Chapter 4 Penrose Carter diagrams

(1964–1965)

(see for example MTW §34.1)

The Penrose-Carter diagrams are spacetime diagrams which are useful for

1. study the asymptotic properties of spacetime and fields (eg g, R, F_{ab}, ...) near "infinity" (not enough to study spatial infinity also need to discuss future infinity, etc.)
   Key: find coordinates which "bring in" infinity into a finite distance

For asymptotically flat spacetimes we have

$i^{\pm}$ F(P)-TL infinity: $t \to \pm\infty$, finite radius r
(TL curves extend towards this region)

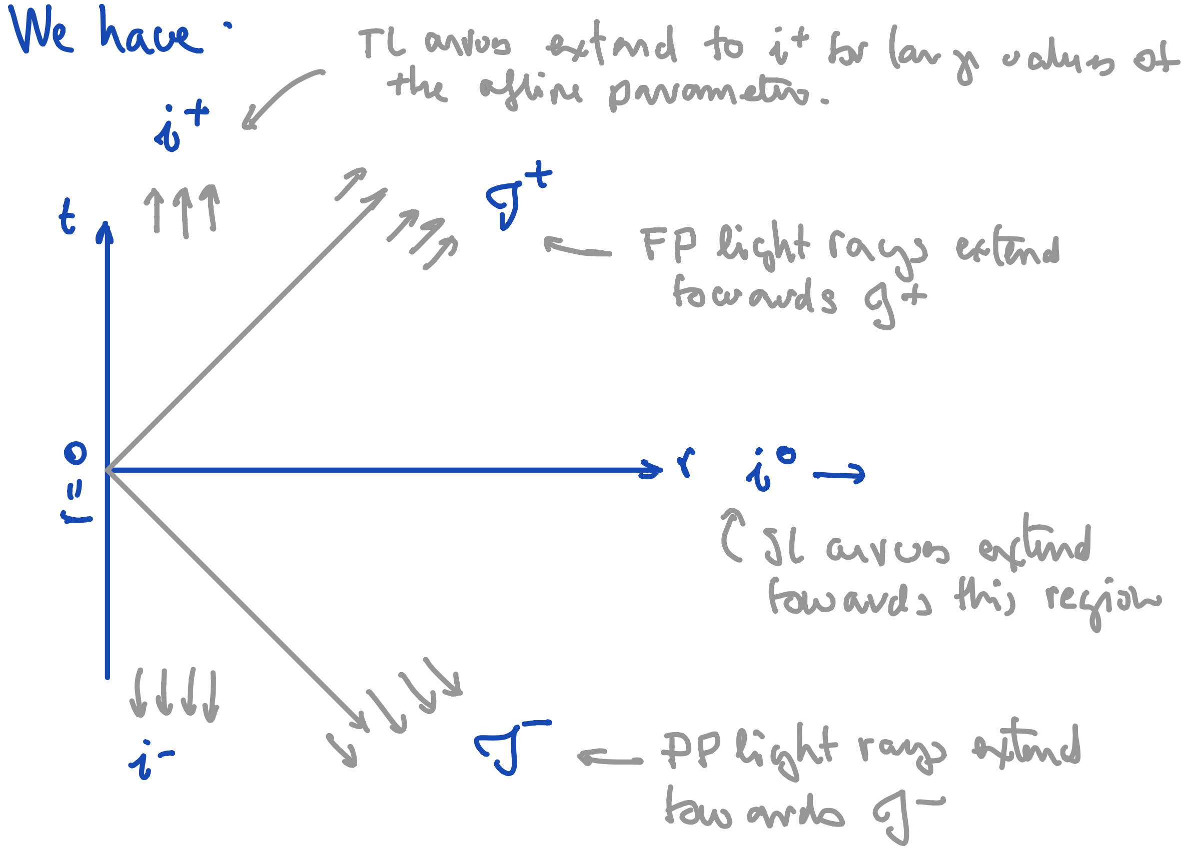
i^{0} SL infinity: $r \to \infty$, finite t
(SL curves extend towards this region)

$\mathcal{J}^{\pm}$ F(P) - Null infinity: $t \pm r \to \pm\infty$
$t \mp r$ finite
(+ outgoing Null curves extend towards $\mathcal{J}^{+}$
– ingoing " " " " $\mathcal{J}^{-}$)

For example: consider flat spacetime with coordinates (t, r, θ, ϕ) and metric

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

We have:



2 study the causal structure of space-time

key: conformal transformations

<u>Definition</u>: A <u>conformal transformation</u> is a map

$$(M, g) \longrightarrow (M, \tilde{g})$$

such that $$g_{ab}(x) = \Lambda^2(x)\, g_{ab}(x)$$

where $\Lambda(x) \neq 0 \ \forall x$, $\Lambda(x)$ smooth.

