Problem Sheet 1 with Hints

1st class, of 4. QUESTIONS 1,2 of PART I.

Note

This hand-out is intended to be a guide for when you find yourself stuck on one of the prescribed problems: it is not intended to be a replacement that you hand in to your tutor or TA. This sheet will provide a general overview of how to obtain the answer for the harder parts; however, it will be up to you to complete as necessary.

To ultimately obtain the best understanding of the lecture material, give a strong attempt on the problem set before consulting these notes.

1) Radius of Gyration, s.

$$s^2 = \frac{1}{(N+1)^2} \sum_{0 \leq i < j \leq N} \underline{r}_{ij}^2$$

$\underline{r}_{ij} = \underline{R}_j - \underline{R}_i$, where

$$s^2 := \frac{1}{N+1} \sum_{i=0}^{N} \underline{s}_i^2$$

with $\underline{s}_i = \underline{R}_i - \underline{S}$ and $\underline{S} = \frac{1}{N+1} \sum_{i=0}^{N} \underline{R}_i$

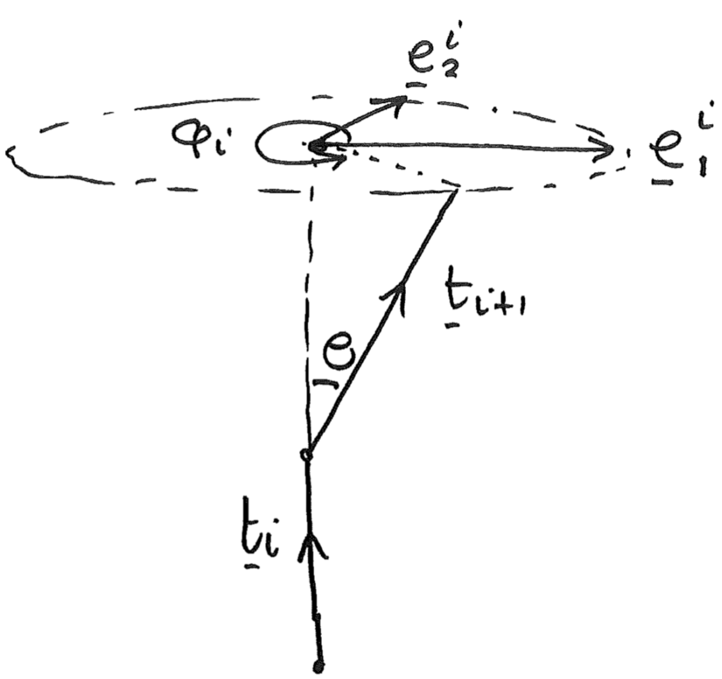
$$\therefore s^2 = \frac{1}{N+1} \sum_{i=0}^{N} (\underline{R}_i - \underline{S})^2 = \frac{1}{N+1} \sum_{i=0}^{N} (\underline{R}_i^2 - 2\underline{R}_i \cdot \underline{S} + \underline{S}^2)$$

$$= \frac{1}{N+1} \left\{ (N+1)\underline{S}^2 - 2\underline{S} \cdot \underbrace{\sum_{i=0}^{N} \underline{R}_i}_{(N+1)\cdot\underline{S}} + \sum_{i=0}^{N} \underline{R}_i^2 \right\}$$

$$= \left(\frac{1}{N+1}\right) \left\{ -(N+1)\underline{S}^2 + \sum_{i=0}^{N} \underline{R}_i^2 \right\} = -\underline{S}^2 + \frac{1}{N+1} \sum_{i=0}^{N} \underline{R}_i^2$$

etc ...

2.



$$t_{i+1} = \cos\theta \, t_i + \sin\theta \left(\cos\varphi_i e_1^i + \sin\varphi_i e_2^i \right)$$

$$t_{i+2} = \cos\theta \, t_{i+1} + \sin\theta \left(\cos\varphi_{i+1} e_1^{i+1} + \sin\varphi_{i+1} e_2^{i+1} \right)$$

$\therefore \quad t_i \cdot t_{i+1} = \cos\theta \quad$ as $\quad t_i \cdot e_1^i = t_i \cdot e_2^i = 0$

by construction

$\therefore \langle t_i \cdot t_{i+1} \rangle = \cos\theta$

$$t_i \cdot t_{i+2} = \cos\theta \, t_i \cdot t_{i+1} + \sin\theta \left(\cos\varphi_{i+1} \, t_i \cdot e_1^{i+1} + \sin\varphi_{i+1} \, t_i \cdot e_2^{i+1} \right)$$

doesn't depend on φ_{i+1} (arrow to t_i)

depend on φ_i but not φ_{i+1} (arrow to e_2^{i+1})

By independence

$$\langle \sin\theta \cos\varphi_{i+1}\, \underline{t}_i \cdot \underline{e}_1^{i+1} \rangle \propto \int_0^{2\pi} d\varphi_{i+1} \cos\varphi_{i+1} = 0$$

$$\langle \sin\theta \sin\varphi_{i+1}\, \underline{t}_i \cdot \underline{e}_2^{i+1} \rangle \propto \int_0^{2\pi} d\varphi_{i+1} \sin\varphi_{i+1} = 0$$

$$\therefore \langle \underline{t}_i \cdot \underline{t}_{i+2} \rangle = \langle \cos\theta \cdot \cos\theta \rangle = \cos^2\theta.$$

<u>Induct</u>

← doesn't depend on φ_{i+n-1}

$$\langle \underline{t}_i \cdot \underline{t}_{i+n} \rangle = \langle \underline{t}_i \cdot \{ \cos\theta\, \underline{t}_{i+n-1} + \sin\theta (\cos\varphi_{i+n-1} \underline{e}_1^{i+n-1} + \sin\varphi_{i+n-1} \underline{e}_2^{i+n-1}) \} \rangle$$

$$= \cos\theta \langle \underline{t}_i \cdot \underline{t}_{i+n-1} \rangle + \{ \dots \} \int_0^{2\pi} d\varphi_{i+n-1} \cos\varphi_{i+n-1} + \{ \dots \} \int_0^{2\pi} d\varphi_{i+n-1} \sin\varphi_{i+n-1}$$

$$= \underline{\cos^n\theta}.$$

$$\therefore \langle \underline{r}_i \cdot \underline{r}_{i+n} \rangle = b^2 \cos^n\theta$$

$$\langle \underline{R}^2 \rangle = \left\langle \sum_{i=1}^{N} \underline{r}_i \cdot \sum_{j=1}^{N} \underline{r}_j \right\rangle = \langle \sum_{i=1}^{N} \underline{r}_i^2 \rangle + \sum_{\substack{i=1\\ j=1\\ i\neq j}}^{N} \langle \underline{r}_i \cdot \underline{r}_j \rangle$$

$$= Nb^2 + 2 \sum_{i=1}^{N} \underbrace{\sum_{j=1}^{i-1}}_{\text{i.e. } j<i} \langle \underline{r}_i \cdot \underline{r}_j \rangle$$

$$\underbrace{\langle \underline{r}_j \cdot \underline{r}_{j+(i-j)} \rangle}_{(\cos\theta)^{i-j}} \quad i-j>0$$

$$= Nb^2 + 2b^2 \sum_{i=1}^{N} \underbrace{\sum_{j=1}^{i-1} (\cos\theta)^{i-j}}_{\sum_{p=1}^{i-1} (\cos\theta)^p}$$

