# Outline for today

- Introduction to the scattering transform: repeated application of a fixed transform
- Translation and deformation as examples of invariance sought for classification
- Wavelet transform as time-frequency tilings
- Properties of the scattering transform: Energy preservation and deformation invariance in the limit
- Examples of scattering transform energy decay and classification

# Scattering Transform (Mallat 12'[1])

The Scattering Transform repeatedly applied a deterministic wavelet transform followed by $\sigma(x) = |x|$ as nonlinear activation

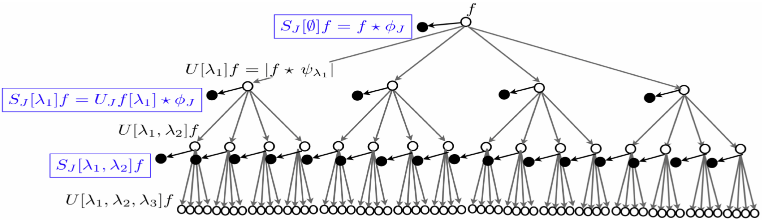



Figure 1: A scattering propagator $U_J$ applied to $f$ computes each $U[\lambda_1]f = |f \star \psi_{\lambda_1}|$ and outputs $S_J[\emptyset]f = f \star \phi_{2^J}$. Applying $U_J$ to each $U[\lambda_1]f$ computes all $U[\lambda_1, \lambda_2]f$ and outputs $S_J[\lambda_1] = U[\lambda_1] \star \phi_{2^J}$. Applying iteratively $U_J$ to each $U[p]f$ outputs $S_J[p]f = U[p]f \star \phi_{2^J}$ and computes the next path layer.

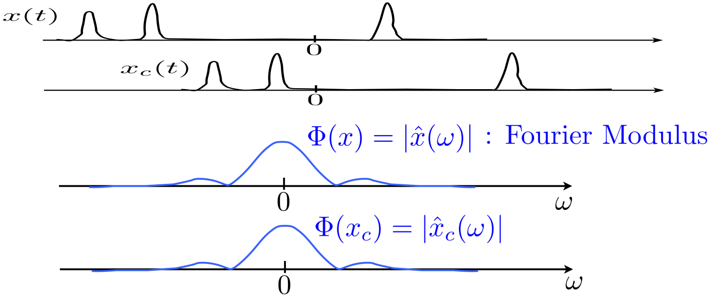
Depth allows the transform to become increasingly invariant to translation and small diffeomorphisms.

[1] https://arxiv.org/pdf/1101.2286.pdf

# Classification as learning invariance (Mallat '13[2])

**Invariance to translations** $x_c(t) = x(t-c)$

$$\forall c \in \mathbf{R} \quad , \quad \Phi(x_c) = \Phi(x) \ .$$




**Lipschitz stable** to deformations $x_\tau(t) = x(t - \tau(t))$

small deformations of $x \implies$ small modifications of $\Phi(x)$

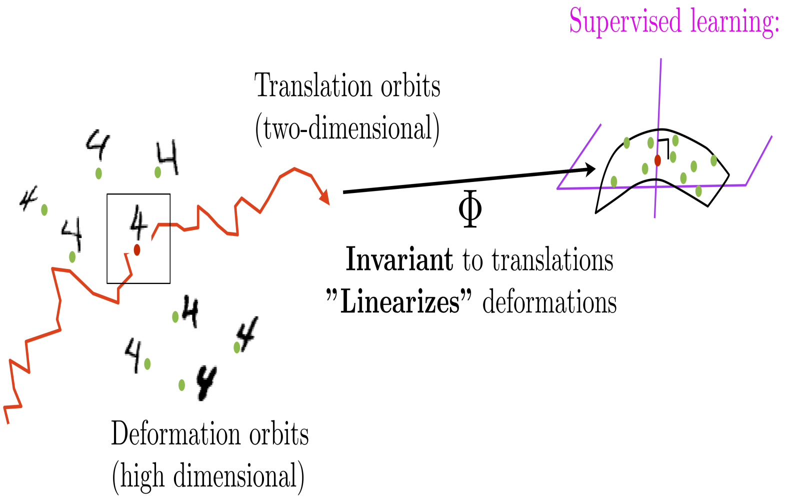
$$\forall \tau \quad , \quad \|\Phi(x_\tau) - \Phi(x)\| \leq C \underbrace{\sup_t |\nabla \tau(t)|}_{\text{deformation size}} \|x\| \ .$$

