantity: <u>energy</u>

$$-E = K_0 \dot{X}^0 = g_{00}\,\dot{t} = -F\dot{t}, \quad \dot{t} = \frac{dt}{ds}$$

- $J_i = -\epsilon_{ijk} x^j \partial_k$ spherical symmetry (space rotations)

  Conserved quantities

$$Q_i = (J_i)_a \dot{x}^a = g_{\theta\theta}\,\dot{\theta}\,J_i^\theta + g_{\phi\phi}\,\dot{\phi}\,J_i^\phi$$

$$= 0 \quad \text{for } \underline{\text{radial geodesics}} \quad \dot{\theta} = \dot{\phi} = 0$$

Note: one can prove that one can always choose motion in the equatorial plane (using spherical symmetry) so $\theta = \pi/2$

Then the equations for radial null geodesics are

$$E = F\dot{t} \qquad (1)$$

$$2L = 0 = -F\dot{t}^2 + F^{-1}\dot{r}^2 \qquad (2)$$

(1) into (2) to eliminate $\dot{t}$

$$0 = -F\left(\frac{E}{F}\right)^2 + F^{-1}\dot{r}^2 \Rightarrow \underline{\dot{r}^2 = E^2}$$

$$\Rightarrow \quad r = r_0 \pm Es \qquad r_0 > 2M$$

$+$ outgoing geodesics
$-$ incoming geodesics

so r : affine parameter

(1) $\quad \dot{t} = \dfrac{dt}{dr}\dot{r} = EF^{-1}$

$$\xRightarrow{\dot{r} = \pm E} \quad \frac{dt}{dr} = \pm F^{-1}$$

$$\Rightarrow \quad t = \text{constant} \pm r_*$$

where $\quad r_* = r + 2M \log\left|\dfrac{r}{2M} - 1\right|$

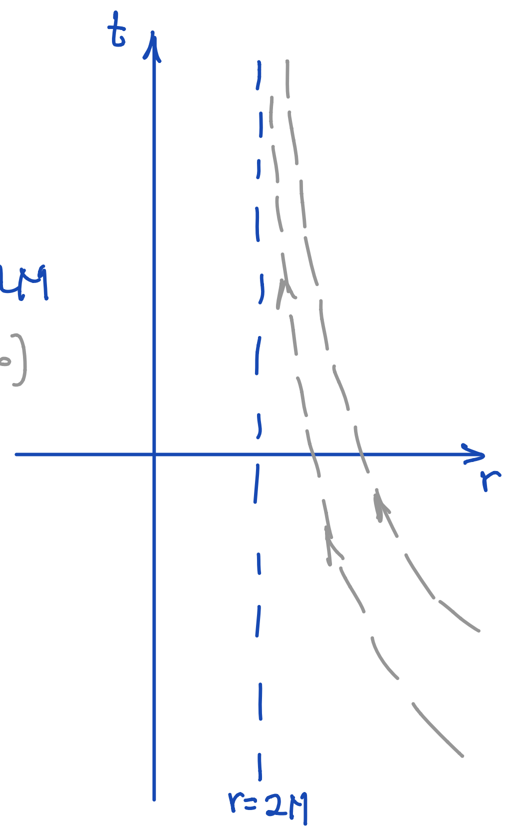
ie $\quad t \mp r_* = \text{const} \quad \begin{array}{l} - \text{ out} \\ + \text{ in} \end{array}$