<u>Key fact</u>: conformal transformations <u>do not</u> change the causal structure of space-time. In fact, if V is a vector on M, then

$$\tilde{g}(V,V) = \Lambda^2 g(V,V)$$

As $\Lambda^2 > 0$, if V is TL-N-SL with respect to g, then it is TL-N-SL (respectively) with respect to $\tilde{g}$.

Note however that geodesics of g are not necessarily geodesics of $\tilde{g}$ <u>unless</u> they are null (exercise)

Solution to the exercise:

Consider an affinely parametrized geodesic in (M,g) with tangent vector V

$$V^a \nabla_a V^b = 0$$

Then, with respect to $\tilde{g}$

$$V^a \tilde{\nabla}_a V^b = V^a \left(\partial_a V^b + \tilde{\Gamma}^b{}_{ac} V^c \right)$$

$$\tilde{\Gamma}^b{}_{ac} = \frac{1}{2} \Lambda^{-2} g^{bd} \left(\partial_a (\Lambda^2 g_{cd}) + \partial_c (\Lambda^2 g_{ad}) - \partial_d (\Lambda^2 g_{ac}) \right)$$

$$= \Gamma^b{}_{ac} + \gamma^b{}_{ac}$$

where $\gamma^b{}_{ac} = \frac{1}{2} \Lambda^{-2} \left(\delta^b_c \partial_a \Lambda^2 + \delta^b_a \partial_c \Lambda^2 - g^{bd} g_{ac} \partial_d \Lambda^2 \right)$

Then

$$V^a \tilde{\nabla}_a V^b = \cancel{V^a \nabla_a} V^b + V^a V^c \gamma^b{}_{ac}$$

$$= \frac{1}{2} \Lambda^{-2} \left(2 V^a V^b \partial_a \Lambda^2 - g(V,V) g^{bd} \partial_d \Lambda^2 \right)$$

$$= \left(\Lambda^{-2} V^a \partial_a \Lambda^2 \right) V^b - g(V,V) g^{bd} \Lambda^{-2} \partial_d \Lambda^2$$

Then the curve is not a geodesic (wrt $\tilde{g}$) unless $g(V,V) = 0$ ie unless it is null (in which case it is a null geodesic wrt $\tilde{g}$ but not necessarily affinely parametrized) //

To construct the Penrose-Carter diagrams we

1. Find a coordinate transformation of (M, g) to bring in infinity (compactify) to a finite coordinate distance.
   As a consequence: we get a new metric which is not regular at "infinity".

2. To resolve this problem, we perform a conformal transformation
   $$g \longrightarrow \tilde{g}$$
   st $\tilde{g}$ is now regular on the "edges"

3. Now we add the points at infinity to get a new space-time $(\tilde{M}, \tilde{g})$
   This new space is called the conformal compactification of (M, g).

The Penrose-Carter diagram is a space time diagram of the conformal compactification of (M, g)

Example: flat Minkowski space-time

coordinates (t, r, θ, ϕ)

metric $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$

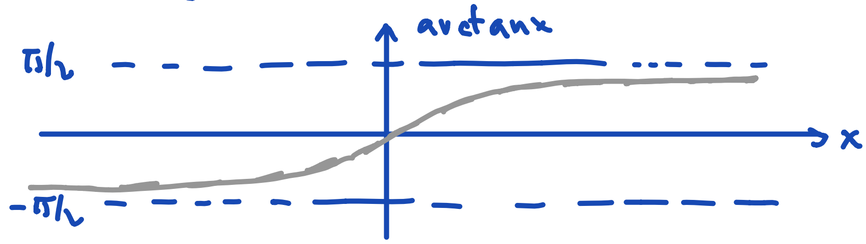
light cone coordinates: $u = t - r$
$v = t + r$

let $(\tilde{u}, \tilde{v})$ be new coordinates st

$u = \tan \tilde{u}$, $v = \tan \tilde{v}$ so $-\frac{\pi}{2} < \tilde{u} < \frac{\pi}{2}$, $-\frac{\pi}{2} < \tilde{v} < \frac{\pi}{2}$

constraint: $r \geq 0 \Leftrightarrow v \geq u \Leftrightarrow \underline{\tilde{v} \geq \tilde{u}}$

