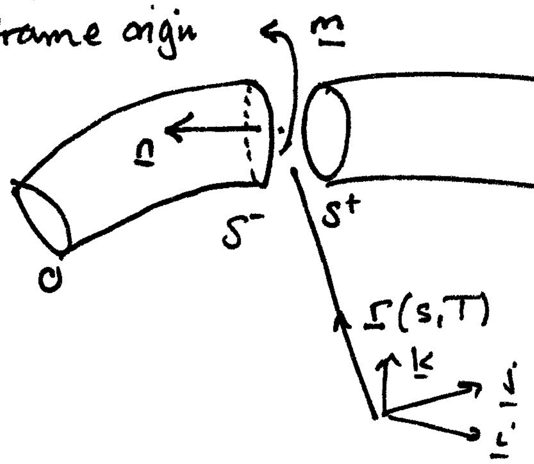
the rod section $(s,L]$ on the rod section $[0,s)$.

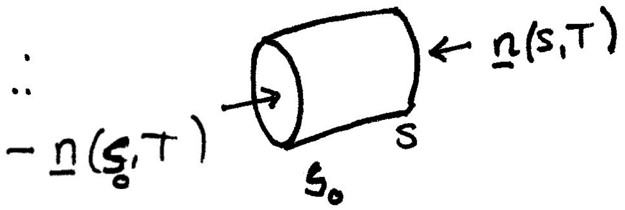
Let $\underline{m}(s,T) + \underline{r}(s,T) \wedge \underline{n}(s,T)$ be the resultant contact moment exerted by the rod section (s,L) on the section $[0,s)$, relative to the inertial frame origin



inertial reference frame.

Balance of Force

$\therefore$



On (s_0,s) with $s > s_0$, the total contact force is

$$\underline{n}(s,T) - \underbrace{\underline{n}(s_0,T)}_{\text{negated by Newton's 3rd}}$$

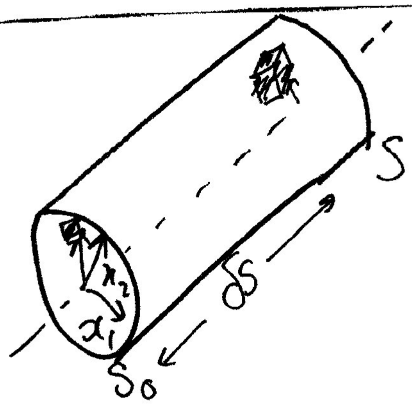
With $s = s_0 + \delta s$

$$\big(\underline{n}(s_0+\delta s,T) - \underline{n}(s_0,T)\big) + \underbrace{\underline{f}}_{\substack{\text{Body}\\ \text{Force per}\\ \text{unit length}}}\delta s = \rho A(s)\,\delta s\,\frac{\partial^2 \underline{r}(s,T)}{\partial T^2} + \text{h.o.t}$$

Dividing by δs and taking limits

$$\boxed{\frac{\partial \underline{n}}{\partial s} + \underline{f} = \rho A \frac{\partial^2 \underline{r}}{\partial T^2}}$$

$$\partial/\partial s\,(\underline{m} + \underline{r}\wedge\underline{n}) + \underline{L} + \underline{r}\wedge\underline{f} = \left(\begin{array}{l}\text{Rate of}\\ \text{Change of AngMtm per unit length}\end{array}\right)$$



$$\underline{r} = \underline{r_c}(s,T) + x_1\,\underline{d_1} + x_2\,\underline{d_2}$$

$$\underline{r}_{TT} = \underset{TT}{\underline{r_c}}(s,T) + x_1\,\underline{d_1}_{TT} + x_2\,d_{2TT}.$$

$$\therefore\quad \text{Ang mtm} = \int ds\left[\overbrace{\int dx_1 dx_2\,\underline{r}\wedge\rho}^{\leftarrow dV \rightarrow}\quad \underline{r}_T\right]$$

$$\therefore\ \frac{\partial}{\partial T}\left(\frac{\text{Ang Mtm}}{\text{Unit length}}\right) = \int dx_1 dx_2\,\rho\,\underline{r}\wedge\underline{r}_{TT}$$