deformation size

[2] http://lcsl.mit.edu/ldr-workshop/Home.html

# Linearising deformations (Mallat '13[3])

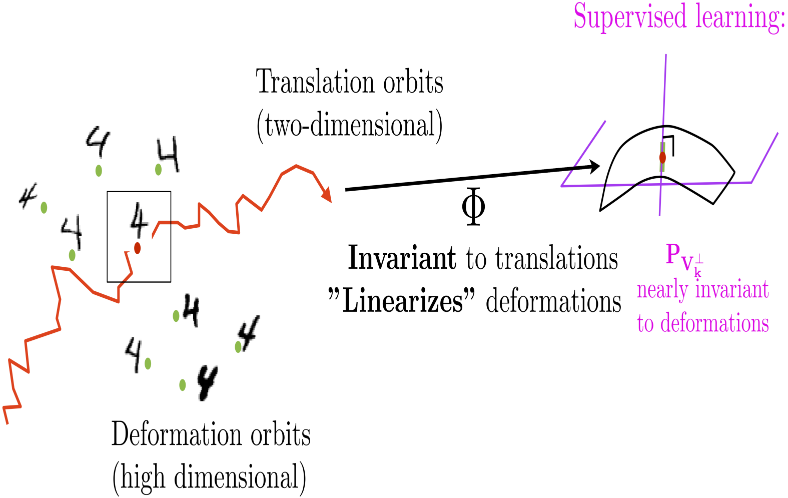
- Specific deformation invariance must be learned.




[3] http://lcsl.mit.edu/ldr-workshop/Home.html

# Linearising deformations (Mallat '13[4])

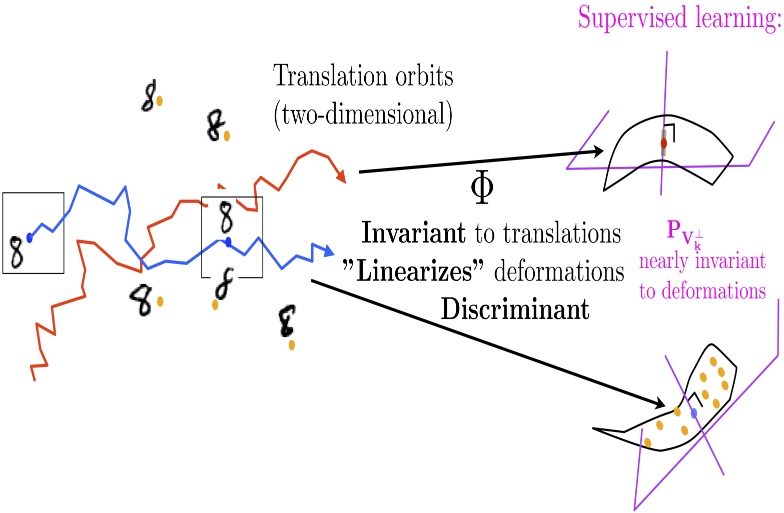
- Specific deformation invariance must be learned.




[4] http://lcsl.mit.edu/ldr-workshop/Home.html

# Linearising deformations (Mallat '13[5])

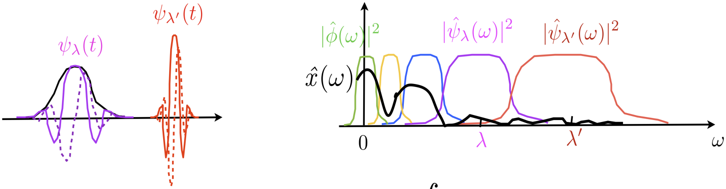
- Specific deformation invariance must be learned.




[5] http://lcsl.mit.edu/ldr-workshop/Home.html

# Wavelet Transform as frequency tiling (Mallat '13[6])

- Complex wavelet: $\psi(t) = \psi^a(t) + i\,\psi^b(t)$
- Dilated: $\psi_\lambda(t) = 2^{-j}\,\psi(2^{-j}t)$ with $\lambda = 2^{-j}$ .



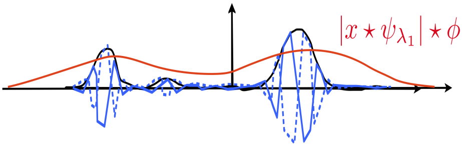

- Wavelet transform: $x \star \psi_\lambda(t) = \int x(u)\,\psi_\lambda(t-u)\,du$

$$Wx = \begin{pmatrix} x \star \phi(t) \\ x \star \psi_\lambda(t) \end{pmatrix}_{t,\lambda}$$

Unitary: $\|Wx\|^2 = \|x\|^2$ .