Let $p = i-j$
$j=1 \; p=i-1$
$j=i-1, \; p=1$

$c_\theta + c_\theta^2 + \dots + c_\theta^{i-1}$

$$= Nb^2 + 2b^2 \sum_{i=1}^{N} \cos\theta \left(\frac{1-\cos^{i-1}\theta}{1-\cos\theta} \right)$$

$$= Nb^2 + b^2 \left[\frac{2\cos\theta}{1-\cos\theta} \sum_{i=1}^{N} (1-\cos^{i-1}\theta) \right]$$

$$= Nb^2 + b^2 \left[\frac{2N\cos\theta}{1-\cos\theta} - \frac{2\cos\theta}{1-\cos\theta} \underbrace{\sum_{i=1}^{N} \cos^{i-1}\theta}_{\frac{1-\cos^N\theta}{1-\cos\theta}} \right]$$

$$= Nb^2 + \left[\frac{2N\cos\theta}{1-\cos\theta} - \frac{2\cos\theta(1-\cos^N\theta)}{(1-\cos\theta)^2} \right] b^2$$

$$\langle s^2 \rangle = \frac{1}{(N+1)^2} \sum_{0 \le i < j \le N} \langle r_{ij}^2 \rangle$$ Starts from zero.

$$\langle R^2 \rangle_{N,\ \text{start from } 0} = \langle R^2 \rangle_{N+1,\ N \text{ upper limit, } 1 \text{ lower limit}} = \frac{b^2(1+\alpha)(1+N)}{1-\alpha} - \frac{2b^2\alpha}{(1-\alpha)^2}\left(1-\alpha^{N+1}\right)$$

with $\alpha := \cos\theta$.

$$\langle r_{ij} \rangle^2 = \langle R_j - R_i \rangle^2 = \langle R^2 \rangle_{j-i,\ \text{starting from } 1}$$

eg $\langle r_{i\,i+2}^2 \rangle = \langle (R_{i+2} - R_i)^2 \rangle = \langle (r_1 + r_2)^2 \rangle$

etc ...

$= \langle R^2 \rangle_{2,\ \text{starting from } 1}$.

$$\therefore \langle s^2 \rangle = \frac{1}{(N+1)^2} \sum_{0 \le i < j \le N} \langle r_{ij}^2 \rangle = \frac{1}{(N+1)^2} \sum_{0 \le i < j \le N} \langle R^2 \rangle_{j-i}$$

etc ...

Problem Sheet 2

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For class 2.
QUESTIONS 5,7,8,9 of PART I,
QUESTION 1 of PART II.

Q5. Radius of Gyration of the Wormlike Chain Model

See next page.

Radius of Gyration WLC

$$\underline{r}_{ij} = \underline{R}_j - \underline{R}_i = \sum_{p=i}^{j} \underline{r}_p$$

$$\langle s^2 \rangle \overset{\text{Lagrange}}{=} \frac{1}{(N+1)^2} \sum_{0 \le i \le j \le N} \langle \underline{r}_{ij}^2 \rangle$$

$$= \frac{1}{(N+1)^2} \sum_{0 \le i \le j \le N} \sum_{p=i}^{j} \sum_{q=i}^{j} \langle \underline{r}_p \cdot \underline{r}_q \rangle$$

etc ...

Q7. Static Kirchhoff Equations in the Local Basis

Below stress free curvature and twist, $\widehat{u}_2$, $\widehat{u}_3$ *in the online Bio-filaments notes, are set equal to zero.* Before we begin to manipulate the static equivalents to equations (71) and (72) from these notes, let us calculate the spatial derivatives of each of the basis vectors. This is done by introducing a strain vector, $\mathbf{u}(s)$, with components $(\mathsf{u}_1(s), \mathsf{u}_2(s), \mathsf{u}_3(s))$ that depend on the arc parameter, s, and using equation (49):

$$\frac{\partial \boldsymbol{d}_1}{\partial s} = \mathsf{u}_3 \boldsymbol{d}_2 - \mathsf{u}_2 \boldsymbol{d}_3 \quad (1)$$

$$\frac{\partial \boldsymbol{d}_2}{\partial s} = \mathsf{u}_1 \boldsymbol{d}_3 - \mathsf{u}_3 \boldsymbol{d}_1. \quad (2)$$

$$\frac{\partial \boldsymbol{d}_3}{\partial s} = \mathsf{u}_2 \boldsymbol{d}_1 - \mathsf{u}_1 \boldsymbol{d}_2 \quad (3)$$

In calculating this, what has been about the basis vectors, and thus the continuum rod.

Let us now take the static version of equation (71) from the notes and substitute $\boldsymbol{n} = n_1\boldsymbol{d}_1 + n_2\boldsymbol{d}_2 + n_3\boldsymbol{d}_3$, where each of our components depends on s:

$$\frac{d\boldsymbol{n}}{ds} + \boldsymbol{f} = \frac{d}{ds}\left(n_1\boldsymbol{d}_1 + n_2\boldsymbol{d}_2 + n_3\boldsymbol{d}_3\right) + \boldsymbol{f} = 0. \quad (4)$$

Using our results from equation (1), (2) and (3), we can calculate the derivative term in equation (4):

$$\begin{aligned}\frac{d}{ds}\left(n_1\boldsymbol{d}_1 + n_2\boldsymbol{d}_2 + n_3\boldsymbol{d}_3\right) &= \frac{dn_1}{ds}\boldsymbol{d}_1 + n_1\frac{\partial \boldsymbol{d}_1}{\partial s} + \frac{dn_2}{ds}\boldsymbol{d}_2 + n_2\frac{\partial \boldsymbol{d}_2}{\partial s} + \frac{dn_3}{ds}\boldsymbol{d}_3 + n_3\frac{\partial \boldsymbol{d}_3}{\partial s} \\ &= \left(\frac{dn_1}{ds} - n_2\mathsf{u}_3 + n_3\mathsf{u}_2\right)\boldsymbol{d}_1 + \left(\frac{dn_2}{ds} + n_1\mathsf{u}_3 - n_3\mathsf{u}_1\right)\boldsymbol{d}_2 + \left(\frac{dn_3}{ds} - n_1\mathsf{u}_2 + n_2\mathsf{u}_1\right)\boldsymbol{d}_3, \quad (5)\end{aligned}$$

remembering that the components of our strain vector were space-dependent.