Remark: graph of arctan



arctan x maps the real line to a finite open interval (note that $\tan(\pm\frac{\pi}{2}) = \pm\infty$ do not belong to space-time) metric in coordinates $(\tilde{u}, \tilde{v}, \phi, \theta)$

$$ds^2 = \frac{1}{(2\cos\tilde{u}\cos\tilde{v})^2}\left(-4\,d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{u}-\tilde{v})\,d\Omega^2\right)$$

diverges as $u, v \to \pm\infty$

ie as $\tilde{u}, \tilde{v} \to \pm\frac{\pi}{2}$

Conformal transformation:

$$d\tilde{s}^2 = (2\cos\tilde{u}\cos\tilde{v})^2\,ds^2$$

$$= -4\,d\tilde{u}\,d\tilde{v} + \sin^2(\tilde{u}-\tilde{v})\,d\Omega^2$$

$\tilde{g}$ is regular at infinity, ie at

$\tilde{u}, \tilde{v} = \pm\frac{\pi}{2}$

Now "bring in infinity": that is extend coordinates

$$-\frac{\pi}{2} \le \tilde{u} \le \frac{\pi}{2}$$

$$-\frac{\pi}{2} \le \tilde{v} \le \frac{\pi}{2}$$

Consider new coordinates

$T = \tilde{u} + \tilde{v}$ range: $0 \leq X \leq \pi$

$X = \tilde{v} - \tilde{u}$ $-\pi \leq T \leq \pi$

$|T| + X \leq \pi$

$$d\tilde{s}^2 = -dT^2 + dX^2 + \sin^2 X\, d\Omega^2 = \Omega^2 ds^2$$

$(\Omega = 2\cos\tilde{u}\cos\tilde{v} = \cos T + \cos X)$

$(\tilde{M}, \tilde{g}) \rightsquigarrow$ conformally compactified Minkowski

(note that it has non-vanishing curvature)

$dX^2 + \sin^2 X\, d\Omega^2$ round metric of S^3 parametrized in polar coordinates (X, θ, ϕ)

(this has constant +ve scalar curvature)

$\rightarrow$ 2-spheres of constant $X \neq 0$ have radius $|\sin X|$

$\rightarrow$ $X = 0, \pi$ poles of S^3

$(\tilde{M}, \tilde{g}) \rightsquigarrow$ finite portion (as $-\pi \leq T \leq \pi$) of Einstein's static universe metric