$$= \rho\int dx_1 dx_2\,\underline{r}\wedge\underline{r}_{TT}$$

$$\begin{aligned}\underline{r}\wedge\underline{r}_{TT} &= \underline{r_c}\wedge\underline{r_c}_{TT} + x_1\,\underline{r_c}\wedge\underline{d}_{1TT} + x_2\,\underline{r_c}\wedge\underline{d_2}_{TT}\\ &+ x_1\,\underline{d_1}\wedge\underline{r_c}_{TT} + x_1^2\,\underline{d_1}\wedge d_{1TT} + x_1x_2\,\underline{d_1}\wedge d_{2TT}\\ &+ x_2\,\underline{d_2}\wedge r_{cTT} + x_1x_2\,\underline{d_2}\wedge d_{1TT} + x_2^2\,\underline{d_2}\wedge\underline{d_2}_{TT}\end{aligned}$$

$$\begin{aligned}\therefore\int dx_1dx_2\,\underline{r}_c\wedge\underline{r}_{cTT} &= A\,\underline{r_c}\wedge\underline{r_c}_{TT} + \left(\int dx_1\,x_1^2\int dx_2\right)\underline{d_1}\wedge d_{1TT}\\ &\quad + \left(\int dx_1dx_2\,x_2^2\right)\underline{d_2}\wedge\underline{d_2}_{TT}\\ &= A\,\underline{r}\wedge\underline{r}_{TT} + I_2\,\underline{d_1}\wedge\ddot{\underline{d_1}} + I_1\,\underline{d_2}\wedge\ddot{\underline{d_2}}\end{aligned}$$

$$\therefore\ \frac{\partial}{\partial s}(\underline{m} + \underline{r}\wedge\underline{n}) + \underline{L} + \underline{r}\wedge\underline{f} = \rho A\,\underline{r}\wedge\ddot{\underline{r}} + \rho I_2\,\underline{d_1}\wedge\ddot{\underline{d_1}} + I_1\,\underline{d_2}\wedge\ddot{\underline{d_2}}\,\rho$$

$$\therefore \left(\frac{\partial \underline{m}}{\partial s} + \frac{\partial \underline{r}}{\partial s} \wedge \underline{n} + \underline{l} \right) + \left(\underline{r} \wedge \overbrace{\left\{ \frac{\partial \underline{n}}{\partial s} + \underline{f} - \rho A \ddot{\underline{r}} \right\}}^{\text{Zero by LinMtm}} \right)$$

$$= \rho I_2 \, \underline{d}_1 \wedge \ddot{\underline{d}}_1 + \rho I_1 \, \underline{d}_2 \wedge \ddot{\underline{d}}_2$$

Balancing moments similarly gives

$$\frac{\partial \underline{m}}{\partial s} + \frac{\partial \underline{r}}{\partial s} \wedge \underline{n} + \underline{l} = \rho \left[I_2 \, \underline{d}_1 \wedge \ddot{\underline{d}}_1 + I_1 \, \underline{d}_2 \wedge \ddot{\underline{d}}_2 \right]$$

where $\dot{} \equiv d/dT$, and $I_1 = \int_{\text{Cross Section}} x_2^2 \, dx_1 dx_2$, $I_2 = \int_{\text{Cross Section}} x_1^2 \, dx_1 dx_2$,

$\int x_1 x_2 \, dx_1 dx_2 = 0$

assuming $\int x_1 \, dx_1 dx_2 = \int x_2 \, dx_1 dx_2 = 0$ by symmetry, and that ρ is constant. [Think circular cross section]

For a circular cross section $\underbrace{I_1 = I_2 = \frac{\pi a^4}{4}}_{:= I \text{ below}}$ where a is the radius.

Constitutive Laws

- Rod is inextensible and unshearable $\therefore$ $\underline{n}$ is not related to extension or shear (it must be found)

- Assume rod does not twist. Recall $\frac{\partial \underline{d}_i}{\partial s} = \underline{u} \wedge \underline{d}_i$

$$= \Big[\underbrace{u_1 \underline{d}_1 + u_2 \underline{d}_2}_{\substack{\text{Cause bending} \\ \text{(change } \underline{d}_3\text{)}}} + \underbrace{u_3 \underline{d}_3}_{\substack{\text{Rotates} \\ \underline{d}_1 \text{ and } \underline{d}_2 \\ \text{about } \underline{d}_3}} \Big] \wedge \underline{d}_i$$