Lastly, by writing $\boldsymbol{f} = (f_1, f_2, f_3)$ in terms of the director basis, we substitute equation (5) into equation (4) and then extract the $\boldsymbol{d}_1$, $\boldsymbol{d}_2$ and $\boldsymbol{d}_3$ components:

$$\frac{dn_1}{ds} - n_2\mathsf{u}_3 + n_3\mathsf{u}_2 + f_1 = 0 \quad (6)$$

$$\frac{dn_2}{ds} + n_1\mathsf{u}_3 - n_3\mathsf{u}_1 + f_2 = 0 \quad (7)$$

$$\frac{dn_3}{ds} - n_1\mathsf{u}_2 + n_2\mathsf{u}_1 + f_3 = 0. \quad (8)$$

We now apply a similar process to the static version of equation (72) from the bio-filaments notes. To close our system of equations, we use the linear constitutive relation for the unstressed reference configuration, as given in equation (73) of the bio-filaments notes, and evaluate using our results from equation (1), (2) and (3) again:

etc ...

Q8. Axon Injury

We begin with the following equation:

$$B\frac{d^4W}{dx^4} + P\frac{d^2W}{dx^2} + kW = 0, \tag{13}$$

and non-dimensionalize it by including a characteristic length-scale, L, and non-dimensionalized variable, $x = XL$. In this instance, we find:

$$\frac{d^4w}{dX^4} + \lambda\frac{d^2w}{dX^2} + \beta w = 0, \tag{14}$$

where $\lambda = \frac{PL^2}{B}$ and $\beta = \frac{kL^4}{B}$ are non-dimensionalized parameters and we have new boundary conditions $w(\pm1) = 0$ and $\frac{dw}{dX}|_{X=\pm1} = 0$.

Using the ansatz $w(X) = e^{i\omega X}$ in the above equation, we find:

$$\omega^4 - \lambda\omega^2 + \beta = 0. \tag{15}$$

Upon using the discriminant (by letting $\alpha = \omega^2$), real solutions will be obtained provided that $\lambda^2 - 4\beta > 0$. This corresponds to the non-trivial solution required in the question, however, you are invited to think about why that might be. What would the solution be if ω is real? What if ω is a repeated root? Imaginary? Hence why is it necessary that $\lambda^2 - 4\beta > 0$?

We can solve the characteristic equation for ω to find 4 real solutions: ω_+, ω_-, $-\omega_+$ and $-\omega_-$, where:

$$\omega_+ = \sqrt{\frac{\lambda + \sqrt{\lambda^2 - 4\beta}}{2}} \tag{16}$$

$$\omega_- = \sqrt{\frac{\lambda - \sqrt{\lambda^2 - 4\beta}}{2}}. \tag{17}$$

As such, the general solution to the non-dimensionalized equation is given by:

$$w(X) = A\cos(\omega_+X) + B\cos(\omega_-X) + C\sin(\omega_+X) + D\sin(\omega_-X), \tag{18}$$

for A, B (not to be confused with the bending stiffness from equation (13)), C and D being arbitrary constants to determine with our boundary conditions. Doing this yields two possible scenarios:

$$w(X) = A_0\left(\frac{\cos(\omega_+X)}{\cos(\omega_+)} - \frac{\cos(\omega_-X)}{\cos(\omega_-)}\right), \tag{19}$$

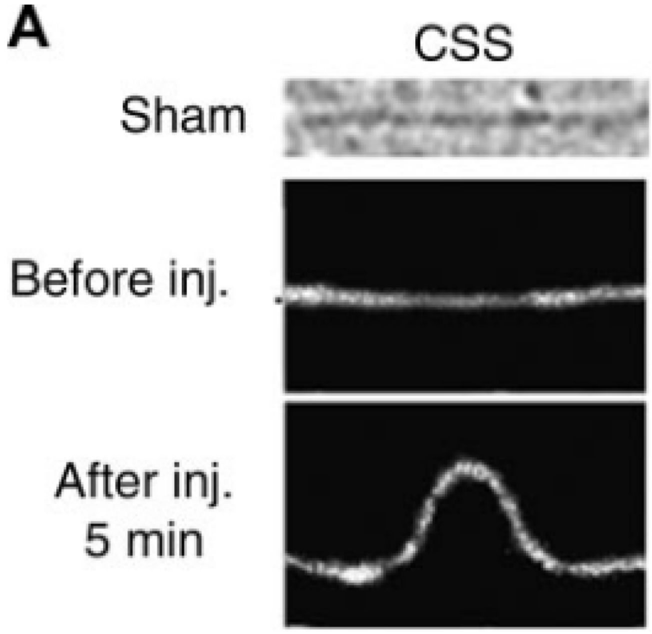
whereby $A_0 = A\cos(\omega_+)$ and where we have the relationship $\omega_+\tan(\omega_+) = \omega_-\tan(\omega_-)$ that must be satisfied, or, alternatively:

$$w(X) = C_0\left(\frac{\sin(\omega_+X)}{\sin(\omega_+)} - \frac{\sin(\omega_-X)}{\sin(\omega_-)}\right), \tag{20}$$

where $B_0 = B\sin(\omega_+)$ and where the relationship $\omega_+\cot(\omega_+) = \omega_-\cot(\omega_-)$ must be satisfied.

Note that A_0 and C_0 are still undefined even after using the boundary conditions. In a biological context, this doesn't make sense; an axon cannot be stretched to infinity. What further constraints can we use to find A_0 or C_0?

Additionally, you may have noticed that P is not defined in the question on the problem sheet. We can, however, solve for P using the relationships $\omega_+\tan(\omega_+) = \omega_-\tan(\omega_-)$ or $\omega_+\cot(\omega_+) = \omega_-\cot(\omega_-)$, but there are infinitely many solutions in this instance. What other constraint is needed to restrict the values P can possibly take?



Find the conditions for which a non-trivial solution occurs and test your solution with the physiological values $B = 6 \times 10^{-19}\text{Nm}^2$, $k = 12\text{Nm}^{-2}$ and $L = 15\mu\text{m}$. Compare your solution to the profile of Fig. 2 and Fig. 3 of the paper [Min D. Tang-Schomer, Ankur R. Patel, Peter W. Baas, and Douglas H. Smith. *Mechanical breaking of microtubules in axons during dynamic stretch injury underlies delayed elasticity, microtubule disassembly, and axon degeneration.* The FASEB Journal, 24(5):1401–1410, 2010.]

Q9. Derivation of Beam Equation

We begin with the general equations for a rod confined to planar motion:

$$\frac{\partial F}{\partial s} + f = \rho A \frac{\partial^2 x}{\partial t^2} \tag{21}$$
$$\frac{\partial G}{\partial s} + g = \rho A \frac{\partial^2 y}{\partial t^2} \tag{22}$$
$$EI\frac{\partial^2 \theta}{\partial s^2} + G\cos\theta - F\sin\theta = \rho I \frac{\partial^2 \theta}{\partial t^2}, \tag{23}$$

where:

$$\frac{\partial x}{\partial s} = \cos\theta \tag{24}$$
$$\frac{\partial y}{\partial s} = \sin\theta. \tag{25}$$

We use the hint from the problem sheet; namely, to consider $\theta \ll 1$. In this case, we can consider the corresponding asymptotic behavior of equation (24) and (25) in the limit of $\theta \to 0$:

$$\frac{\partial x}{\partial s} \sim 1 \Rightarrow x = s \tag{26}$$
$$\frac{\partial y}{\partial s} \sim \theta. \tag{27}$$

So equation (23) becomes:

$$EI\frac{\partial^3 y}{\partial x^3} + G - F\theta = 0, \tag{28}$$

however, differentiating this with respect to x, we find:

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^3 y}{\partial x^3}\right) + \frac{\partial G}{\partial x} - \theta\frac{\partial F}{\partial x} - F\frac{\partial \theta}{\partial x} = \frac{\partial}{\partial x}\left(EI\frac{\partial^3 y}{\partial x^3}\right) - g + f\frac{\partial y}{\partial x} - F\frac{\partial^2 y}{\partial x^2} = 0, \tag{29}$$

where have used equation (27) and the time-independent forms of equation (21) and equation (22).

