Small gradient approximation

We instead impose $\bar{\kappa} = H_0 = 0$ and consider small gradients, dropping sub-leading terms in gradients of h.

$$\therefore \sqrt{\det G} = (1 + h_x^2 + h_y^2)^{1/2} \simeq 1 + \tfrac{1}{2}(h_x^2 + h_y^2) + O(\nabla h^3).$$

$$(2H)^2 = \frac{1}{(\det G)^3}\left[h_{xx} + h_{yy} + O(\nabla h)^4\right]^2 = (h_{xx} + h_{yy})^2 (1 + O(\nabla h^2))$$

$$\therefore \quad \mathcal{E}_{BM} = \int_\Sigma dx\,dy \underbrace{\sqrt{\det G}}_{(1+\frac{1}{2}(\nabla h)^2)} \underbrace{\{\gamma + 2\kappa H^2\}}_{\{\gamma + \frac{\kappa}{2}(\nabla^2 h)^2\}} \, dx\,dy \quad + \text{h.o.t}$$

$$= \int_\Sigma dx\,dy \left\{ \gamma + \frac{\gamma}{2}(\nabla h)^2 + \frac{\kappa}{2}(\nabla^2 h)^2 + \text{h.o.t} \right\}$$

$$= \int_\Sigma dx\,dy \left\{ \gamma + \frac{1}{2}\left[\gamma(\nabla h)^2 + \kappa(\nabla^2 h)^2\right]\right\} \quad + \text{h.o.t.}$$

Again, constants do not alter minimisation
∴ Shift energy

∴ Only need to consider

$$\mathcal{E} = \frac{1}{2}\int_\Sigma dx\,dy \left[\gamma(\nabla h)^2 + \kappa(\nabla^2 h)^2\right]$$

We find extrema of $\mathcal{E}$ for variations in h:

$$\mathcal{E}[h+\delta h]-\mathcal{E}[h] = \frac{1}{2}\int_{\Sigma} dx\,dy \left[\gamma\left\{(\nabla h+\nabla\delta h)^2-(\nabla h)^2\right\} + \kappa\left\{(\nabla^2 h+\nabla^2\delta h)^2-(\nabla^2 h)^2\right\}\right]$$

$$= \frac{1}{2}\int_{\Sigma} dx\,dy \;\; 2\gamma\{\underbrace{\nabla h\cdot\nabla\delta h}_{\nabla\cdot(\nabla h\cdot\delta h)-\nabla^2 h\,\delta h}\} + 2\kappa\{\underbrace{\nabla^2 h\;\nabla^2\delta h}_{\nabla\cdot(\nabla^2 h\;\nabla\delta h)-(\nabla\delta h\cdot\nabla\nabla^2 h)}\} \quad + h.o.t$$

$$= \int_{\Sigma} dx\,dy \;\; -(\nabla^2 h)\delta h\,\gamma - \kappa\,\underbrace{\nabla\delta h\cdot(\nabla\nabla^2 h)}_{\nabla\cdot(\delta h\nabla\nabla^2 h)-\delta h\nabla^4 h} + \oint_{\partial\Sigma} ds \;\; \gamma(\underline{N}\cdot\nabla h)\delta h + \kappa\nabla^2 h\;\underline{N}\cdot\nabla\delta h$$

N is the outer normal, of the projected surface contour, on the x-y plane

$$= \int_{\Sigma} dx\,dy\;\delta h\left[-\gamma\nabla^2 h+\kappa\nabla^4 h\right] + \oint_{\partial\Sigma} ds\;\underline{N}\cdot\Big(-\nabla(\nabla^2 h)\kappa\delta h + \kappa(\nabla\delta h)\nabla^2 h + (\gamma\nabla h)\delta h\Big)$$

Give (A) Gives (B)

∴ For an extremum we require

i) $\nabla^4 h - \frac{1}{\lambda^2}\nabla^2 h = 0 \qquad \lambda = \sqrt{\frac{\kappa}{\gamma}}$

{ Note if $\lambda \ll 1$ we have a singular perturbation problem.

ii) The Boundary conditions.

