Lecture 16

Kerr solution (continued)

[5.1] Kerr solution

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar}{\Sigma} \sin^2\theta \, d\phi \, dt$$
$$+ \frac{1}{\Sigma} \sin^2\theta \, (\Delta\Sigma + 2Mr(r^2 + a^2)) \, d\phi^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2\right)$$

$$\Sigma = r^2 + a^2 \cos^2\theta \qquad \Delta = r^2 - 2Mr + a^2$$

Boyer-Lindquist coordinates:

$$(t, r, \theta, \phi) \qquad 0 \leq \phi < 2\pi \, , \quad 0 \leq \theta < \pi$$

Remarks

- Carter & Robinson: the Kerr solution is the most general solution (of $R_{ab} = 0$) which is stationary, axisymmetric and asymptotically flat.
- Hawking & Wald: demanding a stationary solution with is a BH implies axisymmetric.
- Asymptotic behaviour: as $r \to \infty \quad \Sigma \to r^2$ and $\Delta \to r^2$

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 - \frac{4Mar}{\Sigma}\sin^2\theta\,\mathrm{d}\phi\,\mathrm{d}t$$
$$+\left(1 + \frac{2M}{r}\right)(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)) + \cdots$$

$M =$ total mass , $\quad J = Ma$= angular momentum

choose $a > 0$ as $\phi \to -\phi$ gives $a \to -a$

Remarks (continued)

- $a = 0 \to$ Schwarzschild metric.

  In fact, in this case

$$\Sigma = r^2\,, \quad \Delta = r^2 - 2Mr = r^2 F\,, \quad F = 1 - \frac{2M}{r}$$

$$\mathrm{d}s^2 = -F\mathrm{d}t^2 + F^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + G\,\mathrm{d}\phi^2$$

where

$$G = \frac{1}{r^2}\sin^2\theta(r^4 F + 2Mr^3) = r^2\sin^2\theta$$

Remarks (continued)

- $M = 0 \rightarrow$ Minkowski metric, even if $a \neq 0$.

  In this case: $\Sigma = r^2 + a^2 \cos^2\theta$, $\Delta = r^2 + a^2$

$$ds^2 = -dt^2 + (r^2 + a^2)\sin^2\theta d\phi^2$$
$$+ \frac{r^2 + a^2\cos^2\theta}{r^2 + a^2} dr^2 + (r^2 + a^2\cos^2\theta)d\theta^2$$

  This is Minkowski metric in spheroidal coordinates

$$x = (r^2 + a^2)^{1/2}\sin\theta\cos\phi \, , \quad y = (r^2 + a^2)^{1/2}\sin\theta\sin\phi$$
$$z = r\cos\theta$$

  r= constant surfaces are ellipsoids $\frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} = 1$

  compare: in spherical coordinates $r = constant$ surfaces are round spheres

Singularities

1 $\Sigma = r^2 + a^2 \cos^2\theta = 0 \iff r = 0 \;\; and \;\; \theta = \pi/2$

curvarture singularity (ring singularity)

2 $\Delta = r^2 - 2Mr + a^2 = 0 \iff r_\pm = M \pm \sqrt{M^2 - a^2}$

- $M^2 < a^2$: naked singularity at $r = 0$, $\theta = \pi/2$

  [Cosmic censorship conjecture: naked sigularities cannot form from gravitational collapse]

- $M^2 \geq a^2$: $r_\pm$ these are coordinate singularities,

  in fact, these are event horizons

  ($r_+ \geq r_-$ so we have an outer and an inner horizon when $r_+ > r_-$; only one when $r_+ = r_-$.)

Kerr-Eddington coordinates

To understand singularities consider new coordinates

$$\mathrm{d}T = \mathrm{d}t + \frac{2Mr}{\Delta}\mathrm{d}r\,, \qquad \mathrm{d}\Phi = \mathrm{d}\phi + \frac{a}{\Delta}\mathrm{d}r$$

Eliminating t and ϕ in favor of T and Φ,

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}T^2 + \mathrm{d}r^2 - 2a\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\Phi + \Sigma\,\mathrm{d}\theta^2 \\ &\quad + (r^2 + a^2)\sin^2\theta\,\mathrm{d}\Phi^2 + \frac{2Mr}{\Sigma}(\mathrm{d}T - a\sin^2\theta\,\mathrm{d}\Phi + \mathrm{d}r)^2 \end{aligned}$$

Metric is not singular at $\Delta = 0$ (but still singular at $\Sigma = 0$).