Introducing $y(x) = w(x)$, our final result is:

$$\frac{\partial}{\partial x}\left(EI\frac{\partial^3 w}{\partial x^3}\right) - F\frac{\partial^2 w}{\partial x^2} + f\frac{\partial w}{\partial x} - g = 0, \tag{30}$$

or, if we consider the scenario where there are no loads on our beam and EI is constant (for what sort of materials or rods would this be a good approximation?), then equation (30) reduces to:

$$EI\frac{\partial^4 w}{\partial x^4} - F\frac{\partial^2 w}{\partial x^2} = 0, \tag{31}$$

which is the classical Euler-Bernoulli equation for a beam.

Considering the possible boundary conditions we can impose on our system, clamped boundary conditions correspond to not only fixing the position of the beam at its ends, but also its direction. In other words, $w(x=0)$, $w(x=L)$, $w'(x=0)$ and $w'(x=L)$ are specified. Meanwhile, for a pinned beam, the positions are fixed but are done so in a way where the beam is still free to rotate. In this instance, $w(x=0)$ and $w(x=L)$ are specified along with the condition $EI\frac{d^2w}{dx^2} = 0$. Where does this latter condition come from? Are these boundary conditions sufficient to fully solve the system?

Part II. Question 1 - Invariance of Arclength and Area

As with most questions requiring a proof, there are multiple ways to derive the desired result. Again, you are encouraged to work out a method that makes sense to you, however, we will begin by defining some notation:

Under our original parametrization, we define our surface metric to be g_{ij} with corresponding displacement coordinates ξ^1 and ξ^2. After a change of parametrization, these now become $g_{ij}^{\dagger}$, $\left(\xi^1\right)^{\dagger}$ and $\left(\xi^2\right)^{\dagger}$ respectively. However, to do this transformation, we define a Jacobian, J:

$$J = \begin{bmatrix} \frac{\partial(\xi^1)^{\dagger}}{\partial\xi^1} & \frac{\partial(\xi^1)^{\dagger}}{\partial\xi^2} \\ \frac{\partial(\xi^2)^{\dagger}}{\partial\xi^1} & \frac{\partial(\xi^2)^{\dagger}}{\partial\xi^2} \end{bmatrix}, \tag{32}$$

which we can use as follows to change our coordinates and metric:

$$\left(\xi^i\right)^{\dagger} = J\xi^i \tag{33}$$

$$g_{ij}^{\dagger} = J^T g_{ij} J, \tag{34}$$

where (^{T}) is the transpose.

To prove arclength invariance, it is sufficient to show that $\left(ds^2\right)^{\dagger} = ds^2$. So, beginning with the left hand side:

$$\left(ds^2\right)^{\dagger} = \left[\left(\xi^i\right)^{\dagger}\right]^T g_{ij}^{\dagger} \left(\xi^j\right)^{\dagger} = \left[\left(\xi^i\right)^{\dagger}\right]^T J^T g_{ij} J \left(\xi^j\right)^{\dagger} = \left(\xi^i\right)^T g_{ij}\xi^j = ds^2, \tag{35}$$

where we have used our previous results given in equation (33) and (34).

Let us now prove the invariance of area:

etc ...

Problem Sheet 3

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For class 3.
QUESTIONS 2, 6, 8, 9 of PART II.

Question 2 - Orthogonality

Please see next page

Sheet 2, Q2

$$\langle \bar{v}, Lv \rangle = \langle \bar{v}, G^{-1}Kv \rangle = \bar{v}^T G G^{-1} K v = \bar{v}^T K v,$$

with K symmetric.

$\therefore \quad Lv = \lambda v, \quad L\bar{v} = \bar{\lambda}\bar{v}, \quad v \not\equiv 0.$

$$\therefore \quad \langle \bar{v}, Lv \rangle - \langle v, L\bar{v} \rangle = \lambda \langle \bar{v}, v \rangle - \bar{\lambda} \langle v, \bar{v} \rangle$$

$$= \lambda \bar{v}^T G v - \bar{\lambda} v^T G \bar{v}$$

$$\therefore \quad \underbrace{\bar{v}^T K v - v^T K \bar{v}}_{\text{Zero as K symmetric}} = (\lambda - \bar{\lambda}) \bar{v}^T G v, \text{ noting } G \text{ is symmetric}$$

$\bar{v}^T G v$ → Not zero as G positive definite and $v \not\equiv 0$.

$\therefore \lambda = \bar{\lambda}$, eigvals real.

$\therefore$ v, eigenvectors real.

Orthogonality

etc ...

Question 6 - The Slightly Deformed Sphere

It is highly recommended that you use a symbolic algebra package like Mathematica or Maple to help you complete this problem; the algebra becomes messy very quickly. There are some similarities in computing the curvatures as in question 4, however, the main difference will be consistently working to first order in ϵ.

To begin with, we start with our position vector, defined as:

$$\boldsymbol{x} = R(1 + \epsilon h(\theta, \phi) \left\{\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)\right\}, \tag{1}$$

find the corresponding tangent vectors, $\boldsymbol{r}_\theta$ and $\boldsymbol{r}_\phi$:

$$\begin{aligned}\boldsymbol{r}_\theta = [&R\cos(\phi)\left(\epsilon\sin(\theta)h^{(1,0)}(\theta,\phi) + \epsilon\cos(\theta)h(\theta,\phi) + \cos(\theta)\right), \\ &R\sin(\phi)\left(\epsilon\sin(\theta)h^{(1,0)}(\theta,\phi) + \epsilon\cos(\theta)h(\theta,\phi) + \cos(\theta)\right), \\ &R\epsilon\cos(\theta)h^{(1,0)}(\theta,\phi) - R\sin(\theta)(\epsilon h(\theta,\phi) + 1)],\end{aligned} \tag{2}$$

$$\begin{aligned}\boldsymbol{r}_\phi = [&R\sin(\theta)\left(\epsilon\cos(\phi)h^{(0,1)}(\theta,\phi) - \sin(\phi)(\epsilon h(\theta,\phi) + 1)\right), \\ &R\sin(\theta)\left(\epsilon\sin(\phi)h^{(0,1)}(\theta,\phi) + \epsilon\cos(\phi)h(\theta,\phi) + \cos(\phi)\right), \\ &R\epsilon\cos(\theta)h^{(0,1)}(\theta,\phi)],\end{aligned} \tag{3}$$

and compute the unit normal as we did previously:

$$\boldsymbol{n} = \frac{\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi}{\|\boldsymbol{r}_\theta \times \boldsymbol{r}_\phi\|}. \tag{4}$$

