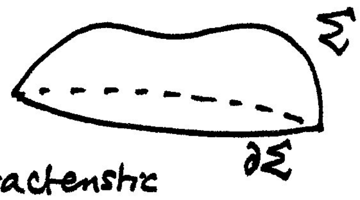
Gauss–Bonnet Theorem

Let Σ be a compact 2d surface with boundary $\partial\Sigma$.

Then

$$\underbrace{\int_{\Sigma}}_{\text{Gaussian Curvature}} K_G \, dS + \int_{\partial\Sigma} \underbrace{k_g}_{\text{Geodesic Curvature}} \, ds = 2\pi \underbrace{\chi(\Sigma)}_{\text{Euler Characteristic of } \Sigma}$$



$\chi(\Sigma) = 2 - 2p$ for surface of genus p.

[Sphere has genus 0, Torus has genus 1, etc.]

For a closed, orientable surface

$$\int_{\Sigma} K_G \, dS = 2\pi \chi(\Sigma) = 4\pi(1-p),$$ where p is the genus of Σ.

3 Fluid Biomembranes

Canhan (1970) Evans (1973) Helfrich (1973)

Many biological membranes are made of lipid bilayers. Mechanically, these structures resist bending, stretching but are fluid in the plane and as such do not resist shear. In this Chapter we consider the model of Canham (1970)-Helfrich (1973)-Evans (1973) to describe the response of membranes under pressure.

3.1 Biomembrane model

Assumptions

- Biomembrane is sufficiently thin, that it can be represented as a surface, Σ.
- Biomembrane offers no resistance to shear (shearless) but resists bending and stretching
- Free energy given by (Helfrich 1973)

Intrinsic mean curvature — saddle-splay modulus

$$\mathcal{E}_{BM} = \int_\Sigma dS \left(\gamma + 2K(H-H_0)^2 + \bar{K} K_G \right)$$

Surface Tension — mean curvature — Gaussian curvature

Bending modulus (Confusing notation, but standard).

Energy density as $H = \frac{k_1+k_2}{2}$ $\sim K/_2 (k_1^2+k_2^2) \sim 2KH^2$

Aside (Discussion only.) We intersect disciplines, Physics, Analytical Mechanics, Engineering, Biology... notation conflicts occur. Above is a biological membrane ... not the same as a membrane in analytical mechanics or engineering... be careful with textbooks. See Notes for more details.

For a closed surface $\int_\Sigma K_G \, dS$ is constant if there is no topological change (or membrane heterogeneity) (from Gauss-Bonnet theorem). Hence, typically $\bar{K} K_G$ term doesn't have a mechanical effect.

Note

$[\kappa] \sim \text{Energy}$

$[\gamma] \sim \dfrac{\text{Energy}}{(\text{Length})^2}$

$\therefore \quad \lambda_{tb} := (\kappa/\gamma)^{1/2}$ characteristic length at which both tension and bending important.

$\therefore \quad L \gg \lambda_{tb}$ Surface tension dominant

Lengthscale of variation in membrane

$L \ll \lambda_{tb}$ Bending dominant.

Typically (lipid bilayers)

$$\left.\begin{aligned} \kappa &\simeq 10^{-19}\,\text{J} \\ \gamma &\simeq 10^{-3}\,\text{N/m} = 10^{-3}\,\text{J/m}^2 \end{aligned}\right\} \lambda_{tb} \simeq \sqrt{\frac{10^{-19}}{10^{-3}}} \simeq 10^{-8}\,\text{m}$$

$\therefore$ For $L > \lambda_{tb} \simeq 10\,\text{nm}$, surface tension energy dominates.

Discuss only. Thus (in the absence of other constraints) the membrane will form an approximate sphere. However, we cannot neglect bending... we have a singularly perturbed problem in general (to be seen).

To find membrane shape... minimise energy subject to any imposed constraints

Example constraints

Constant Volume. Minimise $\mathcal{E}_P = \mathcal{E}_{BM} - P(V - V_0)$

Lagrange Multiplier (identified as pressure as Work $\sim p\,\delta V$).

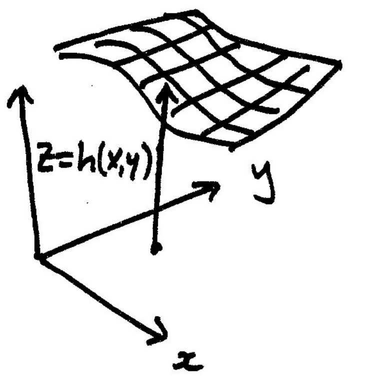
Constant Pressure. Minimise $\mathcal{E}_V = \mathcal{E}_{BM} - VP$

Lagrange Multiplier

$\mathcal{E}_V, \mathcal{E}_P$ share same extrema, but whether they are minima need not coincide.