The Boundary variation must also be zero. We need one condition from (A) and one condition from (B).

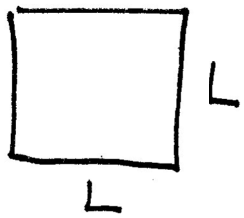
(A) { Either $\delta h = 0$ on $\partial\Sigma$ or $\underline{N}\cdot\left(-\lambda^2\nabla(\nabla^2 h)+\nabla h\right)=0$ on $\partial\Sigma$

(B) { Either $\nabla^2 h = 0$ on $\partial\Sigma$ or $\underline{N}\cdot\nabla\delta h = 0$

Example. Flicker Spectroscopy

- Study biomembrane mechanics by measuring spectrum of thermal undulations via light microscopy.

- Small gradients $\Rightarrow$ Linear equations $\therefore$ Fourier methods
- Consider square membrane (closed surfaces analysed in practice).



$$h = h(x, y) = h(\underline{r}) = \sum_{\underline{q}} e^{i\underline{q}\cdot\underline{r}} h_{\underline{q}}$$

$$\underline{q} = \frac{2\pi}{L}(n_x, n_y) \; ; \quad n_x, n_y \in \mathbb{Z}.$$

$$h \text{ real } \therefore \quad h_{-\underline{q}} = h^*_{\underline{q}}$$

$$\nabla h = \sum i\underline{q}\, h_{\underline{q}}\, e^{i\underline{q}\cdot\underline{r}} \qquad (\nabla h)^2 = \sum_{\underline{q},\underline{q}'} -\underline{q}\cdot\underline{q}'\, h_{\underline{q}} h_{\underline{q}'}\, e^{i\underline{r}\cdot(\underline{q}+\underline{q}')}$$

$$\nabla^2 h = \sum -\underline{q}^2 h_{\underline{q}}\, e^{i\underline{q}\cdot\underline{r}} \qquad (\nabla^2 h)^2 = \sum_{\underline{q},\underline{q}'} \underline{q}^2 \underline{q}'^2 h_{\underline{q}} h_{\underline{q}'}\, e^{i\underline{r}\cdot(\underline{q}+\underline{q}')}$$

$$\therefore \; \mathcal{E} = \frac{1}{2}\int dx\,dy \sum' h_{\underline{q}} h_{\underline{q}'}\, e^{i\underline{r}\cdot(\underline{q}+\underline{q}')} \left\{ \kappa\, \underline{q}^2 \underline{q}'^2 - \gamma\, \underline{q}\cdot\underline{q}' \right\}$$

$$\int dx\,dy\, e^{i\underline{r}\cdot(\underline{q}+\underline{q}')} = L^2 \delta_{\underline{0},\, \underline{q}+\underline{q}'}$$

$$= \frac{L^2}{2} \sum_{\underline{q}} h_{\underline{q}} h_{-\underline{q}} \left\{ \kappa q^4 + \gamma q^2 \right\} = \frac{L^2}{2} \sum_{\underline{q}} |h_{\underline{q}}|^2 (\kappa q^4 + \gamma q^2)$$

as $h_{-\underline{q}} = h^*_{\underline{q}}$

Equipartition

$$\langle |h_p|^2 \rangle = \frac{1}{Z}\int \underbrace{D[h]}_{\text{Not in scope... covers all } h} |h_p|^2 e^{-\beta \mathcal{E}}$$

short-hand $p = \frac{2\pi}{L}(p_1, p_2)$

short-hand $m = \frac{2\pi}{L}(m_1, m_2)$

$f(q) = \kappa q^4 + \gamma q^2$

$$= \frac{1}{Z}\int \prod_m d|h_m| d\theta_m |h_m| \left\{ |h_p|^2 e^{-\frac{\beta L^2}{2}\sum_q |h_q|^2 f(q)} \right\}$$

all h covered by all allowed values of $h_{2\pi/L(1,0)}, h_{2\pi/L(2,0)}, \dots$

with $Z = \int \prod_m d|h_m| d\theta_m |h_m| e^{-\frac{\beta L^2}{2}\sum_q |h_q|^2 f(q)}$