Kerr-Eddington coordinates

Coordinate change obtained by studying radial null geodesics (as for Schwarzschild).

Consider null radial geodesics along $\theta = 0$. Since $L = 0$

$$\mathrm{d}t^2 = \left(\frac{\Sigma}{\Delta}\right)^2 \mathrm{d}r^2 = \left(1 + \frac{2Mr}{\Delta}\right)^2 \mathrm{d}r^2$$

$$\pm\mathrm{d}r = \mathrm{d}t \mp \frac{2Mr}{\Delta}\mathrm{d}t \equiv \mathrm{d}T$$

So, for example for incoming null geodesics

$$\mathrm{d}r = -\mathrm{d}T \quad \Longrightarrow \quad T + r = \mathit{constant}$$

Event horizons

Coordinate singularity $\Delta = 0$, that is $r_\pm = M \pm \sqrt{M^2 - a^2}$

- $r_\pm$ are null hypersufaces

  $r =$ constant hypersufaces become null at $r = r_\pm$.
  Let $n_a = \nabla_a r$ be normal to $r =$constant hypersufaces.
  Then

$$g(n,n) = g^{ab}\,\partial_a r \partial_b r = g^{rr} = -\frac{\Delta\Sigma \sin^2\theta}{\det g}$$

  so $g(n,n) = 0$ on $r = r_\pm$.
  Hypersufaces $r = r_\pm$ are collections of null geodesics (integral curves of n are null geodesics).

Event horizons

Null hypersufaces separate space-time points (events) which are connected to i^0 by a TL path from those which are not.

We have a black hole:
that is, a region separated from $r \to \infty$ (i^0) by an event horizon.

Kerr-Schild coordinates (T, x, y, z)

$$dT = dt + \frac{2Mr}{\Delta} dr \,, \quad x + iy = (r + ia) \sin\theta \, e^{i\Phi} \,, \quad z = r \, \cos\theta$$

$$ds^2 = -dT^2 + dx^2 + dy^2 + dz^2$$

$$+ \frac{2Mr}{\Sigma} \left(\frac{1}{r^2 + a^2} (r(xdx + ydy) - a(xdy - ydx)) + \frac{1}{r} zdz + dT \right)^2$$

- $M = 0$ clearly flat metric
- $r =$ constant hypersurfaces are ellipsoids

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1$$

- surfaces $\theta =$ *constant* are hyperboloids

$$\frac{x^2 + y^2}{a^2 \sin^2\theta} - \frac{z^2}{a^2 \cos^2\theta} = 1$$

asymptotic cones: $(x^2 + y^2)^{1/2} = \pm z \tan\theta$

Kerr-Schild coordinates (T, x, y, z)

- singularity at $r = 0 \,, \;\; \theta = \pi/2$:

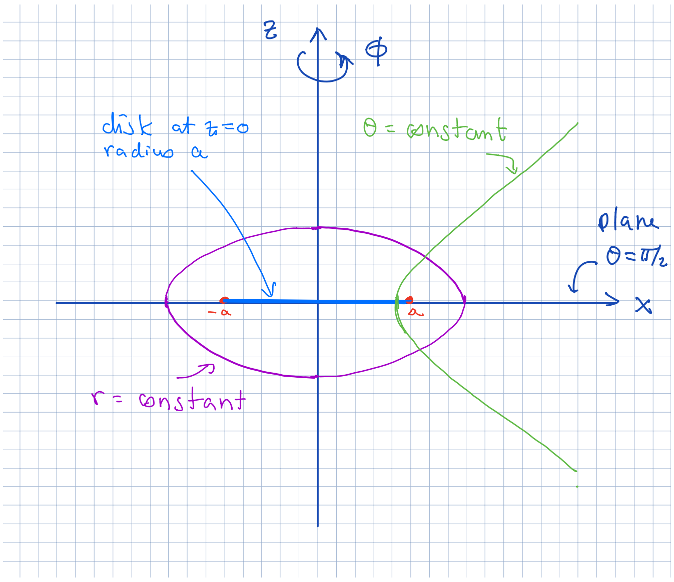
$$r = 0 \qquad \Longleftrightarrow \qquad z = 0 \,, \quad x^2 + y^2 = a^2 \sin^2\theta$$

this is a disk at $z = 0$, radius a.

