Finite Element Methods. QS 0

These problems are for practice and revision purposes. This sheet is not to be turned in.

1. (a) Let $C^1([a,b])$ be the set of continuous differentiable real-valued functions on $[a,b]$. Let $f \in C^1([a,b])$. Show that

$$\int_a^b f(x)g(x)\,\mathrm{d}x = 0 \text{ for all } g \in C^1([a,b]) \text{ such that } g(a) = g(b) = 0$$

if and only if $f = 0$ on $[a,b]$.

Remark: This is the fundamental lemma of the calculus of variations).

(b) Let $f, g \in C^1([a,b])$.

(i) Show that

$$(f,g) = \int_a^b f(x)g(x)\,\mathrm{d}x + \int_a^b f'(x)g'(x)\,\mathrm{d}x,$$

is an inner product on $C^1([a,b])$.

(ii) Show that

$$\|f\| = \int_a^b |f(x)|\,\mathrm{d}x + \int_a^b |f'(x)|\,\mathrm{d}x,$$

defines a norm on $C^1([a,b])$.

2. Let $A \in \mathbb{R}^{m\times n}$. The transpose $A^\top \in \mathbb{R}^{n\times m}$ of A is defined as the matrix that satisfies

$$(y, Ax)_{\mathbb{R}^m} = (A^\top y, x)_{\mathbb{R}^n}$$

for all $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. (Here $(\cdot,\cdot)$ denotes the standard Euclidean inner product, the dot product.)

We define the so-called four fundamental subspaces

$$\begin{aligned}
\textbf{Column space:}\quad & \operatorname{range}(A) = \{y \in \mathbb{R}^m \mid y = Ax,\ x \in \mathbb{R}^n\}\,;\\
\textbf{Nullspace:}\quad & \operatorname{kernel}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}\,;\\
\textbf{Row space:}\quad & \operatorname{range}(A^\top) = \left\{x \in \mathbb{R}^n \mid x = A^\top y,\ y \in \mathbb{R}^m\right\};\\
\textbf{Left Nullspace:}\quad & \operatorname{kernel}(A^\top) = \left\{y \in \mathbb{R}^m \mid A^\top y = 0\right\}.
\end{aligned}$$

(a) Show that the null space is orthogonal to the row space, i.e. $\operatorname{kernel}(A) = \big(\operatorname{range}(A^\top)\big)^\perp$.

(b) Show that the left nullspace is orthogonal to the column space, i.e. $\operatorname{kernel}(A^\top) = (\operatorname{range}(A))^\perp$.

Remark: This is the fundamental theorem of linear algebra.

3. (a) Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Prove that

$$Ax = b \iff (v, Ax) = (v, b) \quad \text{for all } v \in \mathbb{R}^m.$$

(b) Now consider $A \in \mathbb{R}^{n\times n}$, a symmetric matrix. Define

$$a(x, v) = (v, Ax).$$

State necessary and sufficient conditions on the spectrum of A that guarantee that

$$a(x, x) \geq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

4. Familiarise (or refamiliarise) yourself with Lagrange interpolation on an interval and its error bound in terms of higher derivatives of the function to be interpolated. This material will be available in any elementary numerical analysis textbook, e.g. pg. 179–184 of *An Introduction to Numerical Analysis* by Süli & Mayers, Cambridge University Press, 2003.

5. Familiarise (or refamiliarise) yourself with the approximation of integrals by quadrature. This material will be available in any elementary numerical analysis textbook, e.g. pg. 200–208 of *An Introduction to Numerical Analysis* by Süli & Mayers, Cambridge University Press, 2003.