Cover Argand diagram for h_m by $|h_m|$ and $\theta = \arg(h_m)$

$\therefore d|h_m| d\theta |h_m|$

$$\therefore \langle |h_p|^2 \rangle = \frac{\cancel{2\pi}\int d|h_p| \; |h_p|^3 e^{-\beta L^2/2 |h_p|^2 f(p)}}{\cancel{2\pi}\int d|h_p| \{e^{-\beta L^2/2 |h_p|^2 f(p)}\} |h_p|}$$

for a fixed single value of $p = \frac{2\pi}{L}(p_1, p_2)$

$$= \frac{-1}{L^2/2 \, f(p)} \; \frac{\partial/\partial\beta \int d|h_p| \; e^{-\beta L^2/2 |h_p|^2 f(p)} |h_p|}{\int d|h_p| \{e^{-\beta L^2/2 |h_p|^2 f(p)}\} |h_p|}$$

$$= \frac{-1}{L^2/2 \, f(p)} \frac{\partial}{\partial\beta} \underbrace{\ln \int d|h_p| e^{-\beta L^2/2 |h_p|^2 f(p)} |h_p|}_{\frac{\partial}{\partial\beta}\ln\left(\frac{\text{Const}}{\beta}\right) = -\frac{1}{\beta}}$$

$$= \frac{1}{L^2/2 \, f(p)} k_b T$$

$$\therefore \boxed{\langle |h_p|^2 \rangle = \frac{2 k_b T}{L^2(\kappa p^4 + \gamma p^2)}}$$

4 Axisymmetric Membranes and Shells

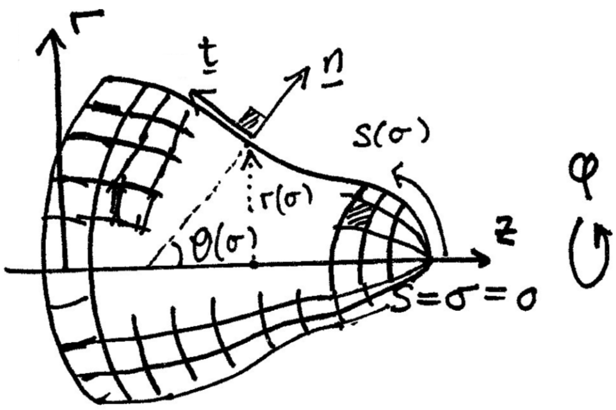
4.1 Elastic membranes with linear constitutive laws.

We consider a membrane which

- is axisymmetric
- is filled with an incompressible viscous fluid under pressure, P
- does not support shear
- remains axisymmetric under deformation.
- is made of an incompressible and elastic material

As usual we consider kinematics, mechanics and constitutive laws for model formulation.

4.1.1 Kinematics



$\underline{n}$: unit normal.

$\underline{t} = \underline{e}_s$, unit vector in direction of increasing $\underline{e}_s$

σ : material parameter, arclength before deformation.

s : Arclength

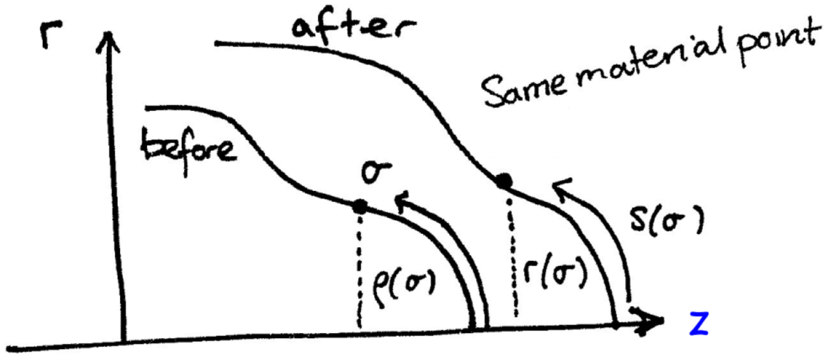
θ: Angle between normal and axis of symmetry