Using equations (14) and (15), we can calculate our first fundamental form, G, however, we only work up to $O(\epsilon)$ and neglect lower ordered terms. This can be done in Mathematica by using the "Series" command and expanding in powers of ϵ. To first order, we find:

$$G = \begin{pmatrix} R^2(2\epsilon h(\theta,\phi) + 1) & 0 \\ 0 & R^2(2\epsilon h(\theta,\phi) + 1)\sin^2(\theta) \end{pmatrix}. \tag{5}$$

Calculating the second fundamental form, K, again up to $O(\epsilon)$ only, we find that the entries of K are given by:

$$K_{11} = R\left(\epsilon h(\theta,\phi) - \epsilon h^{(2,0)}(\theta,\phi) + 1\right) \tag{6}$$

$$K_{12} = K_{21} = \frac{R\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{\sin(\theta)} \tag{7}$$

$$K_{22} = R\left(\epsilon h(\theta,\phi)\sin^2(\theta) + \left(\sin(\theta) - \epsilon\cos(\theta)h^{(1,0)}(\theta,\phi)\right)\sin(\theta) - \epsilon h^{(0,2)}(\theta,\phi)\right). \tag{8}$$

where $h^{(1,0)}(\theta,\phi) \equiv \frac{\partial h}{\partial \theta}$, $h^{(0,1)}(\theta,\phi) \equiv \frac{\partial h}{\partial \phi}$ and so forth.

Lastly, we obtain the entries of the principal curvature matrix, $L = G^{-1}K$ by combining our first order results from equations (17) to (20). We again only take terms to $O(\epsilon)$:

$$
\begin{aligned}
L_{11} &= \frac{1}{R} - \frac{\epsilon\left(h(\theta,\phi) + h^{(2,0)}(\theta,\phi)\right)}{R} && (9)\\
L_{12} &= \frac{\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{R\sin(\theta)} && (10)\\
L_{21} &= \frac{\epsilon\left(\cos(\theta)h^{(0,1)}(\theta,\phi) - \sin(\theta)h^{(1,1)}(\theta,\phi)\right)}{R\sin^3(\theta)} && (11)\\
L_{22} &= \frac{1}{R} - \frac{\epsilon\left(h^{(0,2)}(\theta,\phi)\csc^2(\theta) + h(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi)\right)}{R}. && (12)
\end{aligned}
$$

Computing the Gaussian curvature, K_G, and mean curvature, H, to first order, we find:

$$
\begin{aligned}
H &= \frac{\mathrm{tr}(L)}{2} = \frac{1}{R} - \frac{\epsilon\left(\left(h^{(2,0)}(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi) + \csc^2(\theta)h^{(0,2)}(\theta,\phi) + 2h(\theta,\phi)\right)\right)}{2R} && (13)\\
K_G &= \det(L) = \frac{1}{R^2} - \frac{\epsilon\left(h^{(2,0)}(\theta,\phi) + \cot(\theta)h^{(1,0)}(\theta,\phi) + \csc^2(\theta)h^{(0,2)}(\theta,\phi) + 2h(\theta,\phi)\right)}{R^2}, && (14)
\end{aligned}
$$

which are the required results.

A small Mathematica script which does this computation is provided below. Be sure to understand what the program and the commands actually do before just copying it!

Algorithm 1 A script to compute the Mean and Gaussian curvatures of a slightly deformed sphere to $O(\epsilon)$.

```
x = R (1 + ep*h[t, p])*{Cos[p] Sin[t], Sin[p] Sin[t], Cos[t]}
rt = Simplify[D[x, t]];
rp = Simplify[D[x, p]];
n = Simplify[Cross[rt, rp]/Sqrt[Cross[rt, rp].Cross[rt, rp]], Assumptions -> R > 0];
G = Simplify[{{rt.rt, rt.rp}, {rp.rt, rp.rp}}];
G1 = Simplify[Normal[Series[G, {ep, 0, 1}]]];
K = Simplify[{{-n.D[rt, t], -n.D[rt, p]}, {-n.D[rp, t], -n.D[rp, p]}}];
K1 = Simplify[Normal[Series[K, {ep, 0, 1}]]];
L1 = Normal[Series[Inverse[G1].K1, {ep, 0, 1}]];
GaussK1 = Series[Det[L1], {ep, 0, 1}]
H = 0.5*Simplify[Series[Tr[L1], {ep, 0, 1}]]
```

Question 8 - Shape Equation with Fixed Pressure

We are asked to show that for a membrane under fixed pressure, P, the shape equation, under the small gradient approximation, is given by:

$$\kappa\triangle\triangle h - \gamma\triangle h = P. \tag{15}$$

The idea is that the total energy of the membrane configuration does not only have a contribution from the approximate bending energy, given by equation (35) of the bio-membranes notes:

$$\varepsilon_2 = \frac{1}{2} \quad dxdy \left[\kappa\left(\triangle h\right)^2 + \gamma\left(\nabla h\right)^2\right], \tag{16}$$

but also a contribution from the work done by pressure, given by:

$$\varepsilon_P = -PV = -P \quad {}_{\Omega} \; dV, \tag{17}$$

where the pressure is assumed to be constant and Ω is the integral over the volume of the membrane.

Using the hint from the problem sheet, we introduce the divergence of the position vector, $\boldsymbol{r} = (x, y, h(x, y))$, into equation (29):

$$\varepsilon_P = -P \quad {}_{\Omega} \; dV = -\frac{P}{3} \quad {}_{\Omega} \; \nabla\cdot\boldsymbol{r}dV = -\frac{P}{3} \quad {}_{\Sigma} \; \boldsymbol{r}\cdot\boldsymbol{n}dS, \tag{18}$$

where we have used the divergence theorem and are now finding the integral over the area of the membrane, Σ.

As a reminder, the unit normal vector was given by:

$$\boldsymbol{n} = \frac{1}{\sqrt{1+\left(\nabla h\right)^2}}\left(-h_x, -h_y, 1\right), \tag{19}$$

so evaluating equation (30) in the Monge parametrization, we find:

$$\begin{aligned}\varepsilon_P = -\frac{P}{3} \quad dxdy &\left\{\sqrt{1+\left(\nabla h\right)^2}(x, y, h(x,y))\cdot\frac{1}{\sqrt{1+\left(\nabla h\right)^2}}\left(-h_x, -h_y, 1\right)\right\}\\ &= -\frac{P}{3} \quad dxdy\left\{-xh_x - yh_y + h\right\} = \frac{P}{3} \quad dxdy\left\{\boldsymbol{r}_2\cdot\nabla h - h\right\},\end{aligned} \tag{20}$$

where $\boldsymbol{r}_2 = (x, y)$.