Incoming geodesics

(1) $r = r_0 - Es$ $\qquad r_0 > 2M$ $\qquad (r = \infty$ for $s = -\infty)$

past

$t + r_* = \text{constant} = v_0$



s st $r_0 - 2M = Es$

(1) at finite value of s (into the future) r reaches $2M$

(2) but as $r \to 2M$ $\quad (r_* \to -\infty)$

$t \to \infty$

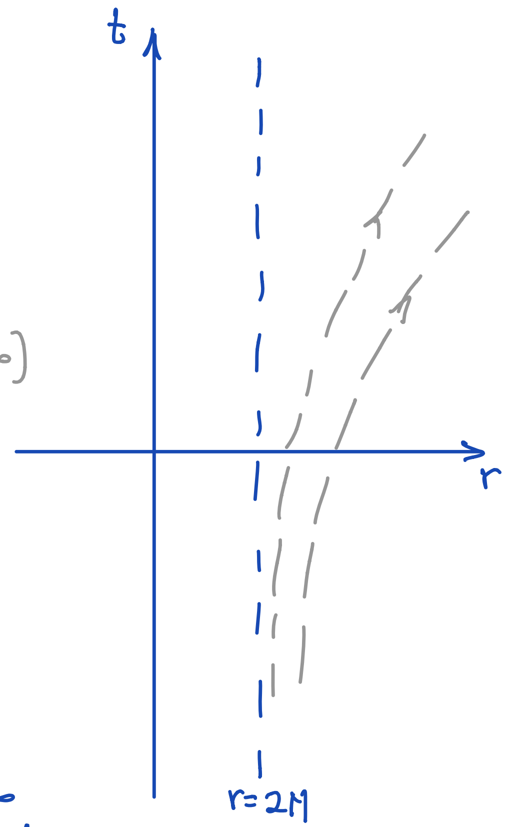
where does the geodesic go?

geodesics run off the coordinate patch at $t = \infty$ at a finite s into the future.

Outgoing geodesics

① $r = r_0 + Es$ $r_0 > 2M$ ($r = \infty$ for $s = \infty$)
future

$t - r_* = \text{constant} = u_0$



In this case geodesics "leave" $r=2M$ at finite s in the past
(s st $-Es = r_0 - 2M$)

② but as $r \to 2M$ ($r_* \to -\infty$)
$t \to -\infty$

where was the geodesic before?

geodesics run off the coordinate patch at $t = -\infty$ at a finite affine parameter s into the past.

3.3 Extension of the SM: The Eddington-Finkelstein coordinates

Problem: geodesics cannot be extended to arbitrarily large values of the affine parameter so we have a geodesically incomplete space-time

New coordinate system: Eddington-Finkelstein

let $\mathcal{U}_1$ be the manifold for $r>2M$ with SM

and note that there is a disconnected manifold for $0<r<2M$ with the same metric.

Question: can we extend $\mathcal{U}_1$ with the same metric?

That is: Is there a "larger" manifold (space-time) $M \supset \mathcal{U}_1$ and metric g on M which coincides with SM on $\mathcal{U}_1$.

Recall: $$r_* = r + 2M \log\left|\frac{r}{2M}-1\right|$$

let

$r>2M$

$v = t + r_*$ constant along incoming null geodesics

$u = t - r_*$ constant along outgoing null geodesics

Then
$$dv = dt + F^{-1}dr$$
$$du = dt - F^{-1}dr$$

Incoming null geodesics:

coordinates (v, r, θ, ϕ)

Eliminating t in favor of v

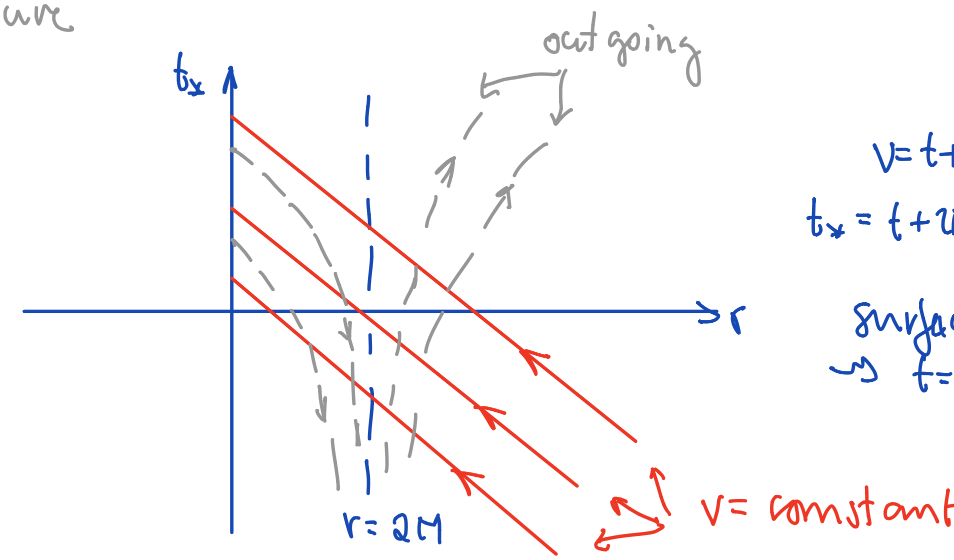
$$ds^2 = -F\,dv^2 + 2\,dv\,dr + r^2\,d\Omega^2$$