Hence, the ring singularity is the boundary of this disk

$$x^2 + y^2 = a^2$$

T = constant diagram



Remark: travel toward $r = 0$. For $\theta = \pi/2$, hit singularity.
For $\theta \neq \pi/2$ does not encounrter singularity, and particle can go through interior of the ring ($x^2 + y^2 < a^2$)

What happens? there is no reason to constrain r to $r > 0$.
In fact, can extend to $r < 0$ to obtain another asymptotically flat region

(Hawking and Ellis: analytic continuation to obtain a maximal extension of the solution)

[5.2] Killing vectors in Kerr

Killing vectors

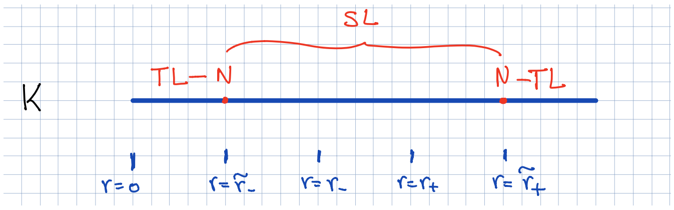
- $K = \partial_t :$ $\qquad g(K,K) = -\left(1 - \frac{2Mr}{\Sigma}\right)\ , \quad \Sigma \geq 0$

  TL: for $\Sigma > 2Mr$

  N: for $\Sigma = 2Mr$, that is for $r = \tilde{r}_\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}$

  ($r = \tilde{r}_\pm$ are not Null hypersurfaces)

  SL: for $\Sigma < 2Mr$



- $L = \partial_\phi$ is SL for all values of r

  $$g(L,L) = \sin^2\theta\left(r^2 + a^2 + \frac{2Ma^2}{\Sigma}\, r\sin^2\theta\right) > 0\ , \quad \forall r \geq 0$$

Killing vectors

Proposition: There is a KV vector

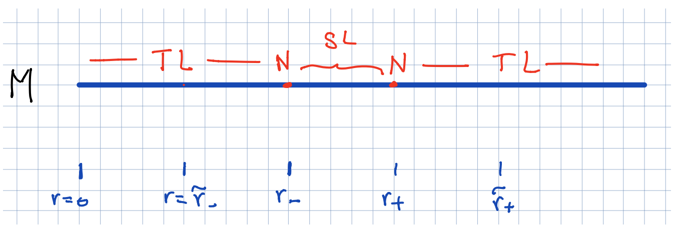
$$M = K + \lambda\, L$$

for some constant λ such that M is

TL for $r > r_+$ and $r < r_-$

N at $r = r_\pm$

SL for $r_- < r < r_+$



Moreover: the null hypersurfaces $\mathcal{N}_\pm$ ($r = r_\pm$) are Killing horizons for M with surface gravity

$$\kappa_\pm = \frac{r_\pm - r_\mp}{2(r_\pm^2 + a^2)}$$

Killing vectors

Proof Consider the metric in coordinates

$$(v, r, \theta, \Phi) \qquad \text{incoming Kerr coordinates}$$

where

$$d\Phi = d\phi + \frac{a}{\Delta} dr \qquad dv = dt + \frac{r^2 + a^2}{\Delta} dr$$

In these coordinates $v = T + r$ is constant along incoming Null radial geodesics, where $dT = dt + \frac{2Mr}{\Delta} dr$

The metric in these coordinates is (eliminating t and ϕ in favor of Φ and v)

$$\begin{aligned} ds^2 = & -\frac{\Delta - a^2 \sin^2\theta}{\Sigma} dv^2 + 2 dv\, dr \\ & - \frac{2a \sin^2\theta}{\Sigma}(r^2 + a^2 - \Delta)\, dv\, d\Phi - 2a \sin^2\theta\, d\Phi\, dr \\ & + \frac{1}{\Sigma} \sin^2\theta((r^2 + a^2)^2 - \Delta a^2 \sin^2\theta)\, d\Phi^2 + \Sigma d\theta^2 \end{aligned}$$

Note: the metric is independent of v and Φ so ∂_v and ∂_Φ are Killing vectors.