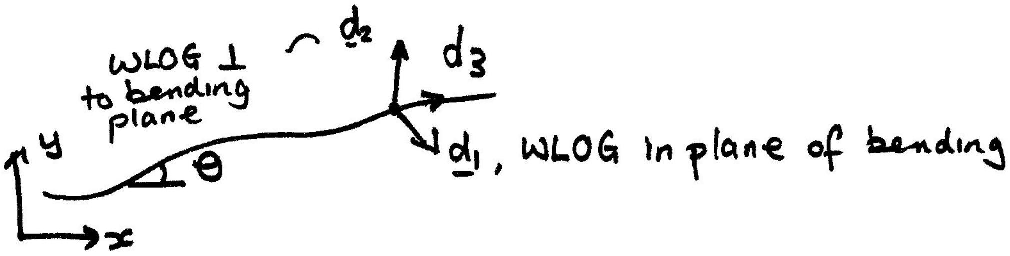
$\therefore$ Fix $u_3 = 0$

- Need to relate $\underset{\text{bending moment}}{\underline{m}}$ to $\underbrace{u_1 \underline{d}_1 + u_2 \underline{d}_2}_{\text{Causes bending}}$

Simplest $\quad \underline{m} = \underbrace{EI}_{\text{bending moment}} \left[u_1 \underline{d}_1 + u_2 \underline{d}_2 \right]$ for a circular... i.e. with symmetry

An example. The Planar Elastica

The rod is (i) Planar (ii) Unstretchable (iii) Unshearable (iv) does not twist, (v) has a circular cross section, and (vi) there is no body force or body moment.



$\therefore \underline{r} = \left(x(s,T), y(s,T), 0\right)$

$\underline{d}_3 = \dfrac{\partial \underline{r}}{\partial s} = \left(\cos(\theta(s,T)), \sin(\theta(s,T)), 0\right)$

$$0 = \frac{\partial \underline{d}_2}{\partial s} = \left[u_1 \underline{d}_1 + u_2 \underline{d}_2\right] \wedge \underline{d}_2 \quad \therefore u_1 \equiv 0 \quad \therefore \underline{u} = u_2 \underline{d}_2$$

$$\underline{d}_2 = \underline{k} \quad \therefore \quad \underline{d}_1 = \underline{d}_2 \wedge \underline{d}_3 = \underline{k} \wedge (\cos\theta, \sin\theta, 0) = (-\sin\theta, \cos\theta, 0)$$

This is also curvature.

$$\frac{\partial \underline{d}_3}{\partial s} = u_2 \underline{d}_2 \wedge \underline{d}_3 = u_2 \underline{d}_1 = \frac{\partial \theta}{\partial s}(-\sin\theta, \cos\theta, 0) \quad \therefore u_2 = \frac{\partial \theta}{\partial s}$$

$\dfrac{\partial \underline{d}_1}{\partial s} = u_2 \underline{d}_2 \wedge \underline{d}_1$ satisfied. Also $\underline{d}_1 \wedge \ddot{\underline{d}}_1 = \ddot{\theta} \underline{d}_2 = \ddot{\theta} \underline{k}$

Summary $\quad \underline{d}_1 = (-\sin\theta, \cos\theta, 0) \quad \underline{d}_2 = (0,0,1) \quad \underline{d}_3 = (\cos\theta, \sin\theta, 0)$

$\underline{u} = (0, \partial\theta/\partial s, 0)$

With $\underline{n} = (F(s,T), G(s,T))$ and $\underline{m} = EI\theta_s \underline{d}_2 = EI\theta_s \underline{k}$

we have

$$F' = \rho A \ddot{x} \quad , \quad G' = \rho A \ddot{y}$$

$$\underline{k}(EI\theta'' + \cos\theta\, G - \sin\theta\, F) = \rho I\, \underline{d}_1 \wedge \ddot{\underline{d}}_1 = \rho I \ddot{\theta} \underline{k}$$

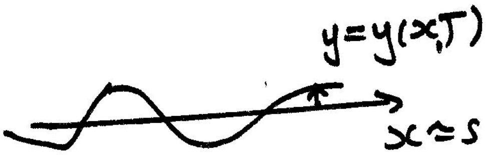
with $\dot{} \equiv d/dT \quad ' \equiv d/ds$

$$\therefore \quad \boxed{EI\,\theta'' + \cos\theta\, G - \sin\theta\, F = \rho I \ddot{\theta}}$$

Small angle reduction to beam equation

$$|\theta| \ll 1 \qquad \cos\theta \simeq 1 \qquad \sin\theta \simeq \theta \qquad \therefore \quad \frac{\partial x}{\partial s} \simeq 1 \quad \therefore \; x \simeq s.$$

$\therefore$ We can write



with $\dfrac{\partial y}{\partial s} \simeq \dfrac{\partial y}{\partial x} \simeq \theta$, and $\ddot{x} \approx \ddot{S} \equiv 0$.