[6] http://lcsl.mit.edu/ldr-workshop/Home.html

# Modulus and averaging in wavelet domain (Mallat '13[7])

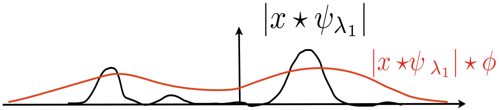



- The modulus $|x \star \psi_{\lambda_1}|$ is a regular envelop
- The average $|x \star \psi_{\lambda_1}| \star \phi(t)$ is invariant to small translations relatively to the support of $\phi$.
- Full translation invariance at the limit:

$$\lim_{\phi \to 1} |x \star \psi_{\lambda_1}| \star \phi(t) = \int |x \star \psi_{\lambda_1}(u)|\, du = \|x \star \psi_{\lambda_1}\|_1$$

[7] http://lcsl.mit.edu/ldr-workshop/Home.html

# Second layer of the scattering transform (Mallat '13[8])




- The high frequencies of $|x \star \psi_{\lambda_1}|$ are in wavelet coefficients:

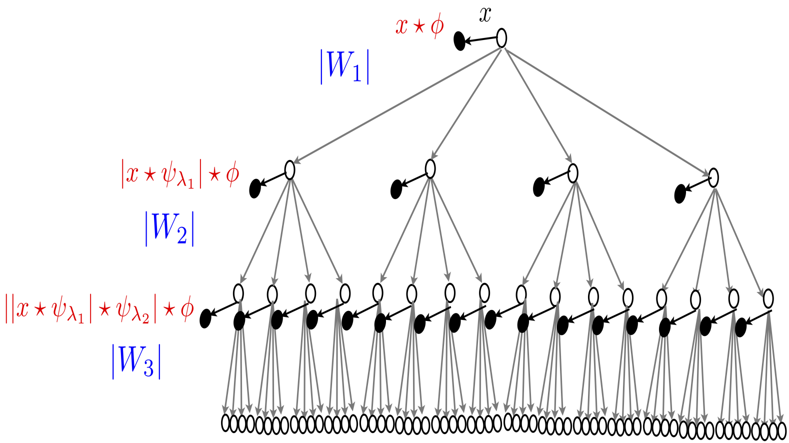
$$W|x \star \psi_{\lambda_1}| = \left( \begin{array}{c} |x \star \psi_{\lambda_1}| \star \phi(t) \\ |x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}(t) \end{array} \right)_{t,\lambda_2}$$

- Translation invariance by time averaging the amplitude:

$$\forall \lambda_1, \lambda_2, \quad ||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi(t)$$

[8] http://lcsl.mit.edu/ldr-workshop/Home.html

# Scattering transform (Mallat '13[9])




- Cascade of contractive operators

$$\| |W_k| x - |W_k| x' \| \leq \|x - x'\| \quad \text{with} \quad \| |W_k| x \| = \|x\| \; .$$

[9] http://lcsl.mit.edu/ldr-workshop/Home.html

# Scattering transform properties(Mallat '13[10])

$$Sx = \begin{pmatrix} x \star \phi\,(u) \\ |x \star \psi_{\lambda_1}| \star \phi(u) \\ ||x \star \psi_{\lambda_1}| \star \psi_{\lambda_2}| \star \phi(u) \\ |||x \star \psi_{\lambda_2}| \star \psi_{\lambda_2}| \star \psi_{\lambda_3}| \star \phi(u) \\ \dots \end{pmatrix}_{u,\lambda_1,\lambda_2,\lambda_3,\dots}$$

**Theorem**: *For appropriate wavelets, a scattering is*

*contractive* $\|Sx - Sy\| \leq \|x - y\|$

*preserves norms* $\|Sx\| = \|x\|$

*stable to deformations* $x_\tau(t) = x(t - \tau(t))$

$$\|Sx - Sx_\tau\| \leq C \sup_t |\nabla\tau(t)|\,\|x\|$$

$\Rightarrow$ linear discriminative classification from $\Phi x = Sx$

[10] http://lcsl.mit.edu/ldr-workshop/Home.html

# Scattering Transform: energy decay (Mallat 12'[11])

**Lemma**

For suitably chosen wavelet transforms (see Theorem 2.6 in footnote) then for all $f \in L^2(\mathbb{R}^d)$

$$\lim_{m\to\infty} \|U[\Lambda_J^m]f\|^2 = \lim_{m\to\infty} \sum_{n=m}^{\infty} \|S_J[\Lambda_J^n]f\|^2 = 0$$

where $U[\lambda]f = |f \star \psi_\lambda|$ and $S_J[\lambda]f = \phi_j \star U[\lambda]f$ and $\|S_J[P_J]f\| = \|f\|$. Morevover, for all $c \in \mathbb{R}^d$

$$\lim_{J\to\infty} \|S_J[P_J]f = S_J[P_J]L_c f\| = 0$$

where $L_c f = f(x-c)$ is the translation operator.