at $r=2M$ $F=0$ however $\det g \neq 0$ & g^{-1} at $r=2M$ is regular

- non singular at $r=2M$ and extends to a larger manifold

extend $\mathcal{M}_1$ towards the future

$\mathcal{M}_2 \supset \mathcal{M}_1$ for which $0<r<\infty$, $-\infty<v<\infty$



$v = t + r_* = t_* + r$

$t_* = t + 2M \log\left|\frac{r}{2M} - 1\right|$

surface $r=2M$ $\rightsquigarrow$ $t=\infty$, $u=+\infty$

- asymmetry between incoming and outgoing geodesics
- $\underline{r=0}$ is in the future of incoming null geodesics
- at $\underline{r=2M}$
  - $\rightarrow$ FP null geodesics go through $r=2M$
  - $\rightarrow$ but not past pointing (out-going)

  (no signal from an event inside $(r<2M)$ can escape)

Outgoing geodesics:

coordinates (u, r, θ, ϕ)

$$ds^2 = -F\,du^2 - 2\,du\,dr + r^2\,d\Omega^2$$

- non singular at $r = 2M$ and extends to a larger manifold

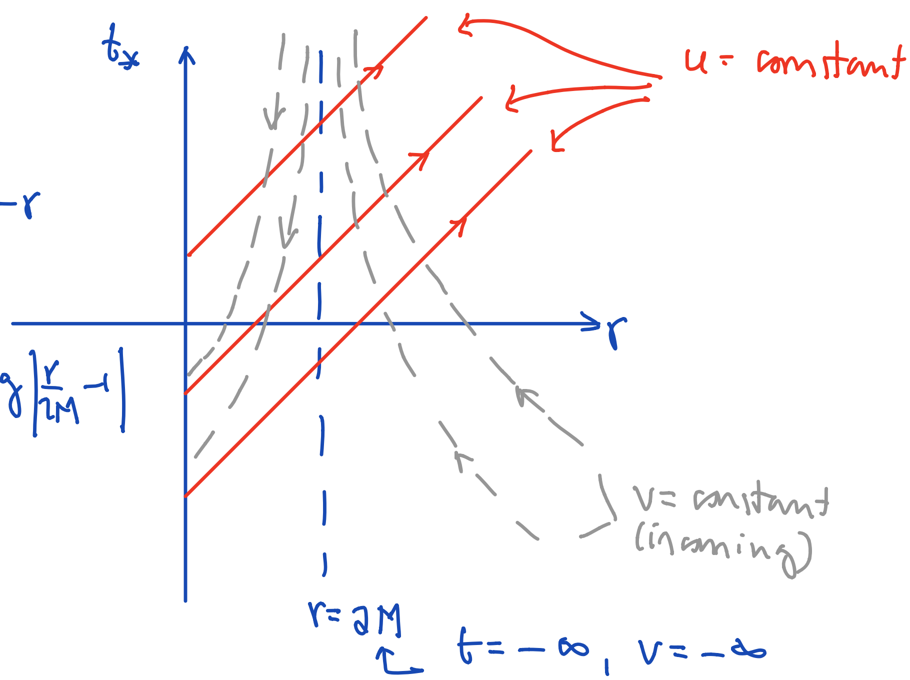
extend $\mathcal{M}_1$ into the past

$\mathcal{M}_3 \supset \mathcal{M}_1$ for which $0 < r < \infty$, $-\infty < u < \infty$

$u = t - r_* = t_* - r$

$v = u + 2r_*$

$t_* = t - 2M \log\left|\frac{r}{2M} - 1\right|$



- asymmetry between in and out geodesics
- $r = 0$ in the past of outgoing null geodesics
- at $r = 2M$