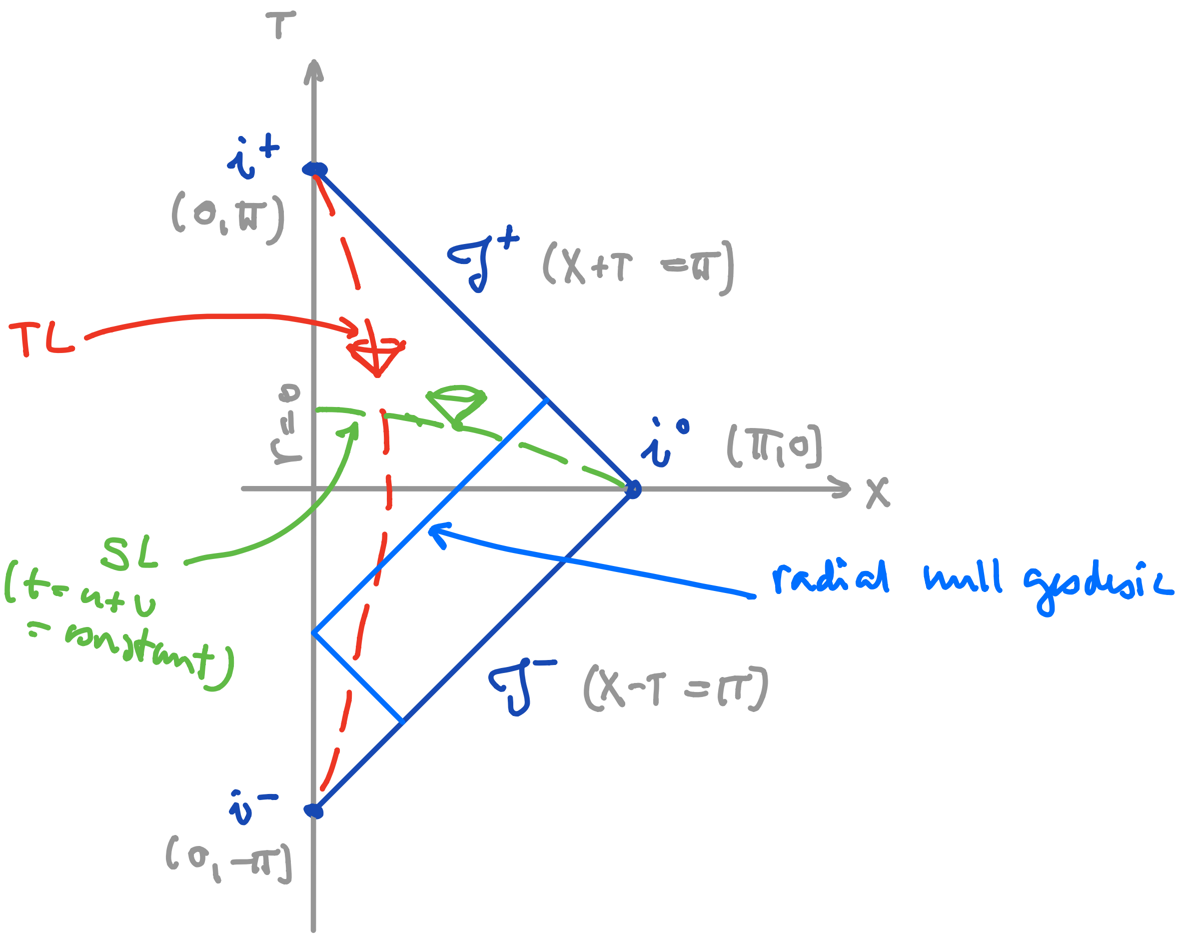
$\rightsquigarrow$ topology $\mathbb{R} \times S^3$

(static universe with spherical spatial slices i.e. $T =$ constant hypersurfaces)

"Infinities" ⑨

i^0	SL-infinity $r \to \infty$, t finite	$u = -\infty$ $v = \infty$	$\tilde{u} = -\pi/2$ $\tilde{v} = \pi/2$	$T = 0$ $X = +\pi$
$i^\pm$	Future (Past) TL infinity $t \to \pm\infty$ r finite	$u \to \pm\infty$ $v \to \pm\infty$	$\tilde{u} = \pm\pi/2$ $\tilde{v} = \pm\pi/2$	$T = \pm\pi$ $X = 0$
$\mathscr{I}^+$	Future Null infinity $t + r \to +\infty$ $t - r \to$ finite	$v \to +\infty$ u finite	$\tilde{v} = \pi/2$ $-\frac{\pi}{2} < \tilde{u} < \frac{\pi}{2}$	$T + X = \pi$
$\mathscr{I}^-$	Past Null infinity $t - r \to -\infty$ $t + r \to$ finite	$u \to -\infty$ v finite	$\tilde{u} = -\pi/2$ $-\frac{\pi}{2} < \tilde{v} < \frac{\pi}{2}$	$T - X = -\pi$

Penrose diagram: space-time diagram for conformally compactified Minkowski spacetime (bounded by $r=0$, $X+T=\pi$, $X-T=\pi$)



- $T =$ constant slices are S^3 with coordinates (χ, θ, ϕ) and each point on the diagram is a S^2 except at

$i^0 \rightsquigarrow$ N pole of S^3 } points!
$i^\pm \rightsquigarrow$ S " " S^3 }
$r = 0 \rightsquigarrow$ polar coordinates are singular there

- $\mathcal{J}^\pm$ topology $\mathbb{R} \times S^2$
- radial null geodesics $\to$ lines at $45°$
- i^0 end point of SL geodesics
- $i^\pm$ Future (past) TL curves end up at $i^\pm$ for large $\pm$ ve values of their affine parameters

Example: Penrose diagram for Kruskal-Szekeres spacetime

Recall

$$ds^2 = -32M^2 \frac{1}{r} e^{-r/2M} du\, dv + r^2 d\Omega^2$$

$$uv = \left(1 - \frac{r}{2M}\right) e^{r/2M} \qquad u^{-1} v = -e^{t/2M}$$

New coordinates:

$$u = \tan \tilde{u} \qquad -\frac{\pi}{2} < \tilde{u}, \tilde{v} < \frac{\pi}{2}$$
$$v = \tan \tilde{v}$$