Killing vectors

Proof (continued)

In fact, in these coordinates

$$K = \partial_v \,, \qquad L = \partial_\Phi$$

Let $M = K + \lambda L$. Then

$$g(M, M) = g(K, K) + 2\lambda\, g(K, L) + \lambda^2 g(L, L)$$

We want $g(M, M) = 0$ on $r = r_\pm$: this is an equation which is a quadratic in λ. The discriminant D is of this quadratic is

$$\begin{aligned} \frac{D}{4} &= g(L, K)^2 - g(K, K) g(L, L) = g_{v\Phi}^2 - g_{vv} g_{\Phi\Phi} \\ &= \cdots = \Delta \sin^2\theta \end{aligned}$$

which vanishes at $r = r_\pm$.

Killing vectors

Proof (continued)

Then on $r = r_\pm$: $\quad g(L,K)^2 = g(K,K)g(L,L) \quad$ and

$$g(M,M)|_{r=r_\pm} = \left[g(L,L)\left(\lambda + \frac{g(K,L)}{g(L,L)}\right)^2\right]_{r=r_\pm} = 0$$

Therefore, the Killing vector

$$M_\pm = K + \lambda_\pm L\,, \qquad \lambda_\pm = -\left[\frac{g(K,L)}{g(L,L)}\right]_{r=r_\pm} = \frac{a}{r_\pm^2 + a^2}$$

is Null on $\mathcal{N}_\pm$.

Killing vectors

Proof (continued)

Now

$$g(M_\pm, M_\pm) = g(K,K) + 2\lambda_\pm\, g(K,L) + \lambda_\pm^2\, g(L,L)$$

$$\frac{D}{4} = \Delta\,\sin^2\theta = \begin{cases} +ve, & \Delta > 0 \;\; ie \;\; r > r_+ \; or \; r < r_- \\ -ve, & \Delta < 0 \;\; ie \quad r_- < r < r_+ \end{cases}$$

Hence (exercise) $M_\pm$ is
TL when $r > r_+ \; or \; r < r_-$
SL when $r_- < r < r_+$.

Killing vectors

Proof (continued)

To prove that $\mathcal{N}_\pm$ are Killing horizons for $M_\pm$, we need to prove that it is normal to $\mathcal{N}_\pm$ on $\mathcal{N}_\pm$.

$$M_\pm = K + \lambda_\pm L = \partial_v + \lambda_\pm \partial_\Phi \;, \quad \lambda_\pm = \frac{a}{r_\pm^2 + a^2}$$

$$M_{\pm\, v} = g_{va} M_\pm^a = 0 \;\; on \;\; r = r_\pm$$

$$M_{\pm\, \Phi} = g_{\Phi a} M_\pm^a = 0 \;\; on \;\; r = r_\pm$$

$$M_{\pm\, r} = g_{ra} M_\pm^a = \frac{1}{r_\pm^2 + a^2} (r_\pm^2 + a^2 \cos^2\theta) \;\; on \;\; \mathcal{N}_\pm$$

Killing vectors

Proof (continued)

On the other hand, the normal $n_\pm$ to $\mathcal{N}_\pm$ is:

$$n_{\pm\, a} = \Psi_\pm \partial_a r$$

so $n_{\pm\, a} = 0$ unless $a = r$ and $n_{\pm\, r} = \Psi_\pm$.

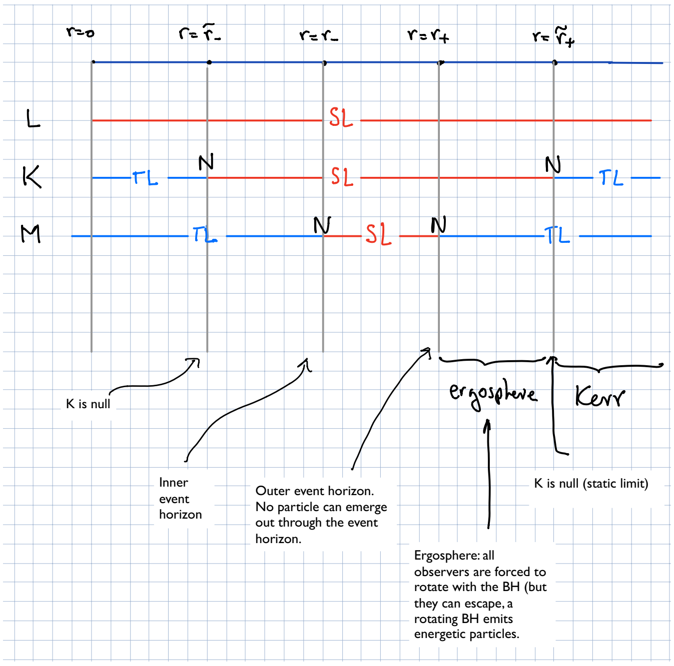
Then $M_\pm|_{r=r_\pm}$ is normal to $\mathcal{N}_\pm$, and we have proved that $\mathcal{N}_\pm$ are Killing horizons of the KV $M_\pm$.