[11] https://arxiv.org/pdf/1101.2286.pdf

# Scattering Transform: energy decay (Mallat 13'[12])

TABLE 1
Percentage of Energy $\sum_{p\in\mathcal{P}_{\downarrow}^{m}} \|S[p]x\|^2/\|x\|^2$ of Scattering Coefficients on Frequency-Decreasing Paths of Length $m$, Depending upon $J$

| $J$ | $m=0$ | $m=1$ | $m=2$ | $m=3$ | $m=4$ | $m\leq 3$ |
|---|---|---|---|---|---|---|
| 1 | 95.1 | 4.86 | - | - | - | 99.96 |
| 2 | 87.56 | 11.97 | 0.35 | - | - | 99.89 |
| 3 | 76.29 | 21.92 | 1.54 | 0.02 | - | 99.78 |
| 4 | 61.52 | 33.87 | 4.05 | 0.16 | 0 | 99.61 |
| 5 | 44.6 | 45.26 | 8.9 | 0.61 | 0.01 | 99.37 |
| 6 | 26.15 | 57.02 | 14.4 | 1.54 | 0.07 | 99.1 |
| 7 | 0 | 73.37 | 21.98 | 3.56 | 0.25 | 98.91 |

*These average values are computed on the Caltech-101 database, with zero mean and unit variance images.*

[12] https://www.di.ens.fr/data/publications/papers/pami-final.pdf

# Scattering Transform: MNIST classification (Mallat 13'[13])

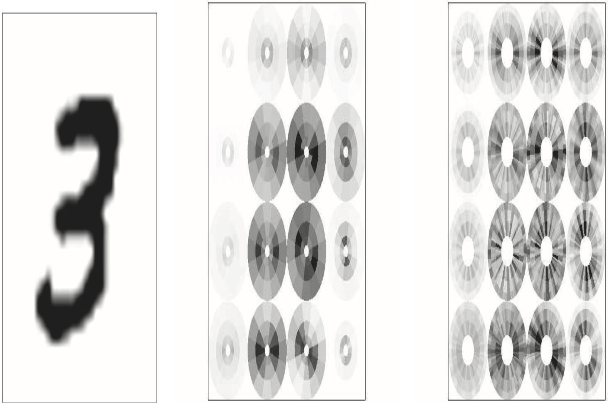
TABLE 4

Percentage of Errors of MNIST Classifiers, Depending on the Training Size

| Training size | $x$ PCA | $x$ SVM | Wind. Four. PCA | Wind. Four. SVM | Scat. $\overline{m}=1$ PCA | Scat. $\overline{m}=1$ SVM | Scat. $\overline{m}=2$ PCA | Scat. $\overline{m}=2$ SVM | Conv. Net. |
|---|---|---|---|---|---|---|---|---|---|
| 300 | 14.5 | 15.4 | 7.35 | 7.4 | 5.7 | 8 | **4.7** | 5.6 | 7.18 |
| 1000 | 7.2 | 8.2 | 3.74 | 3.74 | 2.35 | 4 | **2.3** | 2.6 | 3.21 |
| 2000 | 5.8 | 6.5 | 2.99 | 2.9 | 1.7 | 2.6 | **1.3** | 1.8 | 2.53 |
| 5000 | 4.9 | 4 | 2.34 | 2.2 | 1.6 | 1.6 | **1.03** | 1.4 | 1.52 |
| 10000 | 4.55 | 3.11 | 2.24 | 1.65 | 1.5 | 1.23 | 0.88 | 1 | **0.85** |
| 20000 | 4.25 | 2.2 | 1.92 | 1.15 | 1.4 | 0.96 | 0.79 | **0.58** | 0.76 |
| 40000 | 4.1 | 1.7 | 1.85 | 0.9 | 1.36 | 0.75 | 0.74 | **0.53** | 0.65 |
| 60000 | 4.3 | 1.4 | 1.80 | 0.8 | 1.34 | 0.62 | 0.7 | **0.43** | 0.53 |

[13] https://www.di.ens.fr/data/publications/papers/pami-final.pdf

# Scattering Transform: MNIST digit 3 (Mallat 13'[14])



(a) (b) (c)

Fig. 7. (a) Image $X(u)$ of a digit "3." (b) Arrays of windowed scattering coefficients $S[p]X(u)$ of order $m = 1$, with $u$ sampled at intervals of $2^J = 8$ pixels. (c) Windowed scattering coefficients $S[p]X(u)$ of order $m = 2$.

[14] https://www.di.ens.fr/data/publications/papers/pami-final.pdf