The problem now becomes a matter of minimizing the functional:

$$\varepsilon_{Total} = \varepsilon_2 + \varepsilon_P = \frac{1}{2} \quad dxdy\left[\kappa\left(\triangle h\right)^2 + \gamma\left(\nabla h\right)^2\right] + \frac{P}{3} \quad dxdy\left\{\boldsymbol{r}_2\cdot\nabla h - h\right\}. \tag{21}$$

Let us minimize ε_P by considering the first variation with respect to h:

$$\delta\varepsilon_P = \frac{P}{3} \quad dxdy\left\{\boldsymbol{r}_2\cdot\nabla\left(\delta h\right) - \delta h\right\}, \tag{22}$$

however, to remove the $\nabla\left(\delta h\right)$ term, we realize that $\nabla\cdot\left(\boldsymbol{r}_2\delta h\right) = \left(\nabla\cdot\boldsymbol{r}_2\right)\delta h + \boldsymbol{r}_2\cdot\nabla\left(\delta h\right)$, so equation (34) becomes:

$$\delta\varepsilon_P = \frac{P}{3} \quad dxdy\left\{\nabla\cdot\left(\boldsymbol{r}_2\delta h\right) - \left(\nabla\cdot\boldsymbol{r}_2\right)\delta h - \delta h\right\} = \frac{P}{3} \quad dxdy\left\{\nabla\cdot\left(\boldsymbol{r}_2\delta h\right) - 3\delta h\right\}, \tag{23}$$

due to the fact that $\nabla\cdot\boldsymbol{r}_2 = 2$ from our previous definition.

Let us minimize ε_2 now:

$$\delta\varepsilon_2 = \frac{1}{2} \quad dxdy \left\{ 2\kappa\triangle h \triangle \left(\delta h\right) + 2\gamma \nabla h \cdot \nabla \left(\delta h\right) \right\}, \tag{24}$$

however, using a similar trick that allowed us to go from equation (34) to equation (35), we find that, upon substituting $\nabla \cdot \left(\triangle h \nabla \left(\delta h\right)\right) = \nabla \triangle h \cdot \nabla \left(\delta h\right) + \triangle h \triangle \left(\delta h\right)$ and $\nabla \cdot \left(\nabla h \delta h\right) = \triangle h \delta h + \nabla h \cdot \nabla \left(\delta h\right)$, we obtain:

$$\delta\varepsilon_2 = \quad dxdy \left\{ -\kappa \nabla \triangle h \cdot \nabla \left(\delta h\right) - \gamma \triangle h \delta h + \nabla \cdot \left[\kappa \triangle h \nabla \left(\delta h\right) + \gamma \nabla h \delta h\right] \right\}. \tag{25}$$

Lastly, substituting $\nabla \cdot \left(\nabla \triangle (h) \delta h\right) = \triangle\triangle h \delta h + \nabla \triangle h \cdot \nabla \left(\delta h\right)$:

$$\delta\varepsilon_2 = \quad dxdy \left\{ \left[\kappa \triangle\triangle h - \gamma \triangle h\right] \delta h + \nabla \cdot \left[\kappa \triangle h \nabla \left(\delta h\right) + \gamma \nabla h \delta h - \kappa \nabla \triangle (h) \delta h\right] \right\}. \tag{26}$$

Combining equation (35) and equation (38), we demand that $\delta\varepsilon_{Total} = 0$, so that we end up with:

$$0 = \quad dxdy \left\{ \left[\kappa \triangle\triangle h - \gamma \triangle h - \frac{3P}{3}\right] \delta h + \nabla \cdot \left[\kappa \triangle h \nabla \left(\delta h\right) + \gamma \nabla h \delta h - \kappa \nabla \triangle (h) \delta h + \frac{P}{3} \boldsymbol{r}_2 \delta h\right] \right\}, \tag{27}$$

so that our shape equation is now given by:

$$\kappa \triangle\triangle h - \gamma \triangle h = P, \tag{28}$$

as required. However, what constraints or boundary conditions do we now need in order for equation (40) to be true?

Question 9 - Shape Equation over a Step Function

We are asked to find the shape of a membrane that is hanging over a step-edge. To do this, we work in $1D$ using the corresponding shape equation derived in lectures:

$$\frac{d^4h}{dx^4} - \frac{1}{\lambda^2}\frac{d^2h}{dx^2} = 0, \tag{29}$$

with boundary conditions given by $h(x = 0) = h_0$, $h(L) = 0$, $h'(0) = 0$ and $h'(L) = 0$.

However, let us introduce the scaling $\hat{x} = \frac{x}{\lambda}$, so that equation (41) now becomes:

$$\frac{d^4h}{d\hat{x}^4} - \frac{d^2h}{d\hat{x}^2} = 0, \tag{30}$$

with new boundary conditions given by: $h(\hat{x} = 0) = h_0$, $h(\hat{L} = \frac{L}{\lambda}) = 0$, $h'(0) = 0$ and $h'(\hat{L}) = 0$.

The general solution of this non-dimensionalized equation is then given by:

$$h(\hat{x}) = A + B\hat{x} + C\sinh(\hat{x}) + D\cosh(\hat{x}), \tag{31}$$

and, by substituting the boundary conditions, we find that the resulting shape can be written in the following form (of course, this is not the only form that it could be written in):

$$h(\hat{x}) = h_0 - h_0\left[\frac{\left(\cosh\left(\hat{L}\right) - 1\right)(\cosh(\hat{x}) - 1) - \sinh\left(\hat{L}\right)(\sinh(\hat{x}) - \hat{x})}{\left(\cosh\left(\hat{L}\right) - 1\right)\left(\cosh\left(\hat{L}\right) - 1\right) - \sinh\left(\hat{L}\right)\left(\sinh\left(\hat{L}\right) - \hat{L}\right)}\right], \tag{32}$$

Plotting this solution for $h_0 = 1$, $L = 1$ and $\lambda = \frac{1}{\sqrt{5}}$, we obtain the following:

We note from our definition of $\hat{x}$ that we have two length scales: L and λ. Plot the solution for various values of L and λ. What do you notice about the solution when $L \gg \lambda$? What about when $\lambda \gg L$? What sort of energy is minimized in both scenarios?

Problem Sheet 4

Eamonn Gaffney.

QUESTIONS 2, 3, 4, 5 of PART III. For class 4, of 4.

Part 3. Question 2. Ciliary Pumping.

Detailed calculation is not necessary to deduce the expression for U. Modes decouple at the first non-trivial order the second derivatives acting on mode number n just induce a factor of n^2. Thus one can determine the contribution to U from the mode

$$x_e - x = \epsilon(-b_n \cos(n[x+t]), \quad y_e = \epsilon c_n \sin(n[x+t]) \tag{1}$$

by identifying

$$a = -nb_n, \quad b = nc_n \tag{2}$$

in the lecture note result

$$U_2 = \frac{1}{2}\left(b^2 + 2ab - a^2\right)$$

to obtain the contribution from this mode. How can one consider the remaining modes without detailed calculation?