3.2 Minimisation for the Monge Representation.

Simple, Important case



Surface can be represented as

$$z = h(x,y) \in C^2,$$

i.e. the Monge representation.

$$\therefore \quad \underline{r} = (x, y, h(x,y)) \qquad (x,y) \in U \subset \mathbb{R}^2$$

$$\frac{\partial \underline{r}}{\partial x} = (1, 0, h_x) \qquad \frac{\partial \underline{r}}{\partial y} = (0, 1, h_y)$$

sign by choice.

$$\frac{\frac{\partial \underline{r}}{\partial x} \wedge \frac{\partial \underline{r}}{\partial y}}{\left|\frac{\partial \underline{r}}{\partial x} \wedge \frac{\partial \underline{r}}{\partial y}\right|} = -\underline{n} = \frac{(-h_x, -h_y, 1)}{\sqrt{1 + h_x^2 + h_y^2}}$$

$$G = \begin{pmatrix} \left(\frac{\partial \underline{r}}{\partial x^1}\right)^2 & \frac{\partial \underline{r}}{\partial x^1}\cdot\frac{\partial \underline{r}}{\partial x^2} \\ \frac{\partial \underline{r}}{\partial x^1}\cdot\frac{\partial \underline{r}}{\partial x^2} & \left(\frac{\partial \underline{r}}{\partial x^2}\right)^2 \end{pmatrix} \qquad \begin{aligned} x^1 &\equiv x \\ x^2 &\equiv y \end{aligned}$$

$$\therefore \quad G = \begin{pmatrix} 1+h_x^2 & h_x h_y \\ h_x h_y & 1+h_y^2 \end{pmatrix} \qquad G^{-1} = \frac{1}{1+h_x^2+h_y^2}\begin{pmatrix} 1+h_y^2 & -h_x h_y \\ -h_x h_y & 1+h_x^2 \end{pmatrix}$$

$$\therefore \; -\underline{n} = \frac{-\nabla h + \underline{e}_z}{\sqrt{\det G}} \qquad \det G = 1+h_x^2+h_y^2 .$$

contracted and similarly for other components.

$$\therefore K_{ij} = -\underline{n}\cdot\frac{\partial^2 \underline{x}}{\partial x^i \partial x^j} = -\frac{(h_x, h_y, -1)}{\sqrt{1+h_x^2+h_y^2}} \cdot \left[\begin{pmatrix} & \\ & 0 & \\ & \end{pmatrix}_{ij}, \begin{pmatrix} & \\ & 0 & \\ & \end{pmatrix}_{ij}, \begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}_{ij} \right]$$

$$= \frac{1}{\sqrt{1+h_x^2+h_y^2}}\begin{pmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{pmatrix}_{ij}$$

$$\therefore L = G^{-1}K = \frac{1}{(1+h_x^2+h_y^2)^{3/2}} \left(\begin{array}{c:c} h_{xx}(1+h_y^2) - h_{xy}h_x h_y & h_{xy}(1+h_x^2) - h_{xx}h_x h_y \\ \hdashline h_{xy}(1+h_x^2) - h_{xx}h_x h_y & h_{yy}(1+h_x^2) - h_{xy}h_x h_y \end{array}\right)$$

$$K_G = \det L = \det G^{-1}K = \det G^{-1}\det K = \frac{\det K}{\det G}$$

$$= \frac{1}{(\det G)^2}\left(h_{xx}h_{yy} - h_{xy}^2\right)$$

$$H = \frac{1}{2}\operatorname{tr} L = \frac{1}{2(\det G)^{3/2}}\left[h_{xx}(1+h_y^2) + h_{yy}(1+h_x^2) - 2h_{xy}h_x h_y\right]$$

Also with $\nabla := \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y}$, 2D gradient, we have

$$\det G = 1 + (\nabla h)^2 \qquad \underline{n} = \frac{-(-\nabla h, 1)}{\sqrt{1+(\nabla h)^2}}$$

From here can solve
Eg Sheet 2 Q7

Simple Case. Surface Tension only

If $\kappa = \bar{\kappa} = H_0 = 0$ and γ is constant, the energy is the energy of a surface with constant surface tension:

$$\mathcal{E} = \gamma \int_\Sigma dS .$$

Minimisation (Monge representation)

$$\mathcal{E} = \gamma \int \sqrt{\det G}\, dx^1 dx^2 = \gamma \int \underbrace{\sqrt{1 + h_{x^1}^2 + h_{x^2}^2}}_{\mathcal{L}(h, h_x, h_y)}\, dx^1 dx^2$$

Euler Lagrange ($x^1 \equiv x$, $x^2 \equiv y$)

$$\left.\frac{\partial}{\partial x}\right|_y \left(\frac{\partial \mathcal{L}}{\partial (h_x)} \right) + \left.\frac{\partial}{\partial y}\right|_x \left(\frac{\partial \mathcal{L}}{\partial (h_y)} \right) - \frac{\partial \mathcal{L}}{\partial h} = 0 .$$

$$\therefore \quad \frac{\partial}{\partial x}\left(\frac{h_x}{\sqrt{1 + h_x^2 + h_y^2}} \right) + \frac{\partial}{\partial y}\left(\frac{h_y}{\sqrt{1 + h_x^2 + h_y^2}} \right) = 0$$

$$\therefore \quad \nabla \cdot \left(\frac{\nabla h}{\sqrt{\det G}} \right) = 0$$ where $\nabla = \underline{e}_x \frac{\partial}{\partial x} + \underline{e}_y \frac{\partial}{\partial y}$ is the 2D nabla

$\therefore$ $H = 0$ or equivalently $\nabla \cdot \underline{n} = 0$ $\therefore$ Surface has zero mean curvature

discuss N.B a solution a priori may be extremal but not minimal.