r: Distance of surface from axis of symmetry

By trigonometry $\frac{dr}{ds} = \cos\theta$, $\frac{dz}{ds} = -\sin\theta$

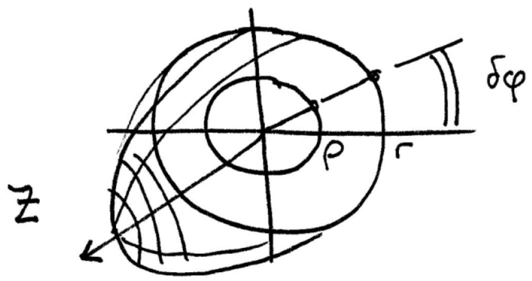
The principal curvatures are $\kappa_s = \frac{\partial\theta}{\partial s}$, $\kappa_\varphi = \frac{\sin\theta}{r}$.

Stretch Variables



$$\lambda_s = \frac{\partial s}{\partial \sigma}$$, Stretch ratio in s direction

Note



Stretch ratio in φ direction is $\frac{r\,\delta\varphi}{\rho\,\delta\varphi} = \frac{r}{\rho} = \lambda_\varphi$

Hence $$\lambda_\varphi = \frac{r(\sigma)}{\rho(\sigma)}$$, Stretch ratio in φ direction

4.1.2 Mechanics

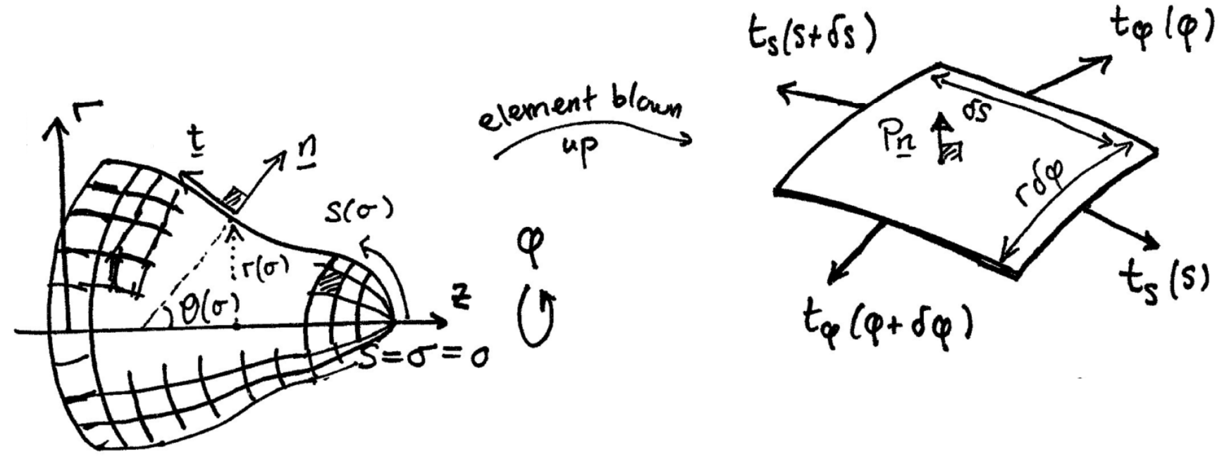
Consider an element of the membrane associated with $[s, s+\delta s]$ and $[\varphi, \varphi+\delta\varphi]$, subject to

i) Pressure, $P\underline{n}$, due to pressurised internal fluid.

ii) Tension on surface in direction $\underline{t} \equiv \underline{e}_s$, denoted t_s

iii) Tension on surface in direction $\underline{e}_\varphi$, unit vector in direction of increasing φ and denoted t_φ.

iv) A force per unit area, eg. due to exterior fluid movement. By axisymmetric this is of the form $f\underline{e}_s$.



