Finite Element Methods. QS 1

Hand-in deadline: 9.00 am, 28 Jan, week 3.
Class: 30 Jan, week 3.

1. Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and associated norm $\|\cdot\|$. The vectors $x, y \in \mathcal{H}$ are said to be *orthogonal* if $(x, y) = 0$. This is denoted by $x \perp y$. The orthogonal complement $A^\perp$ of $A \in \mathcal{H}$ is the set of vectors orthogonal to A,

$$A^\perp = \{x \in \mathcal{H} \mid x \perp y \text{ for all } y \in A\}.$$

Let $\mathcal{M}$ be a closed linear subspace of $\mathcal{H}$.

(a) Show that for each $x \in \mathcal{H}$ there is a unique $y \in \mathcal{M}$ such that $\|x-y\| = \inf_{w\in\mathcal{M}} ||x-w\|$.

(b) Show that $y \in \mathcal{M}$ is the unique element such that $(x - y) \perp w$ for all $w \in \mathcal{M}$.

(c) Show that for every $x \in \mathcal{H}$ we can uniquely decompose $x = y + z$ for $y \in \mathcal{M}$ and $z \in \mathcal{M}^\perp$.

2. Let $\Omega \subset \mathbb{R}^d$ be a volume with boundary $\partial\Omega$. Assume Ω is smooth enough to apply the divergence theorem.

(a) Use the divergence theorem to show that

$$\frac{1}{d}\int_{\partial\Omega} n \cdot r \, \mathrm{d}S = |\Omega|,$$

where r is the position vector, n is the outward-facing unit normal, and $|\Omega|$ is the volume of Ω.

(b) Let $S \subset \mathbb{R}^3$ be the surface of a sphere of radius R. You are told that the area of S is $4\pi R^2$. Use the expression in (a) to find the volume of the ball enclosed by S.

3. Let V be a real vector space. A bilinear form $a(u, v)$ is said to be *skew-symmetric* if $a(u, v) = -a(v, u)$. It is said to be *alternating* if $a(u, u) = 0$ for all $u \in V$.

(a) Show that every bilinear form on V may be written uniquely as the sum of a symmetric bilinear form and a skew-symmetric bilinear form.

(b) Show that a bilinear form on V is alternating if and only if it is skew-symmetric.

4. Let V be a real vector space and let (u, v) be an inner product on V. Let $\|\cdot\|$ be the induced norm. By considering the expansion of $\|u + v\|^2$, express the inner product purely in terms of norms.

Remark: this is the polarisation identity, and may be used to recover the inner product on the dual of a Hilbert space. (Recall that we only defined the norm on the dual space in lectures.)

5. Suppose that $\Omega \subset \mathbb{R}$ is bounded and that $1 \leq p \leq q \leq \infty$. We know from Theorem 2.5.10 of the notes that $L^q(\Omega) \subseteq L^p(\Omega)$.

(a) Give an example to show that the inclusion is strict if $p < q$.

(b) Give an example to show that the inclusion is false if Ω is not bounded.

6. Suppose V is a Banach space, and that it has a coercive bounded bilinear form $a : V \times V \to \mathbb{R}$. Show that V is in fact a Hilbert space.

Remark: this shows that coercivity is *essentially* a property of Hilbert spaces; if a Banach space has a coercive bounded bilinear form it is a Hilbert space in disguise. Later we will study more general conditions of well-posedness that apply to Banach spaces that are genuinely not Hilbert spaces.