The mode

$$x_e - x = \epsilon a_n \sin(n[x+t]), \quad y_e = \epsilon(-d_n \cos(n[x+t]))$$

is simply a phase shift of the mode in equation (1). By considering a shifted time coordinate

$$\bar{t} = t + \frac{1}{\omega n}\frac{\pi}{2}$$

we can determine the contribution from this mode by the subsitution

$$d_n \to c_n, \quad a_n \to -b_n.$$

followed by the identification (2). Hence we use

$$a = na_n, \quad b = nd_n$$

in the lecture note result

$$U_2 = \frac{1}{2}\left(b^2 + 2ab - a^2\right)$$

to obtain the contribution from this mode.

Summing all contributions, and noting $U = \epsilon^2 U_2$ to leading order, gives

$$U = \frac{1}{2}\epsilon^2 \sum_{n=1}^{\infty} n^2[c_n^2 + d_n^2 - a_n^2 - b_n^2 + 2(a_n d_n - c_n b_n)].$$

We now determine power optimal strokes, defined as those maximising absolute velocity, subject to the constraint of a fixed power consumption W using Lagrange multipliers with the above leading order expressions. Thus we consider

$$L[\{a_n, b_n, c_n, d_n\}] = U[\{a_n, b_n, c_n, d_n\}] - \lambda(P[\{a_n, b_n, c_n, d_n\}] - W) \tag{3}$$

and the extremal conditions

$$\frac{\partial L}{\partial a_n} = \frac{\partial L}{\partial b_n} = \frac{\partial L}{\partial c_n} = \frac{\partial L}{\partial d_n} = 0. \tag{4}$$

Thus

$$a_n^2 + 2a_n d_n - d_n^2 = 0\ ,\ b_n^2 - 2b_n c_n - c_n^2 = 0.$$

and hence $a_n = (-1 \pm \sqrt{2})d_n$ and $b_n = (1 \pm \sqrt{2})c_n$, which yields

$$U = \epsilon^2 \sum_{n=1}^{\infty} n^2 \left((2 \pm 2\sqrt{2})c_n^2 + (2 \mp 2\sqrt{2})d_n^2 \right) \tag{5}$$

$$P = \epsilon^2 \sum_{n=1}^{\infty} n^3 \left((4 \pm 2\sqrt{2})c_n^2 + (4 \mp 2\sqrt{2})d_n^2 \right). \tag{6}$$

How might the optimal solution be found?

Therefore the optimal stroke is achieved when $a_n = b_n = c_n = d_n = 0$ for $n \geq 2$. Without loss of generality, we can set $b_1 = 0$ as it is just a phase difference, and we finally have the optimal strokes $a_1 = (-1 \pm \sqrt{2})d_1$ with the extremal velocity,

$$U = \mp \frac{W}{\sqrt{2}}.$$

Part 3. Question 3. Ciliate Motility.

Take a reference frame comoving with the swimmer oriented such that the direction of the swimmer velocity is given by $\mathbf{U} = U\mathbf{e}_z$. The non-dimensional Stokes equations are

$$\nabla^2 \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0,$$

with

$$\mathbf{u} = -U\mathbf{e}_z, \quad \text{as} \quad |\mathbf{x} \to \infty, \qquad \mathbf{u} = \epsilon \frac{\mathrm{d}\beta_1}{\mathrm{d}t} V_1(\cos\theta)\mathbf{e}_\theta = \epsilon\dot{\beta}_1 V_1(\cos\theta)\mathbf{e}_\theta, \quad \text{on} \quad r = 1,$$

where $\mathbf{e}_\theta$ is the unit vector in the direction of increasing spherical polar θ, where $r = |\mathbf{x}|$ and $z = r\cos\theta$, $x = r\sin\theta\cos\varphi$ for instance.

Explain why there is no fluid flow in the $\mathbf{e}_\varphi$ direction.

Show that

$$\mathbf{u} = \left[-U(t) + \frac{Q(t)}{r^3}\right]\cos\theta\mathbf{e}_r + \left[U(t) + \frac{P(t)}{r^3}\right]\sin\theta\mathbf{e}_\theta, \qquad p = \text{Const}$$

is a solution of the Stokes equation for $Q(t) = 2P(t)$. To do this, you will need to consider the vector Lapalacian of $\mathbf{u}$. This is non-trivial in non-Cartesian coordinates and you may wish to consider using a symbolic algebra package such as Mathematica.

```
f1[r_, theta_, phi_]:=(−U + Q/r^3) ∗ Cos[theta]

f2[r_, theta_, phi_]:=(U + P/r^3) ∗ Sin[theta]

FullSimplify[Laplacian[f1[r, theta, phi], {r, theta, phi}, "Spherical"] − 2 ∗ f1[r, theta, phi]/r^2−
2/(r^2 ∗ Sin[theta]) ∗ (f2[r, theta, phi] ∗ Cos[theta] + Sin[theta] ∗ D[f2[r, theta, phi], theta])]
```

$$\frac{2(-2P + Q)\text{Cos[theta]}}{r^5}$$

```
FullSimplify[Laplacian[f2[r, theta, phi], {r, theta, phi}, "Spherical"]+
2 ∗ D[f1[r, theta, phi], theta]/r^2 − 1/(r^2 ∗ Sin[theta] ∗ Sin[theta]) ∗ f2[r, theta, phi]]
```

$$\frac{2(2P - Q)\text{Sin[theta]}}{r^5}$$

Given $Q(t) = 2P(t)$, show that $U(t) = 2\epsilon\dot{\beta}_1/3$.

For the more difficult part of the question, not expected except as an optional exploration of the theory, see J. R. Blake, A spherical envelope approach to ciliary propulsion, J . Fluid Mech. (1971), vol. 46, part 1, pp. 199-208.

Part 3. Question 4. Rotating sphere in Stokes flow.

With $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$, $r = |\mathbf{x} - \mathbf{x}_0|$ and

$$G_{ij} = \frac{\delta_{ij}}{r} + \frac{\hat{x}_i \hat{x}_j}{r^3},$$

the rotational dipole $\mathbf{G}^c$ is defined by

$$G^c_{im} := \frac{1}{2}\epsilon_{mlj}\frac{\partial G_{ij}}{\partial x_{0,l}},$$

where

$$\epsilon_{mlj} := \begin{cases} +1 & \text{if} \quad (m,l,j) = (1,2,3) \text{ or } (3,1,2) \text{ or } (2,3,1) \\ -1 & \text{if} \quad (m,l,j) = (1,3,2) \text{ or } (2,1,3) \text{ or } (3,2,1) \\ 0 & \text{if} \quad \text{any of } i,j,k \text{ are equal} \end{cases}$$

Part a. We have

$$\begin{aligned} \frac{1}{2}\epsilon_{mlj}\frac{\partial}{\partial x_{0,l}}G_{ij} &= \frac{1}{2}\epsilon_{mlj}\frac{\partial}{\partial x_{0,l}}\left[\frac{\delta_{ij}}{r} + \frac{\hat{x}_i\hat{x}_j}{r^3}\right], \\ &= \frac{1}{2}\epsilon_{mlj}\left[\delta_{ij}\frac{\partial}{\partial x_{0,l}}\left(\frac{1}{r}\right) - \frac{1}{r^3}\left(\delta_{lj}\hat{x}_j + \delta_{jl}\hat{x}_i\right) + \hat{x}_i\hat{x}_j\frac{\partial}{\partial x_{0,l}}\left(\frac{1}{r^3}\right)\right], \\ &= \epsilon_{mli}\frac{\hat{x}_l}{r^3}. \end{aligned}$$

Part b. Thus $G^c_{im}q_m$ is a solution of Stokes equations for any constant vector $\mathbf{q}$ and it decays at spatial infinity. With $\mathbf{x}_0$ the centre of the sphere, the sphere is given by $r = a$, where we have

$$a^3 G^c_{im}\Omega_m = a^3\epsilon_{iml}\Omega_m\frac{\hat{x}_l}{r^3} = \epsilon_{iml}\Omega_m\hat{x}_l,$$

which is the velocity on a rotating sphere.