  $\rightarrow$ only outcoming (PP null) geodesics go through ie escape $r = 2M$

So we have a problem:

at $\underline{r = 2M}$ $<$
- $u = +\infty$ incoming direction: only FP-N geodesics go through
- $v = -\infty$ outgoing direction: only PP-N geodesics go through

at $\underline{r = 0}$ $<$
- in future of incoming N geodesics
- in past of outgoing N geodesics

Include **both** extensions $\mathcal{U}_2$ & $\mathcal{U}_3$ in one simultaneously

3.4 The Kruskal-Szekeres spacetime

Kruskal + Szekeres coordinates (1960)

Write metric in terms of (v, u, θ, ϕ)
(eliminate t & r in favor of v, u)

Then

$$ds^2 = -F\, du\, dv + r^2 d\Omega^2$$

where

$$u = t - r_* \qquad v = t + r_*$$

and

$$r_* = r + 2M \log\left|\frac{r}{2M} - 1\right|$$

so

$$t = \frac{1}{2}(u + v)$$

$$r_* = \frac{1}{2}(u - v)$$

gives r in terms of (u, v)

u constant along null out-geodesics

v constant along null in-geodesics

let $\boxed{\mathcal{U} = -e^{-u/4M}, \quad \mathcal{V} = e^{v/4M}}$ $\quad r > 2M$

Then

$$\mathcal{U}\mathcal{V} = -e^{(v-u)/4M} = -e^{-\frac{1}{2M}r_*}$$

defines r implicitly in terms of $\mathcal{U}\mathcal{V}$

$$= -\frac{1}{2M} e^{\frac{r}{2M}} (r-2M) = -\frac{r e^{\frac{r}{2M}}}{2M} F$$

$$\mathcal{U}^{-1}\mathcal{V} = -e^{(u+v)/4M} = -e^{t/2M}$$

Remarks:

* $r = 2M \iff \mathcal{U}$ or $\mathcal{V}$ vanishes
* $\mathcal{U} < 0, \mathcal{V} > 0$ Schwarzschild spacetime

Metric:

$$d\mathcal{U} = -\frac{1}{4M}\mathcal{U}\,du, \quad d\mathcal{V} = \frac{1}{4M}\mathcal{V}\,dv$$

$$F\,du\,dv = F(-16M^2)\frac{1}{\mathcal{U}\mathcal{V}}\,d\mathcal{U}\,d\mathcal{V}$$

$$= -16M^2 \cancel{F}\left(-\frac{2M}{r} e^{-r/2M} \cancel{F^{-1}}\right) d\mathcal{U}\,d\mathcal{V}$$

$$= 32M^3 \frac{e^{-r/2M}}{r}\,d\mathcal{U}\,d\mathcal{V}$$

$$\boxed{ds^2 = -32 M^3 \frac{e^{-r/2M}}{r} dU\, dV + r^2 d\Omega^2 \quad \text{(KSM)}}$$

regular at $r = 2M$ (U or V vanishes)
singular at $r = 0$ ($UV = 1$)

Extend Schwarzschild spacetime
to a manifold M

analytic continuation

$-\infty < V < \infty$, $-\infty < U < \infty$

with the KSM

(U, V, θ, ϕ) are called Kruskal-Szekeres coordinates

[maximal unique extension
↳ simply connected,
TL geodesically complete]

U (and u) is const along N out-geodesics
V (and v) is const along N in-geodesics

let

$$T = \frac{1}{2}(\mathcal{U}+\mathcal{V}), \quad X = \frac{1}{2}(\mathcal{V}-\mathcal{U})$$

$\underline{r>2M}$

$$T = \frac{1}{2}\left(-e^{-u/4M} + e^{v/4M}\right)$$

$$= \frac{1}{2}\left(-e^{-t/4M} + e^{t/4M}\right) e^{r_*/4M}$$

$$= \cosh\left(\frac{t}{4M}\right) e^{r/4M} \left(\frac{r}{2M}-1\right)^{1/2}$$

$$X = \sinh\left(\frac{t}{4M}\right) e^{r/4M} \left(\frac{r}{2M}-1\right)^{1/2}$$

The metric is well defined for $0<r<2M$ but coordinates become imaginary.