Then:

$$ds^2 = -32M \frac{1}{r} e^{-r/2M} \frac{d\tilde{u}\, d\tilde{v}}{\cos^2\tilde{u} \cos^2\tilde{v}} + r^2 d\Omega^2$$

$$= \frac{1}{(2\cos\tilde{u}\cos\tilde{v})^2}\Big(-4 \cdot 32M \frac{1}{r} e^{-r/2M}$$
$$+ 4r^2 \cos^2\tilde{u} \cos^2\tilde{v}\, d\Omega^2\Big)$$

diverges as $\tilde{u}, \tilde{v} \to \pm\frac{\pi}{2}$

(ie $u, v \to \pm\infty$)

Let

$$d\tilde{s}^2 = -4 \cdot 32 M \frac{1}{r} e^{-r/2M} + 4 r^2 \cos^2 \hat{u} \cos^2 \hat{v} \, d\Omega^2$$

regular at $\tilde{u}, \tilde{v} \to \pm \frac{\pi}{2}$

(ie $u, v \to \pm \infty$)

Now add the points at infinity.

$$-\frac{\pi}{2} \leq \tilde{u}, \tilde{v} \leq \frac{\pi}{2}$$

(but be careful with curvature singularity at $r=0$ which cannot be added to spacetime)

Curvature singularity $r=0$

$uv = 1$ iff $\tan \hat{u} \tan \hat{v} = 1$

iff $\sin \hat{u} \sin \hat{v} = \cos \hat{u} \cos \hat{v}$

iff $\cos(\hat{u} + \hat{v}) = 0$

Then $\tilde{u} + \tilde{v} = \pm \frac{\pi}{2}$

$T = \tilde{u} + \tilde{v}$

$X = \tilde{v} - \tilde{u}$

or $T = \pm \frac{\pi}{2}$, $-\frac{\pi}{2} \leq X \leq \frac{\pi}{2}$

(these points cannot be added)

Construct Kruskal-Szekeres Penrose diagram

Boundaries: $X = \tilde{v} - \tilde{u}$, $T = \tilde{v} + \tilde{u}$

(I) $\tilde{u} \le 0$, $\tilde{v} \ge 0$ $(r > 2M)$

so $0 \le X \le \pi$

boundaries: $\tilde{u}^-$ axis, $\tilde{v}^+$ axis, $T = \pi + X$, $T = \pi - X$

event horizons $r=2M$

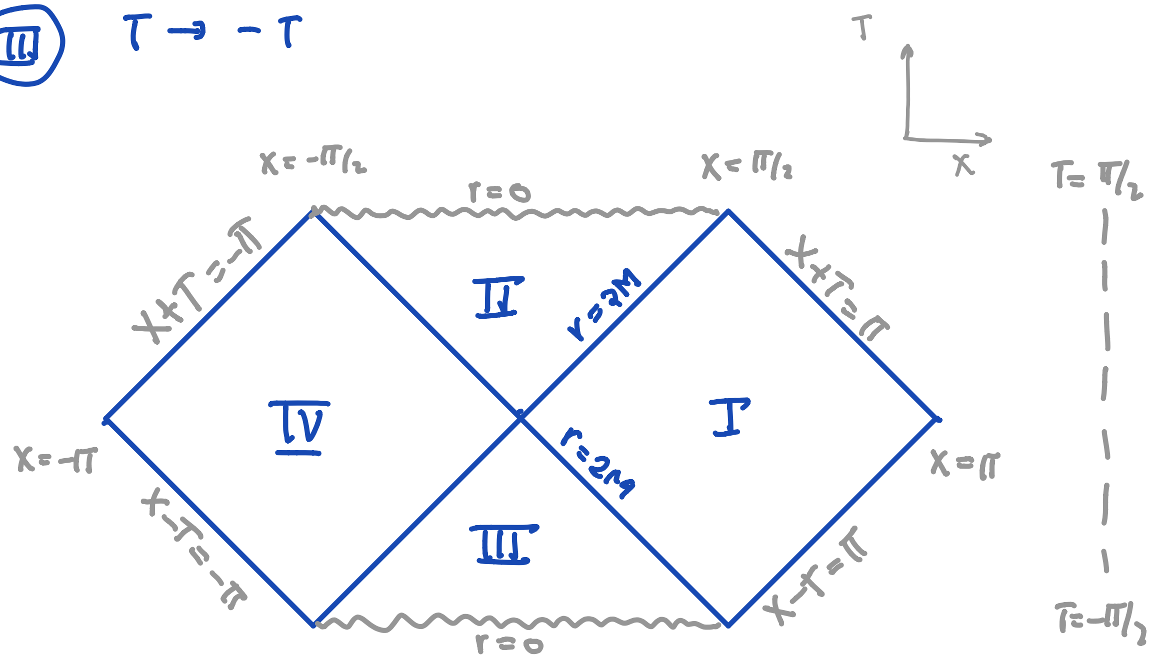
(IV) $(\tilde{u}, \tilde{v}) \longrightarrow (-\tilde{u}, -\tilde{v})$