$\therefore F = \text{Const},$

$$EI y''' + G - y'F = \rho I \ddot{y}'$$

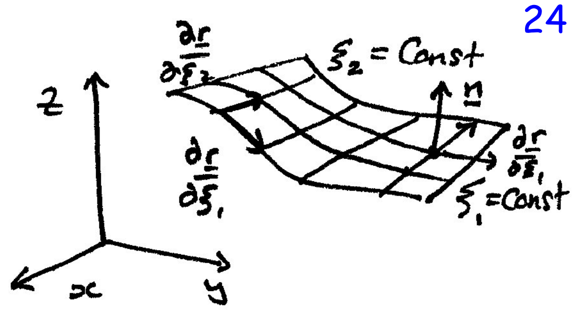
$\partial/\partial x$

$$EI y'''' + \rho A \ddot{y} - F y'' = \underbrace{\rho I \ddot{y}''}$$

Not often considered ... usually negligible ... inertia of cross section.

2. Biomembranes

2.1 Geometry of Surfaces.



Consider an orientable parametrised surface, Σ, via

$$\underline{x} = \underline{x}(\xi^1, \xi^2) \in \mathbb{R}^3 \quad \text{with } (\xi^1, \xi^2) \in M \subset \mathbb{R}^2.$$

We assume x is at least of class C^2 and such that

$$\underline{r}_i := \frac{\partial \underline{x}}{\partial \xi^i}$$

are linearly independent for all $(\xi^1, \xi^2) \in M$.

Since Σ is orientable we can define a normal

$$\underline{n} = \frac{\underline{r}_1 \wedge \underline{r}_2}{|\underline{r}_1 \wedge \underline{r}_2|}$$

and $\{\underline{r}_1, \underline{r}_2, \underline{n}\}$ form a basis.

<u>Surface Area</u> $\quad A = \int_\Sigma dS$

Recall (1^{st} year) $\quad d\underline{S} = \left(\frac{\partial \underline{r}}{\partial \xi_1} \wedge \frac{\partial \underline{r}}{\partial \xi_2}\right) d\xi_1 d\xi_2 = \underline{r}_1 \wedge \underline{r}_2 \, d\xi_1 d\xi_2$

$$dS = |d\underline{S}| = |\underline{r}_1 \wedge \underline{r}_2| \, d\xi_1 d\xi_2$$

Using the identity $(\underline{r}_1 \wedge \underline{r}_2)^2 = (\underline{r}_1 \wedge \underline{r}_2)\cdot(\underline{r}_1 \wedge \underline{r}_2) = \underline{r}_1^2 \underline{r}_2^2 - (\underline{r}_1 \cdot \underline{r}_2)^2$

we have
$$dS = \sqrt{\underline{r}_1^2 \underline{r}_2^2 - (\underline{r}_1 \cdot \underline{r}_2)^2}\, d\xi_1 \, d\xi_2$$

Definition Let $g_{ij} := \underline{r}_i \cdot \underline{r}_j = \frac{\partial \underline{x}}{\partial \xi^i} \cdot \frac{\partial \underline{x}}{\partial \xi^j}$. This is the metric tensor. Also define $G = (g_{ij})$ the matrix of metric tensor.

Then
$$dS = \sqrt{g_{11} g_{22} - g_{12}^2}\, d\xi_1 \, d\xi_2 = \sqrt{\det G}\, d\xi_1 \, d\xi_2$$

and
$$A = \int_M \sqrt{\det G}\, d\xi_1 \, d\xi_2$$

Arclength

$$ds^2 = \left| \underline{x}(\xi^1 + d\xi^1, \xi^2 + d\xi^2) - \underline{x}(\xi^1, \xi^2) \right|^2$$

unit
$$= \left| \underline{r}_1 d\xi^1 + \underline{r}_2 d\xi^2 \right|^2 = \underline{r}_1^2 (d\xi^1)^2 + \underline{r}_2^2 (d\xi^2)^2 + 2\underline{r}_1 \cdot \underline{r}_2 (d\xi^1 d\xi^2)$$

unit
$$= g_{ij}\, d\xi^i d\xi^j \quad \text{(with summation convention)}$$

known as first fundamental form

$$\therefore \quad L = \int_{t_{initial}}^{t_{final}} \sqrt{g_{ij} \frac{d\xi^i}{dt} \frac{d\xi^j}{dt}}\, dt$$

gives the arclength of curve $\underline{x}(\xi^1(t), \xi^2(t))$