For $r<2M$ we have (see MTW §3.14, §3.15)

$$T = \sinh\left(\frac{t}{4M}\right) e^{r/4M} \left(1-\frac{r}{2M}\right)^{1/2}$$

$$X = \cosh\left(\frac{t}{4M}\right) e^{r/4M} \left(1-\frac{r}{2M}\right)^{1/2}$$

However $\underline{\text{for any}}$ $0<r<\infty$

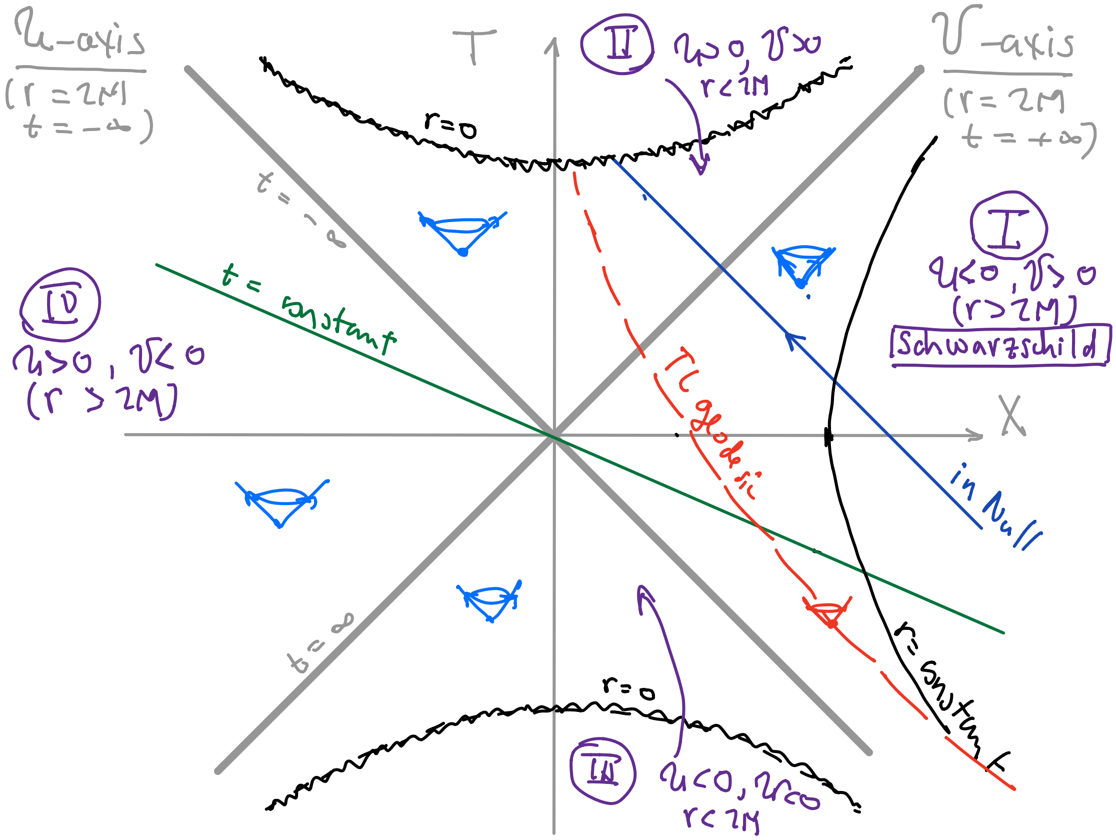
$$\mathcal{U}\mathcal{V} = -\frac{1}{2M}(r-2M)\, e^{r/2M}$$

$$\mathcal{U}^{-1}\mathcal{V} = -e^{t/2M}$$

lecture #12

23

Kruskal-Szekeres spacetime: causal structure of (max) extended Schwarzschild metric
(each point on the plane → 2-sphere)



* u = constant along N-out geodesics
  v = constant along N-in geodesics

* Plot u, v axes at 45°: these correspond to $r=2M$
* Surfaces r = constant
  hyperbolas uv = constant (asymptotic to $u=0$ & $v=0$)
  In particular: $r=0$ is $uv=1$ (curvature singularity)