Exercise: compute the surface gravity: $M^a \nabla_a M_b = \kappa_\pm M^b$

end of proof

Remark: one can prove that in the region $r_- < r < r_+$, there is no TL Killing vector $M = K + \lambda L$ for any λ.

Killing vectors



$r \leq \tilde{r}_+$

At $r = \tilde{r}_+$, K becomes null ("static limit").

A stationary observer at constant (r, θ) sees an unchanging space-time geometry. Hence its 4-velocity U must be a Killing vector. The observer is static if also ϕ= constant.
The angular velocity Ω of the observer is

$$\Omega = \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{U^\phi}{U^t}$$

So an observer is stationary when

$$U = U^t\partial_t + U^\phi\partial_\phi = U^t(K + \Omega\, L)$$

The observer is static iff $\Omega = 0$.

Now U must be TL ($g(U, U) < 0$) and this happens for Ω s.t.

$$\Omega_{min} < \Omega < \Omega_{max}$$

$g(U, U) = (U^t)^2(g_{tt} + 2\Omega\, g_{t\phi} + \Omega^2\, g_{\phi\phi}) < 0$

$r \leq \tilde{r}_+$

Then

- $r > \tilde{r}_+$: can have static ($\Omega = 0$) observers (in fact $\Omega_{min} < 0$ and $\Omega_{max} > 0$)
- at $r = \tilde{r}_+$ one finds $\Omega_{min} = 0$ ("static limit": no static observers for $r < \tilde{r}_+$)
- for $r_+ < r < \tilde{r}_+$ (inside the ergosphere) one has $\Omega_{min,max} > 0$ and $|\Omega_{max} - \Omega_{min}|$ smaller as $r \to r_+$
- at $r = r_+$ there are no stationary observers

ergosphere: no observers can remain at rest.
Moreover, they are forced to rotate with the BH (Ω has the same sign as $J = Ma$)

$r < r_+$

- theoretical region
- events cannot influence exteriof of the BH
- maximal extension: non physical (even if theoretical interesting)
- in particular, can extend to $r < 0$; one finds closed TL curves; possible to pass through the ring between regions $r < 0$ and $r > 0$ avoiding singularities.

Penrose diagram

Harder! Kerr solution is not spherically symmetric. Draw a diagram for $\theta = \pi/2$ and another for $\theta \neq \pi/2$ (one has a singularity at $r = 0$, the other doesn't).
Procedure:

1. coordinate transformation to (u, v, θ, ϕ) where

$$u = t - r_* \,, \qquad v = t + r_* \,, \qquad \mathrm{d}r_* = \frac{r^2 + a^2}{\Delta}\,\mathrm{d}r$$

2. Define Kruskal-type coordinates $\mathcal{U}^\pm$ and $\mathcal{V}^\pm$

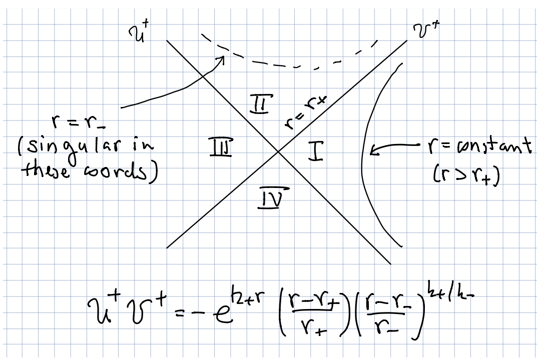
$$\mathcal{U}^\pm = -e^{\kappa_\pm u} \,, \qquad \mathcal{V}^\pm = e^{\kappa_\pm v}$$

Penrose diagram

2. (continued) For + (these coordinates do not cover $r \leq r_-$)

$$\mathrm{d}s^2 = -\frac{r_+ r_-}{\kappa_+^2} \frac{e^{2\kappa_+ r}}{r^2} \left(\frac{r_-}{r - r_-}\right)^{\kappa_+/\kappa_- - 1} \mathrm{d}\mathcal{U}^+ \mathrm{d}\mathcal{V}^+ + r^2 \mathrm{d}\Omega^2$$

