Finite Element Methods. QS 2

Hand-in deadline: 9.00 am, 11 Feb, week 5.
Class: 13 Feb, week 5.

1. Why is

$$u(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

not in $W^{1,p}(-1,1)$ for any $1 \le p \le \infty$?

2. Consider the problem

$$-u'' - k^2 u = f, \quad u(0) = 0 = u(\pi)$$

where $f \in L^2(\Omega)$ and $k^2 \in \mathbb{R}$.

(a) Cast the problem in variational form, stating carefully the spaces employed.

(b) For what values of k is the problem not well-posed? (Hint: take $f = 0$ and look for nonzero solutions.)

(c) For small values of k, the problem is coercive. For large values of k, the problem is not coercive. For what value of k does the problem lose coercivity?

3. Consider the variational formulation: find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \alpha \int_{\partial\Omega} uv \,\mathrm{d}s = \int_\Omega vf \,\mathrm{d}x \quad \text{for all } v \in H^1(\Omega),$$

where $f \in L^2(\Omega)$ and real $\alpha > 0$.

(a) Show that a solution u satisfying the Poisson equation $-\nabla^2 u = f$ in Ω and a Robin condition $\alpha u + \frac{\partial u}{\partial n} = 0$ on the boundary $\partial\Omega$ also satisfies the above weak formulation.

(b) Show that $c(u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \alpha \int_{\partial\Omega} uv \,\mathrm{d}s$ defines an inner product over $H^1(\Omega)$, and hence establish that a solution of the weak fomulation is uniquely defined.

4. Let $a(u,v) = \int_0^1 (u'v' + u'v + uv)\,\mathrm{d}x$ and $V = \{v \in H^1(\Omega) \mid v(0) = v(1) = 0\}$. Show that $a(v,v) = \int_0^1 [(v')^2 + v^2]\,\mathrm{d}x$ for all $v \in V$. (Hint: write $vv' = \frac{1}{2}(v^2)'$.)

5. (a) Suppose that Ω has boundary Γ, and that $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$.

Consider the quadratic functional $J : v \in H^1(\Omega) \mapsto J(v) \in \mathbb{R}$ defined by

$$J(v) = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 + v^2\right)\, \mathrm{d}x + \frac{1}{2}\int_\Gamma v^2 \,\mathrm{d}s - \int_\Omega f v \,\mathrm{d}x - \int_\Gamma g v \,\mathrm{d}s.$$

(i) Show that if $u \in H^1(\Omega)$ is such that $J(u) \le J(v)$ for all $v \in H^1(\Omega)$, then there exist a bilinear functional $a(\cdot,\cdot)$ defined on $H^1(\Omega) \times H^1(\Omega)$ and a linear functional $F(\cdot)$ defined on $H^1(\Omega)$, such that

$$a(u, v) = F(v) \quad \forall v \in H^1(\Omega). \tag{P}$$

(ii) State the strong form of the PDE.

(b)

(i) Given a subspace $V_h \subset H^1(\Omega)$, state the Galerkin approximation of P.
(ii) Show that the Galerkin approximation is equivalent to a linear system, and give formulae for the associated matrix and vector.
(iii) Show that the associated linear system has a unique solution u_h.
(iv) Show that $J(u) \le J(u_h) \le J(v_h)$ for all $v_h \in V_h$.

6. Let $\Omega = (0,1)^2$ with boundary $\partial\Omega$. Let $\Gamma_D = \{(x, y) \in \partial\Omega : x = 0\}$, and let $\Gamma_N = \partial\Omega \backslash \Gamma_D$. Consider the boundary value problem

$$\begin{aligned} -\nabla^2 u + 2u + \sin(x) u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ \nabla u \cdot n &= 0 \quad \text{on } \Gamma_N, \end{aligned}$$

where n denotes the outward unit normal to Γ_N.

Write this boundary value problem as:
(i) a variational problem over a function space V, to be defined;
(ii) the minimisation of an energy functional $J : V \to \mathbb{R}$;
(iii) an equation over elements of V^*, carefully introducing any necessary operators;
(iv) an equation over elements of V.

7. (a) Formulate the following differential equation as a variational problem on $H_0^1(0,1)$:

$$-u'' + u' + u = f, \quad u(0) = 0 = u(1).$$

(b) Show that the bilinear form from this variational problem is coercive and bounded.

(c) State and prove Céa's Lemma.

(d) Let V_h be the continuous piecewise linear finite element space corresponding to a subdivision of $[0,1]$ with maximum width h. Let u_h be the solution to the Galerkin approximation of the variational problem using V_h, and let $I_h : H^2(0,1) \to V_h$ be the interpolation operator onto V_h. Assuming $u \in H^2(0,1)$, and the following result,

$$\|u - I_h u\|_{H^1(0,1)} \leq h\|u''\|_{L^2(0,1)},$$

show that

$$\|u - u_h\| \leq Dh\|u''\|_{L^2(0,1)},$$

and provide a numerical value for D.