* surfaces $t =$ constant
  lines $u v^{-1} =$ constant through origin

* null geodesics
  lines $\begin{cases} u = \text{constant} & \text{(incoming)} \\ v = \text{constant} & \text{(outgoing)} \end{cases}$

Regions: there are four regions of KS-spacetime depending on the signs of u, v

$\boxed{\text{boundaries } r = 2M}$

Regions I & II :

I $\quad u<0 \quad v>0 : \quad r>2M$

is Schwarzschild spacetime

* A radially infalling observer or light signal will cross line
$$u=0 \qquad (r=2M)$$ at finite proper time
and enter region II.
Once there, within a finite proper time, it will fall into the singularity at $r=0$.
* any light signals sent from II will remain in II and will fall into the singularity (consider light cones!)

We call the surface $r=2M$ an $\boxed{\text{event horizon}}$ and region II a $\boxed{\text{black hole}}$ (BH).

Remark: I $\cup$ II $= \mathcal{U}_2$

incoming Eddington-Finkelstein space time

Region I & III

$$\text{I} \cup \text{III} = \mathcal{U}_3$$

outgoing Eddington–Finkelstein space time

Any signal in III must have originated in the singularity $r=0$ and within a finite time must leave III

We call region III a white hole

↖ hypotetical !

Region IV   this is new

$u > 0$, $v < 0$ $r > 2M$

This region is isometric to Schwarzschild
(In fact, $(u, v) \to (-u, -v)$
is an isometry of the Kruskal-Szekers metric)
and it is another asymptotically flat region of space-time outside the surface $r = 2M$

Geodesics crossing from I to IV must be spacelike.
Any light signals sent from I will end up at $r=0$ so an observer in I cannot communicate with one in IV.

Consider the hypersurfaces Σ_t with

$$t = \text{constant}.$$

These are lines through the origin

$$u v^{-1} = \text{constant}.$$

Recall Schwarzschild metric in isotropic coordinates $\underline{(t, \rho, \theta, \phi)}$

$$ds^2 = -\frac{1-M/2\rho}{1+M/2\rho}\, dt^2 + \left(1+\frac{M}{2\rho}\right)^4 \underbrace{(d\rho^2 + \rho^2 d\Omega^2)}_{\substack{dx^2+dy^2+dz^2 \\ \text{flat metric on } \mathbb{R}^3}}$$

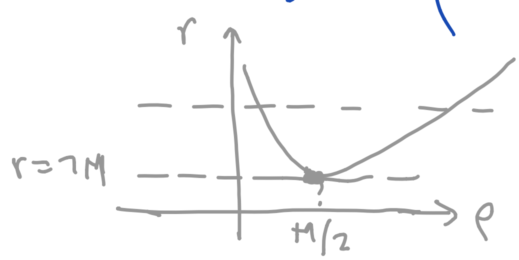
where

$$r = \rho\left(1+\frac{M}{2\rho}\right)^2 \qquad (\rho^2 = x^2+y^2+z^2)$$

The metric on $t = \text{constant}$ hypersurfaces Σ_t is <u>conformally flat</u>

$$ds^2_{\Sigma_t} = \left(1+\frac{M}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\Omega^2)$$

- These coordinates cover I $\cup$ IV as $r > 2M$
- Note that there are two values of ρ for each $r > 2M$



In these coordinates, the map

$$\rho \longrightarrow \hat{\rho} = \frac{M^2}{4\rho}$$

is an isometry which corresponds to

$$(u, v) \longrightarrow (-u, -v)$$

(of the KS spacetime), and which leaves "fixed"

$$\rho = \frac{M}{2}$$

This is a surface of radius <u>$r = 2M$</u>

Consider now the geometry of Σ_t <u>near</u> $r = 2M$ $(\rho = M/2)$ on both sides, that is, consider the geometry of spacetime as we approach the origin $u = v = 0$

We have:

$$ds^2_{\Sigma_t} = \underbrace{\left(1 + \frac{M}{2\rho}\right)^4}_{1 + \frac{2M}{\rho} + \cdots} \left(d\rho^2 + \rho^2 d\Omega^2\right)$$