If this is not clear, without loss take $\boldsymbol{\Omega} = \Omega\mathbf{e}_z$, whereupon the velocity on the rotating sphere is, using spherical polars,

$$a\Omega\sin\theta\mathbf{e}_\phi = -a\Omega\sin\theta\sin\varphi\mathbf{e}_x + a\Omega\sin\theta\cos\varphi\mathbf{e}_y$$

and compare with $\Omega\mathbf{e}_3 \wedge \hat{\mathbf{x}}$.

Hence

$$v_i := a^3 G^c_{im}\Omega_m$$

is a solution for the Stokes flow for a rotating sphere with radius a and angular velocity $\boldsymbol{\Omega}$. Note this is *the* solution as Stokes flows solutions are unique (why? This is a little tricky and one may be better referring to a text rather than directly attempting this, eg Pozrikidis, Ch. 1, Boundary integral and singularity methods for linearized viscous flow, CUP).

Part c. One can use brute force, though that would be on the long side. Alternatively, let the stress field of the Stokes solution

$$u_i = \frac{1}{8\pi\mu}G_{ij}g_j$$

be given by $\sigma_{ij} = T_{ijp}g_p$. Why must it be of this form?

You can show

$$\frac{\partial T_{ijs}}{\partial x_j} = -\delta_{is}\delta(\hat{\mathbf{x}}),$$

from the momentum balance for u_i.

Then the stress field associated with $\mathbf{v} = a^3\mathbf{G}^c \cdot \boldsymbol{\Omega}$ is given by (why?)

$$\Sigma_{ij} = (8\pi\mu)\left(\frac{a^3\Omega_m}{2}\right)\epsilon_{mlj}\frac{\partial}{\partial x_{0,l}}T_{ijp}.$$

Hence the p^{th} component of the moment of the sphere due to the fluid is given by

$$M_p = \int_{Sphere} \epsilon_{pqr}x_q\Sigma_{rs}n_s\mathrm{d}S$$

which simplifies to the required answer.

Part 3. Question 5. Resistive force theory.

As in the notes, we have

$$6\pi\mu a\mathbf{U} = (\text{Drag force on flagellum}). \tag{7}$$

and $\mathbf{e}_t = (-1, \epsilon h_s)$, $\mathbf{e}_n = (\epsilon h_s, 1)$ and the velocity of the flagellum element is given by $\mathbf{U} = (U, V + \epsilon h_t)$. Hence the drag force per unit length on the element ds is given by

$$\begin{aligned}
\mathbf{f} &= -\left[C_N \mathbf{e}_n \cdot \mathbf{U}\mathbf{e}_n + C_T \mathbf{e}_t \cdot \mathbf{U}\mathbf{e}_t\right] = -\left[(C_N - C_T)\mathbf{e}_n \cdot \mathbf{U}\mathbf{e}_n + C_T\mathbf{U}\right] \\
&= -[(C_N - C_T)\mathbf{e}_n \otimes \mathbf{e}_n + C_T\mathbf{I}]\mathbf{U} \\
&= -\left[(C_N - C_T)\begin{pmatrix} \epsilon^2 h_s^2 & \epsilon h_s \\ \epsilon h_s & 1 \end{pmatrix} + C_T\mathbf{I}\right]\begin{pmatrix} U \\ \epsilon h_t + V \end{pmatrix} \\
&= -(C_N - C_T)\begin{pmatrix} \epsilon^2 h_s^2 U + \epsilon^2 h_s h_t + \epsilon h_s V \\ \epsilon h_s U + \epsilon h_t + V \end{pmatrix} - C_T\begin{pmatrix} U \\ \epsilon h_t + V \end{pmatrix}
\end{aligned}$$

Integrating over the flagellum length, $s \in [0, L]$, and using equation (7) we have

$$\begin{aligned}
6\pi\mu a\begin{pmatrix} U \\ V \end{pmatrix} &= -U\begin{pmatrix} C_T L + \epsilon^2(C_N - C_T)\int_0^L \mathrm{d}s h_s^2 \\ \epsilon(C_N - C_T)\int_0^L \mathrm{d}s h_s \end{pmatrix} - V\begin{pmatrix} \epsilon(C_N - C_T)\int_0^L \mathrm{d}s h_s \\ C_N L \end{pmatrix} \\
&\quad - \begin{pmatrix} \epsilon^2(C_N - C_T)\int_0^L \mathrm{d}s h_s h_t \\ \epsilon C_N \int_0^L \mathrm{d}s h_t \end{pmatrix}
\end{aligned}$$

Clearly the term $\epsilon^2(C_N - C_T)\int_0^L \mathrm{d}s h_s^2$ is a lower order than $C_T L$ and hence the former is dropped. Thus

$$(6\pi\mu a + C_T L)U = -(C_N - C_T)\left[\epsilon^2 \int_0^L \mathrm{d}s h_s h_t + \epsilon V \int_0^L \mathrm{d}s h_s\right].$$

Also

$$V = -\frac{1}{6\pi\mu a + C_N L}\left[\epsilon C_N \int_0^L \mathrm{d}s h_t + \epsilon U(C_N - C_T)\int_0^L \mathrm{d}s h_s\right].$$

Subsituting the expression for V into the expression for U we have

$$(6\pi\mu a + C_T L + O(\epsilon^2))U = -\epsilon^2(C_N - C_T)\left[\int_0^L \mathrm{d}s h_s h_t - \frac{C_N}{6\pi\mu a + C_N L}\int_0^L \mathrm{d}s h_s \int_0^L \mathrm{d}s h_t\right].$$

We can drop the $O(\epsilon^2)$ on the left as it is asymptotically small relative to $C_T L$.

Hence we have at leading order

$$U = \epsilon^2 \frac{C_T - C_N}{6\pi\mu a + C_T L}\left[\int_0^L \mathrm{d}s h_s h_t - \frac{C_N}{6\pi\mu a + C_N L}\int_0^L \mathrm{d}s h_s \int_0^L \mathrm{d}s h_t\right]$$

and we recover the expression in the lecture notes provided

$$\frac{C_N}{6\pi\mu a + C_N L}\frac{\int_0^L \mathrm{d}s h_s \int_0^L \mathrm{d}s h_t}{\int_0^L \mathrm{d}s h_s h_t} \ll 1.$$