Cover 4 regions

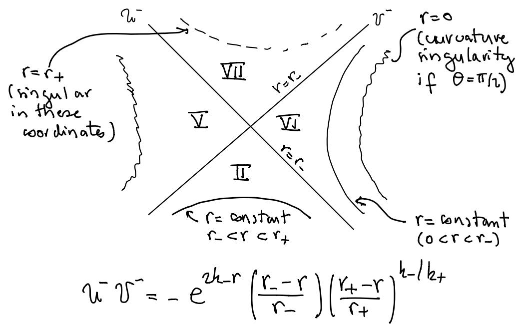


Penrose diagram

2 (continued) For $-$ (these coordinates cover $0 < r \leq r_+$)

$$ds^2 = -\frac{r_+ r_-}{\kappa_-^2} \frac{e^{2\kappa_- r}}{r^2} \left(\frac{r_+}{r - r_+} \right)^{\kappa_-/\kappa_+ - 1} d\mathcal{U}^- d\mathcal{V}^+ + r^2 d\Omega^2$$

Cover 4 regions



Rergion VII must be connected to another region, etc
Leads to an infinite sequence of space time!

Penrose diagram

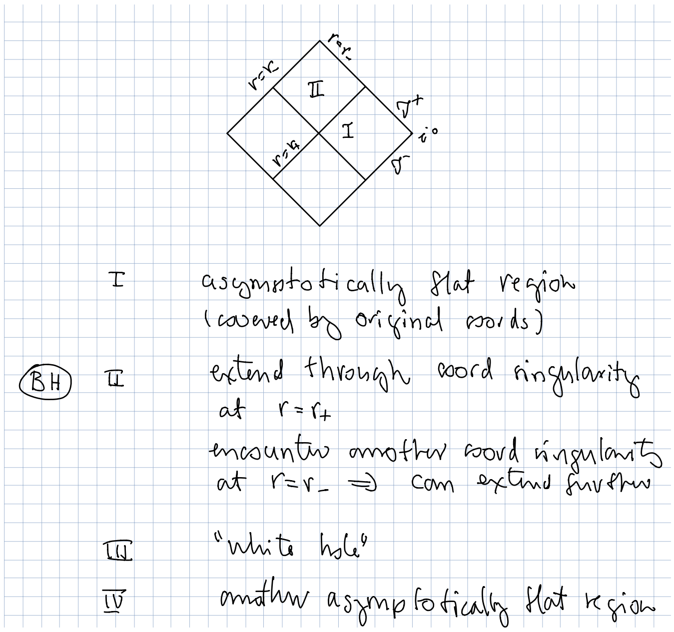
3 Conformal transformation: new coordinates

$$\mathcal{U}^\pm = \tan \tilde{\mathcal{U}}^\pm \,, \qquad \mathcal{V}^\pm = \tan \tilde{\mathcal{V}}^\pm$$

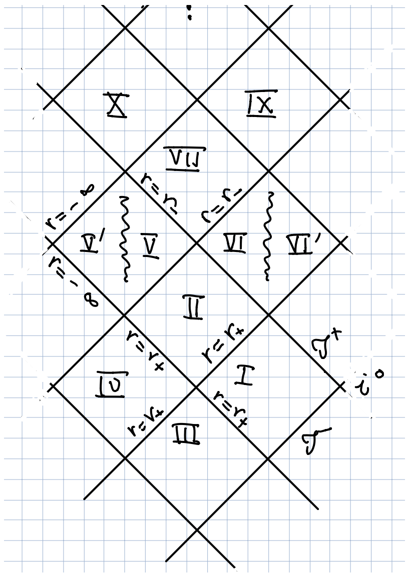
and include "infinity" as before

Penrose diagram

regions I, II, III and IV



Penrose diagram



Extending from II through $r = r_-$ $\longrightarrow$ V, VI
which contain a singularity at $r = 0$ if $\theta = \pi/2$ but no singularity if $\theta \neq \pi/2$

can pass through ring singularity to another asymptotically flat region as $r \to -\infty$

With respect to region I all this lies inside the BH

Final remark

Charged Black holes: Kerr-Newman
This is the most general solution of Einstein's equations which is stationary and axisymmetric, coupled to the electromagnetic field.

- Gravitational collapse of a realistic star produces a Kerr BH in a region of space time
- characterized uniquely by (M, J, Q) only (BH have no hair).

THE END