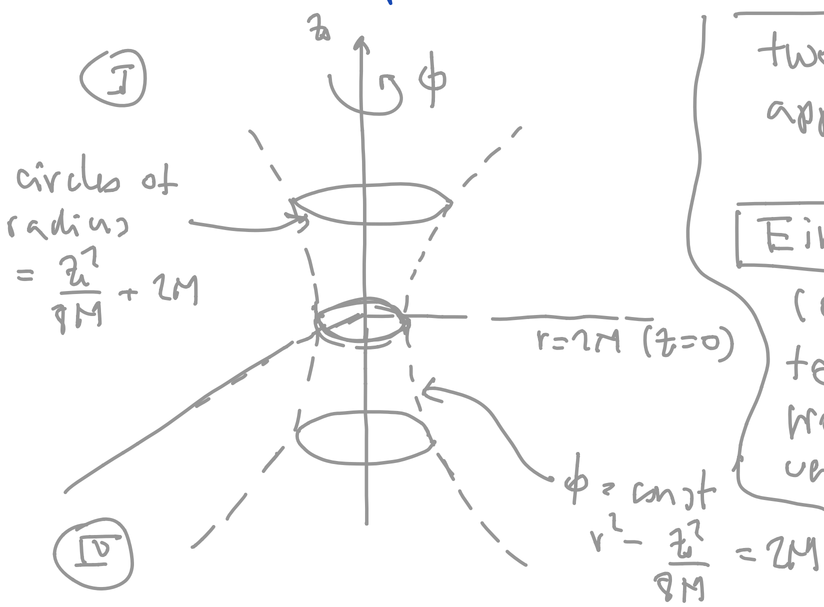
Let $\theta = \pi/2$ to draw a picture of the equatorial plane.

One can prove that this is the metric on the surface

$$x^2 + y^2 = \left(\frac{z^2}{8M} + 2M\right)^2 \quad \text{in } \mathbb{E}^3$$

where $x = r\cos\phi$, $y = r\sin\phi$ and

$$z = \sqrt{8M(r - 2M)}$$



two identical spin appended at $r = 2M$ $(u = v = 0)$

$\boxed{\text{Einstein-Rosen bridge}}$

(or "worm hole")

to cross the wormhole from I to IV (or vice-versa) the trajectory has to be spacelike

3.5 Killing vectors, null-hypersurfaces and event horizons

Killing vectors

Recall: (SM is static)

$K = \partial_t$ is a HSO, TL-Killing vector

In terms of u & v:

$$K = \partial_t u\, \partial_u + \partial_t v\, \partial_v = \frac{1}{4M}\left(-u\,\partial_u + v\,\partial_v\right]$$

and

$$g(K,K) = 2\, g_{uv}\, K^u K^v = +\frac{2M}{r} e^{-r/2M} uv$$

$$= -\left[1 - \frac{2M}{r}\right]$$

- K is TL when $uv < 0$ ie $r > 2M$ ie I & IV (FPTL geodesic in I & PPTL geodesic in IV)
- K is Null when $uv = 0$ ie $r = 2M$
  (ie $u=0$ or $v=0$)
- K is SL when $uv > 0$ ie $r < 2M$ ie II & III

! The KSM is not static in these regions

Is there a TL-Killing vector in these regions?

Null hypersurfaces

Let $\varphi(x)$ be a smooth function on spacetime. Consider a family of hypersurfaces Σ defined by

$$\varphi(x) = \text{constant}$$

Vector fields normal to Σ are: $n_a = \psi \, \partial_a \varphi$

or $n = \psi \, g^{ab} (\partial_a \varphi) \partial_b$

Definition: We say that Σ is

spacelike	iff	$g(n,n) > 0$
null	iff	$g(n,n) = 0$
timelike	iff	$g(n,n) < 0$

For the KS-spacetime, consider the hypersurfaces

$$r = \text{constant}$$

$$n = \psi \, g^{ab} (\partial_a r) \partial_b = \psi \, F \, \partial_r$$

$$= \psi F \left(\partial_r u \, \partial_u + \partial_r v \, \partial_v \right) = \frac{\psi}{4M} \left(u \, \partial_u + v \, \partial_v \right)$$

