Chapter 5

Stationary axisymmetric metrics

Solutions of EFEqs for the exterior gravitational field of an isolated rotating body

$$R_{ab} = 0$$

* Stationary: there is a TL Killing vector K
  We do not require the metric to be static so K is not HSO (want a rotating body after all)
  $K_t \longrightarrow$ time translation symmetry
  metric independent of t

* Axisymmetric: there is a SL Killing vector L st its integral curves are closed (orbits of the 1-parameter group of isometries are closed SL curves)
  $L \longrightarrow$ rotation about an axis
  periodic period 2π

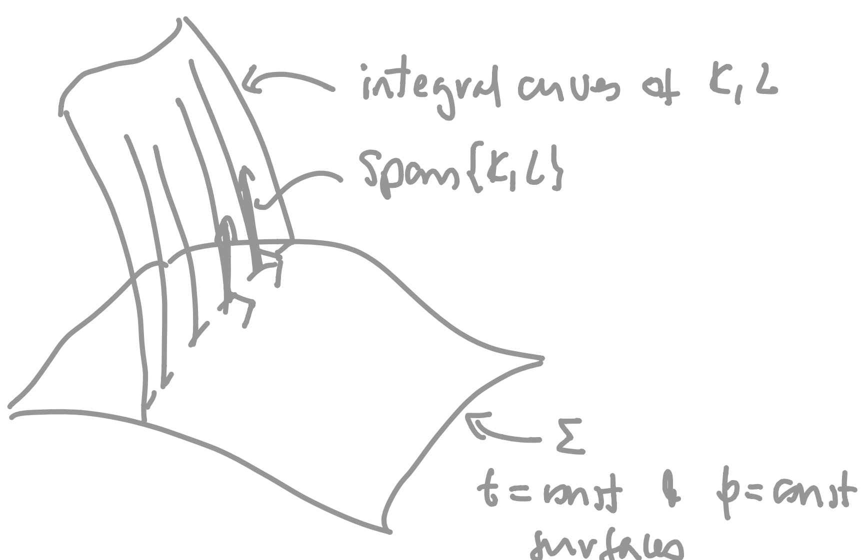
* Assume further that $[K, L] = 0$

Then we can choose coordinates
$$(t, \phi, \rho, z)$$
st $\quad K = \partial_t \quad$ and $\quad L = \partial_\phi$

and in these coordinates
$$g_{ab} = g_{ab}(\rho, z)$$
independent of (t, ϕ)

Moreover: K and L are perpendicular to surfaces Σ
$t = \text{const}, \quad \phi = \text{constant}.$

(a pair of linearly independent vectors are orthogonal to a family of 2 surfaces iff
$L_{[a} K_b \nabla_c L_{d]} = 0$ and $L_{[a} K_b \nabla_c K_{d]} = 0$)



③

The most general metric for a stationary axisymmetric object

$$ds^2 = -V\,dt^2 + 2W\,dt\,d\phi + U\,d\phi^2 + \Omega^2(d\rho^2 + dz^2)$$

metric on Σ in isothermal coords (ρ, z)

where V, W, U, Ω are functions of (ρ, z) only

K is TL $\Rightarrow$ $\underset{g_{tt}}{\underset{\parallel}{g(K,K)}} < 0$ $\Rightarrow$ $V > 0$

L is SL $\Rightarrow$ $\underset{g_{\phi\phi}}{\underset{\parallel}{g(L,L)}} > 0$ $\Rightarrow$ $U > 0$

<u>Also</u>: demand that g is asymptotically flat at large distances

so $V \to 1$ at large distances

With this metric

$R_{ab} = 0$ iff $\frac{1}{\rho}\partial_\rho(\rho\, g^{-1}\partial_\rho g) + \partial_z(g\,\partial_z g) = 0$

where $g = \begin{pmatrix} U & W \\ W & -V \end{pmatrix}$ (see Wald p162)

15.1 The Kerr solution (1963)

Carter & Robinson: the most general soln which is stationary, axisymmetric and asymptotically flat

(Hawking + Wald: stationary BH soln $\Rightarrow$ axisymmetric)

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar}{\Sigma} \sin^2\theta \, d\phi \, dt + \frac{1}{\Sigma} \sin^2\theta \left(\Delta\Sigma + 2Mr(r^2+a^2)\right) d\phi^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2\right)$$

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2$$

M, a constants

<u>Boyer-Lindquist coordinates</u> (t, r, θ, ϕ)

$$0 \le \phi < 2\pi, \quad 0 \le \theta < \pi$$

Asymptotic behaviour: as $r \to \infty$

$\Sigma \to r^2 \qquad \Delta \to r^2$

$$ds^2 = \left(1 - \frac{2M}{r} + \cdots\right) dt^2 - \left(\frac{4Ma}{r} \sin^2\theta\right) dt\, d\phi$$
$$+ \left(1 + \frac{2M}{r} + \cdots\right) \left(dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)\right)$$
$$+ \cdots$$

M = total mass

$J = Ma$ = angular momentum

Choose (WLOG) $a > 0$ $\quad (\phi \to -\phi$, $a \to -a)$

$a = \frac{J}{M}$ "specific" angular momentum

Remarks

1. $a = 0 \implies$ Schwartzschild

$$\Big[\quad a=0 \qquad \Sigma = r^2$$

$$\Delta = r^2 - 2Mr = r^2 F \, , \qquad F = 1 - \frac{2M}{r}$$

$$ds^2 = -F dt^2 + \frac{1}{r^2}\sin^2\theta\,(r^4 F + 2Mr^3)\, d\phi^2$$

$$+ r^2\left(\frac{1}{r^2} F^{-1} dr^2 + d\theta^2\right)$$

$$\frac{1}{r^2}\sin^2\theta\,(r^4 F + 2Mr^3) = r\sin^2\theta\,(r - \cancel{2M} + \cancel{2M})$$

Then

$$ds^2 = -F dt^2 + F^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\Big]$$

(2) $M=0 \implies$ Minkowski metric even if $a \neq 0$!

Proof: When $M=0$

$$\Sigma = r^2 + a^2\cos^2\theta \qquad \Delta = r^2 + a^2$$

$$ds^2 = -dt^2 + (r^2+a^2)\sin^2\theta\, d\phi^2 + \frac{r^2+a^2\cos^2\theta}{r^2+a^2}\, dr^2 + (r^2+a^2\cos^2\theta)\, d\theta^2$$

This is Minkowski metric in spheroidal coords

$$x = (r^2+a^2)^{1/2}\sin\theta\cos\phi$$
$$y = (r^2+a^2)^{1/2}\sin\theta\sin\phi$$
$$z = r\cos\theta$$

$r=$ constant surfaces are ellipsoids

$$\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$$

(compare: in spherical coordinates $r=$ const surfaces are round spheres)