(II) $\tilde{u} \ge 0$ $\tilde{v} \ge 0$ $(0 < r \le 2M)$

so $-\frac{\pi}{2} \le X \le \frac{\pi}{2}$, $0 \le T < \pi/2$

boundaries: $\tilde{u}^+$-axis, $\tilde{v}^+$-axis and $r=0$

(III) $T \to -T$

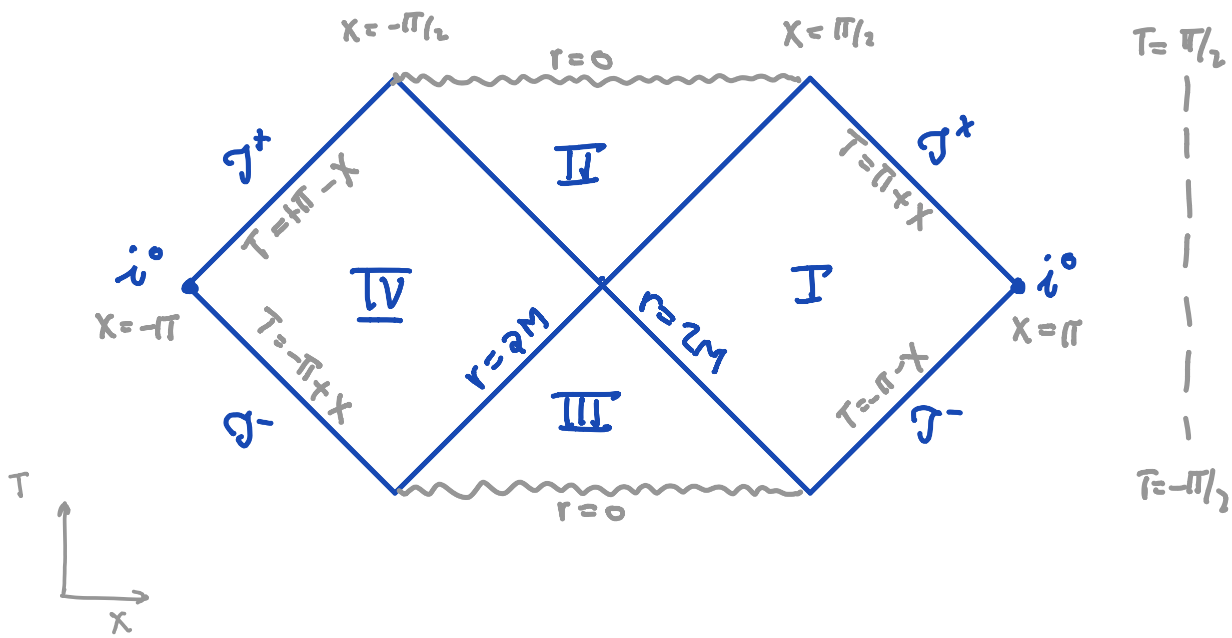


The KS spacetime is asymptotically flat. Hence $(\tilde{M}, \tilde{g})$ must approach the conformally compactified Minkowski spacetime as $r \to \infty$ (for any t) in regions I and IV

So we can add the points

$$i^0 \quad \& \quad \mathscr{I}^{\pm}$$

in both regions



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What happens to $i^\pm$?

$t \to \pm\infty$ so $\tan \tilde{u} \to 0$

so $\tilde{u} \to 0$

i^+ $\quad X = \frac{\pi}{2} = T$

in I

i^- $\quad X = \frac{\pi}{2} = -T$

In IV $\quad X = -\frac{\pi}{2}$ $\quad i^\pm = \pm\frac{\pi}{2}$

$i^\pm$ cannot be added as they meet the singularity $r=0$