$$g(n,n) = g_{ab} n^a n^b = g_{rr} \psi^2 F^2 = \psi^2 F$$

$$= - \psi^2 \frac{2M}{r} e^{-r/2M} u v$$

So <u>r = constant hypersurfaces</u> are

* spacelike when $r > 2M$ (I & IV)
* null when $r = 2M$ $u = 0$ or $v = 0$ (on <u>event horizons</u>)
* timelike when $r < 2M$ (II & III)

n is HSO but not Killing, so these regions are dynamical

Let Σ be a <u>null hypersurface</u> with normal

$$n = \psi\, g^{ab} (\partial_b \varphi)\, \partial_a$$

A vector V tangent to Σ satisfies

$$g(n, V) = 0$$

Hence: <u>n is itself a tangent vector to Σ</u>

(Lorentzian signature!)

So, for some null curve $x^a(s)$ in Σ

$$n^a = \frac{dx^a}{ds}$$

ie integral curves of n lie in Σ

Moreover: The integral curves of n are (null) geodesics

Proof:

$$\nabla_n n_b = n^a \nabla_a n_b = n^a \nabla_a (\psi \partial_b \varphi)$$

$$= n^a ((\nabla_a \psi) \partial_b \varphi + \psi \nabla_a \partial_b \varphi)$$

$$= (\psi^{-1} \nabla_a \psi) n^a n_b + \psi n^a \nabla_b \partial_a \varphi)$$

$$= (\psi^{-1} \nabla_n \psi) n_b + \psi n^a \nabla_b (\psi^{-1} n_a)$$

$$= (\psi^{-1} \nabla_n \psi) n_b + (\psi \nabla_b \psi^{-1}) n^a n_a + \frac{1}{2} \nabla_b (n^a n_a)$$

As $g(n, n) = 0$ on Σ $(\nabla_b n^2 \neq 0 \;!)$

$$\nabla_n n_b = (\psi^{-1} \nabla_n \psi) n_b + \frac{1}{2} \nabla_b (n^a n_a) \quad \underline{\text{on } \Sigma}$$

Now: $\nabla_b (n^a n_a) = f n_b$ for some f $\underline{\text{on } \Sigma}$

(as $n^a n_a$ is constant on Σ, $\nabla_b (n^a n_a)$ must be normal to Σ)

Hence: $$\nabla_n n_b = \left(\psi^{-1} \nabla_n \psi + \frac{1}{2} f\right) n_b$$

(note that geodesics are not necessarily affinely parametrized) //

One can, of course, find affinely parametrized geodesics, ie one can find a function $\tilde{\Psi}$ st $\tilde{n} = \tilde{\Psi} n$ satisfies $\tilde{n}^a \nabla_a \tilde{n}_b = 0$.

Definition: null affinely parametrized geodesics are called the generators of the null hypersurface Σ.

For the KS-spacetime:

the event horizon $u = 0$ $(r = 2M)$ has normal

$$n = \frac{1}{4M} \psi \, v \, \partial_v = \psi \, \partial_v$$

It is not too hard to prove that

$$\tilde{n} = \partial_v$$

are affinely parametrized null geodesics (and the affine parameter is V) and hence are the generators of the event horizon

Definition: A null hypersurface Σ is a Killing horizon of a Killing vector field K if K is normal to Σ on Σ.

Suppose n is normal to Σ and that the corresponding geodesics are affinely parametrized

ie $$n^a \nabla_a n_b = 0$$

A KV normal to Σ has the form $K = fn$ on Σ

for some function f.

Then $$K^a \nabla_a K^b = \kappa K^b$$

$$K^a \nabla_a K^b = K^a \nabla_a (f n^b) = K^a((\nabla_a f) n^b + \cancel{f \nabla_a n^b})$$
$$= (f^{-1} \nabla_K f) K^b$$

so $$\kappa = f^{-1} \nabla_K f$$

Definition: κ is called the hypersurface gravity

For the KS-spacetime: $u = 0$ $(r = 2M)$ is a Killing horizon of the Killing vector field $K = \partial_t$. We can compute
$$\kappa = 1/4M$$

Remarks:

1. k is constant along integral curves of K, and hence it is constant on the null hypersurface Σ
2. k is interpreted as the force that an observer at "infinity" needs to apply to a unit-mass test particle to stay at the horizon.