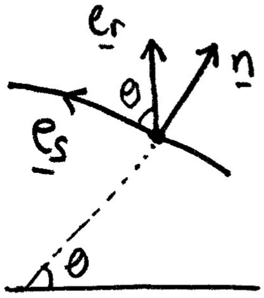
Force Balance, neglecting inertia (i.e. accelerations cf. $\underline{F} = \cancel{m\underline{a}} = 0$)

$$\delta s\,\delta\varphi\left[\frac{\partial}{\partial s}\left(r t_s \underline{e}_s\right) + \frac{\partial}{\partial \varphi}\left(t_\varphi \underline{e}_\varphi\right) + rP\underline{n} + rf\underline{e}_s\right] = 0$$

(neglecting higher order terms).

Note $\quad \dfrac{\partial t_\varphi}{\partial \varphi} = 0$

$$\frac{\partial \underline{e}_\varphi}{\partial \varphi} = -\underline{e}_r, \text{ unit vector in direction of decreasing } r$$



$$= -\cos\theta\, \underline{e}_s - \sin\theta\, \underline{n}$$

$$\frac{\partial \underline{e}_s}{\partial s} = \frac{\partial \underline{t}}{\partial s} = -K_s \underline{n}$$

by definition of curvature, using $\underline{t} = (-\sin\theta, \cos\theta)$, $\underline{n} = (\cos\theta, \sin\theta)$ in z–r coordinates.

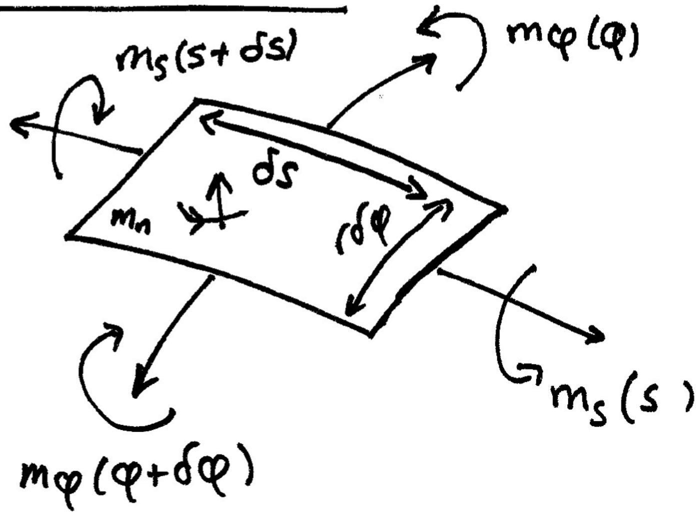
$$\therefore \quad \underline{e}_s \left[\frac{\partial}{\partial s}(r t_s) - t_\varphi \cos\theta + r f \right] + \underline{n} \left[-r t_s K_s - t_\varphi \sin\theta + r P \right] = 0$$

$$\therefore \quad P = t_s K_s + t_\varphi \frac{\sin\theta}{r} = t_s K_s + t_\varphi K_\varphi$$

$$\frac{\partial}{\partial s}(r t_s) = t_\varphi \cos\theta - r f = t_\varphi \frac{\partial r}{\partial s} - r f$$

give the momentum balances.

Moment balance



$$\delta s \delta\varphi \left[\frac{\partial}{\partial s}(r m_s \underline{e}_s) + \frac{\partial}{\partial \varphi}(m_\varphi \underline{e}_\varphi) + r m_n \underline{n} \right] = 0$$

In $\underline{e}_s$ direction, noting $\dfrac{\partial \underline{e}_\varphi}{\partial \varphi} = -\cos\theta \, \underline{e}_s - \sin\theta \, \underline{n}$

$$\boxed{\frac{\partial}{\partial s}(r m_s) - m_\varphi \cos\theta = 0}$$

$\underline{e}_\varphi$ direction

$\dfrac{\partial m_\varphi}{\partial \varphi} = 0$

$\underline{n}$ direction gives m_n in terms of m_s, m_φ